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Variational formulation of the electrodynamics of fluids and its application to the radiation pressure problem

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A covariant formulation of classical electrodynamics is given for nonviscous, compressible, nondispersive, polarizable, and magnetizable fluids. The part of the Lagrangian density accounting for the various interactions to be existent in such fluids under the influence of the electromagnetic fields is constructed rather intuitively by introducing two four-vector variables, one to describe the states of polarization and the other to describe those of magnetization. The energy-momentum conservation law can be derived separately for the material and the field subsystems. The energy-momentum tensor of the total system cannot be split in a unique way into the material and the field parts. However, physically reasonable ways of splitting can be found so as to give one and the same result as to the electromagnetic radiation pressure in electromagnetic fluids. This pressure is shown to be $(\vec{D}\times\vec{E})/4\pi(\mu\nu)^{1/2}$, in agreement with experiment. This indicates that the pressure observed in experiment consists almost entirely of the "true" electromagnetic radiation pressure. For the sake of consistency, the mechanical pressure arising both from the electro- and magnetostrictive forces and the forced motions of the fluid particles under the influence of the electromagnetic radiation fields is also estimated and found, in fact, to be negligible in ordinary experimental conditions.

I. INTRODUCTION

There has been a long-standing controversy as to the proper identification of the electromagnetic momentum density in polarizable and magnetizable media. The question is whether this momentum density has the form $(\vec{D}\times\vec{B})/4\pi c$, as suggested by Minkowski,[1] or $(\vec{E}\times\vec{H})/4\pi c$, as suggested by Abraham.[2] The problem has been discussed on numerous occasions, but here only a few examples of such discussions are referred to. Laue[3] and Møller[4] support the Minkowski form, while Landau and Lifshitz[5] adopt the Abraham expression for the reason that this conforms to a symmetric energy-momentum tensor. Pauli appears to have supported Minkowski in his latest publication on relativity.[6]

In more recent times, most authors are inclined to adopt Abraham's form. However, Jones and Richards's observation[7] of the light pressure on objects immersed in refractive liquids was in excellent agreement with the prediction based on Minkowski's momentum density. More recent experiments of Ashkin and Dziedzic[8] on the radiation pressure exerted on a free liquid surface also provided a result in support of Minkowski's assumption. Haus[9] and Gordon,[10] who accept Abraham's electromagnetic momentum density, argue that the mechanical momentum supported by the material may have appreciable contribution to the light forces in liquids through material-material contacts. Burt and Peierls,[11] who accept Abraham's assumption but reject the possible existence of measurable contributions from the atomic motions in liquids in the case of Jones and Richards's experiments, leave the discrepancy between theory and experiment unsolved.

In order to settle these issues, an exact formulation of electrodynamics of moving media is indispensable. The first part of this paper will be devoted to the establishment of the theory of nonviscous, compressible, nondispersive, polarizable, and magnetizable fluids. Application of Hamilton's principle proves to be the best way of tackling this kind of subject, in view of consistency and freedom from errors. The only disadvantage of this method is the difficulty in finding a logical means for setting up the Lagrangian density. Therefore the Lagrangian density, which is proposed and used in this paper, had to be found rather intuitively. For the system under consideration, however, the situation is not very fatal, because some of the basic equations here are known beforehand and can be utilized *a posteriori* for the justification of the postulated Lagrangian density.

The main results obtained from the basic equations thus derived in this paper are as follows: The energy-momentum conservation law can be derived independently for the material and the field subsystem. This enables one to define the energy-momentum tensors for the respective subsystems, although not uniquely. The tensor for the total system is obtained as the sum of those for subsystems. In spite of some sort of indefiniteness of individual subsystem tensors, the density of the electromagnetic momentum flux and therefore the radiation pressure on opaque bodies immersed in material media can be found to equal Minkowski's momentum density multiplied by the velocity of light in media, in accordance with experiment. Abraham's energy-momentum tensor can be accepted as well and can predict the same radiation pressure as Minkowski's tensor, although Abraham's field momentum density differs from Minkowski's. This is because the same density of the electromagnetic momentum flux as Minkowski's can be found to result if Abraham's tensor is assumed and because the radiation pressure is determined by the density of the flux, not by the density of the momentum.

II. LAGRANGIAN DENSITY

The Lagrangian density in this problem consists formally of three parts: the material part $L^{(m)}$, the field part $L^{(f)}$, and the part owing to interactions, $L^{(i)}$. Interactions between dipole moments induced under the influence of the fields are included in the last part, together with the interactions between the material and the fields.

First, the material part is considered. The motion of the particle (infinitesimal part) of the fluid can be described in a four-dimensional space by introducing a position four-vector $x_\mu = (\vec{x}, ict)$. Greek subscripts are used throughout this paper to indicate four-vector components or four-tensor elements, and Latin subscripts indicate three-vector components. The velocity of the particle $\vec{v}(x_\mu)$ multiplied by the mass density $\rho(x_\mu)$, together with the fourth component $i\rho c$, forms a four-vector of mass flow density,

$$J_\mu = (\rho\vec{v}, i\rho c). \tag{1}$$

Let ρ_0 be the density in the local rest frame; then ρ is related to this by

$$\rho = \gamma\rho_0, \quad \gamma = [1-(v/c)^2]^{-1/2}. \tag{2}$$

The Lagrangian density $L^{(m)}$ then takes the form

$$L^{(m)} = -(-J_\mu^2)^{1/2} - \epsilon_i(\rho_0, s), \tag{3}$$

where $\epsilon_i(\rho_0, s)$ is the specific internal energy of nonelectromagnetic nature and is assumed to be a function of ρ_0 and the specific elastic entropy s. The specific entropy is assumed to be a four-scalar function of x_μ. On the right-hand side of (3) the summation over μ is implied for J_μ^2. Repetition of a Greek index will always imply the summation from 1 to 4.

Next, the form of the field part $L^{(f)}$ will be established. In a relativistically covariant formulation, the most convenient field variables are those of the potential four-vector

$$A_\mu = (\vec{A}, i\phi),$$

where $\vec{A}$ and ϕ are the usual vector and scalar potentials. The field intensities $\vec{E}$ and $\vec{B}$ are expressed in terms of potentials in the form

$$\vec{E} = -\frac{1}{c}\frac{\partial\vec{A}}{\partial t} - \vec{\nabla}\phi, \quad \vec{B} = \text{curl}\vec{A}. \tag{4}$$

The first pair of Maxwell's equations,

$$\text{curl}\vec{E} = -\frac{1}{c}\frac{\partial\vec{B}}{\partial t}, \quad \text{div}\vec{B} = 0, \tag{5}$$

then follows automatically. With the help of the four-tensor defined by

$$F_{\mu\nu} = \frac{\partial A_\nu}{\partial x_\mu} - \frac{\partial A_\mu}{\partial x_\nu}, \tag{6}$$

Maxwell's equations (5) can be cast into a single tensor equation,

$$\frac{\partial F_{\mu\nu}}{\partial x_\gamma} + \frac{\partial F_{\nu\gamma}}{\partial x_\mu} + \frac{\partial F_{\gamma\mu}}{\partial x_\nu} = 0.$$

It is obvious from (4) that

$$F_{\mu\nu} = \begin{bmatrix} 0 & B_3 & -B_2 & -iE_1 \\ -B_3 & 0 & B_1 & -iE_2 \\ B_2 & -B_1 & 0 & -iE_3 \\ iE_1 & iE_2 & iE_3 & 0 \end{bmatrix}. \tag{7}$$

As is well known, the field part of the Lagrangian density is now written in the form[12]

$$L^{(f)} = -(1/16\pi)F_{\gamma\delta}^2. \tag{8}$$

The final and most important task in this section is to find the form of the contributions from interactions. The following is a rather intuitively derived expression of this part of the Lagrangian: If the material is polarizable and magnetizable, then it is necessary to introduce the variables which are capable of describing the states of polarization and magnetization. Here, a four-vector p_μ is introduced to describe polarization, and another four-vector m_μ to describe magnetization. The Lagrangian density $L^{(i)}$ is now assumed to be given in the following form:

$$L^{(i)} = A_\mu j_\mu + \frac{1}{4\alpha(\rho_0, s)}P_{\mu\nu}^2 - A_\mu\frac{\partial P_{\mu\nu}}{\partial x_\nu} - \frac{1}{4\beta(\rho_0, s)}M_{\mu\nu}^{*2} - A_\mu\frac{\partial M_{\mu\nu}^*}{\partial x_\nu}, \tag{9}$$

where the first term indicates the interaction between free charges and the fields, the second and third the electric dipole-dipole and dipole-field interactions, and the fourth and fifth the magnetic dipole-dipole and dipole-field interactions, respectively. The charge-current four-vector for free charges is denoted by $j_\mu = (\rho_E\vec{v}, i\rho_E c)$, where ρ_E is the charge density and the ratio ρ_E/ρ is assumed to be a constant. There have been introduced two new tensors $P_{\mu\nu}$ and $M_{\mu\nu}$, which are defined by the equations

$$P_{\mu\nu} = (1/c)\,(J_\mu p_\nu - J_\nu p_\mu)\,,$$

$$M_{\mu\nu} = (1/ic)(J_\mu m_\nu - J_\nu m_\mu)\,. \qquad (10)$$

Both are antisymmetric tensors, and $M^*_{\mu\nu}$ appearing in (9) is the pseudotensor dual to $M_{\mu\nu}$, i.e.,

$$M^*_{\mu\nu} = \tfrac{1}{2} e_{\mu\nu\gamma\delta} M_{\gamma\delta}\,. \qquad (11)$$

Here, the four-dimensional permutation symbol $e_{\mu\nu\gamma\delta}$ equals $+1$ or -1, depending on whether $\mu\nu\gamma\delta$ is an even or odd permutation of 1, 2, 3, and 4, and 0 if any of the subscripts are equal. Note that

$$M^{*2}_{\gamma\delta} = M^2_{\gamma\delta}\,.$$

The tensor $P_{\mu\nu}$ may be called the polarization tensor and $M_{\mu\nu}$ the magnetization tensor. The parameters α and β, which are assumed to be a function of ρ_0 and s, characterize the linear constitutive law to be expected for the material under consideration, and later will be related, respectively, to the electric and magnetic rest-frame permeabilities.

The total Lagrangian density is given by the sum

$$L = L^{(m)} + L^{(f)} + L^{(i)}\,. \qquad (12)$$

Absence of any explicit dependence of L on x_μ is evident and implies that the system under consideration is closed.

III. EQUATIONS OF "MOTION"

Hamilton's principle will be now applied to derive equations of "motion" with due consideration for a number of constraints. If Herivel-Lin's constraints are taken into account by introducing a necessary number of Lagrange multipliers, then all 17 variables, i.e., 16 four-vector components J_μ, p_μ, m_μ, and A_μ and one four-scalar, s, can be varied without restrictions. The standard procedures of variational calculus with the proper boundary conditions taken into consideration enable one to obtain all of the basic equations of the system. Details of variational principles applied to fluid mechanics can be found in Refs. 13–15.

Herivel's constraints can be written in the form

$$\frac{\partial J_\nu}{\partial x_\nu} = 0\,, \qquad (13)$$

$$J_\nu \frac{\partial s}{\partial x_\nu} = 0\,. \qquad (14)$$

The first equation states the law of conservation of mass (the continuity law) and the second the condition for the adiabatic flow. In order to write Lin's constraint explicitly, it is necessary to introduce a four-vector $X_\mu = (\vec{X}, icT)$ as a label of the fluid particles. The three-vector $\vec{X}$ indicates the position of a particle at some particular time T. Here, T may be chosen differently for different particles. Conservation of the identity of particles requires

$$J_\nu \frac{\partial X_\mu}{\partial x_\nu} = 0\,. \qquad (15)$$

This is the covariant expression of Lin's constraint first introduced by Penfield.[14]

Hamilton's principle now reads

$$\delta \int L\, d\Omega - \delta \int \varphi \frac{\partial J_\nu}{\partial x_\nu} d\Omega - \delta \int \zeta J_\nu \frac{ds}{\partial x_\nu} d\Omega - \delta \int \lambda_\nu J_\gamma \frac{\partial X_\nu}{\partial x_\gamma} d\Omega = 0\,,$$

$$d\Omega = dx_1 dx_2 dx_3 dx_4\,, \qquad (16)$$

where φ, ζ, and λ_ν are Lagrange multipliers introduced to account for the constraints (13), (14), and (15), respectively. Separate variations with respect to all independent variables now yield the following equations:

For δJ_μ,

$$c^2\rho_0 u_\mu + \epsilon_i \rho_0 u_\mu + \left(\frac{\partial \epsilon_i}{\partial \rho_0}\right)_s \rho_0^2 u_\mu - \frac{1}{4}\left(\frac{\partial(1/\alpha)}{\partial \rho_0}\right)_s P^2_{\gamma\delta}\rho_0 u_\mu + \frac{1}{4}\left(\frac{\partial(1/\beta)}{\partial \rho_0}\right)_s M^2_{\gamma\delta}\rho_0 u_\mu + \frac{1}{\alpha} P_{\mu\nu}\rho_0 p_\nu - \frac{1}{i\beta} M_{\mu\nu}\rho_0 m_\nu$$

$$- F_{\mu\nu}\rho_0 p_\nu - \frac{1}{i} F^*_{\mu\nu}\rho_0 m_\nu + \rho_0 \frac{\partial \varphi}{\partial x_\mu} - \zeta\rho_0 \frac{\partial s}{\partial x_\mu} - \lambda_\nu \rho_0 \frac{\partial X_\nu}{\partial x_\mu} + \rho_{E0} A_\mu = 0\,; \qquad (17)$$

for δs,

$$-\rho_0\left(\frac{\partial \epsilon_i}{\partial s}\right)_{\rho_0} + \frac{1}{4}\left(\frac{\partial(1/\alpha)}{\partial s}\right)_{\rho_0} P^2_{\gamma\delta} - \frac{1}{4}\left(\frac{\partial(1/\beta)}{\partial s}\right)_{\rho_0} M^2_{\gamma\delta} + \frac{\partial}{\partial x_\nu}(\rho_0 u_\nu \zeta) = 0\,; \qquad (18)$$

for δX_μ,

$$\rho_0 u_\nu \frac{\partial \lambda_\mu}{\partial x_\nu} = 0\,; \qquad (19)$$

for δp_μ,

$$[F_{\mu\nu} - (1/\alpha)P_{\mu\nu}]u_\nu = 0\,; \tag{20}$$

for δm_μ,

$$[F^*_{\mu\nu} + (1/\beta)M_{\mu\nu}]u_\nu = 0\,; \tag{21}$$

and for δA_μ,

$$\frac{\partial}{\partial x_\nu}(F_{\mu\nu} + 4\pi P_{\mu\nu} + 4\pi M^*_{\mu\nu}) - 4\pi\rho_{E0}\,u_\mu = 0\,. \tag{22}$$

In the derivation of (17), use is made of the relations

$$\rho_0 = c^{-1}(-J_\mu^2)^{1/2}, \quad \delta\rho_0 = -(\rho_0 c^2)^{-1}J_\mu\,\delta J_\mu\,.$$

Here and throughout the remaining part of this paper the variable J_μ is replaced by $\rho_0 c u_\mu$ for ease of understanding, u_μ being the dimensionless four-velocity $c^{-1}(\gamma\vec{v}, i\gamma c)$. In the application of variational principles J_μ has been a more pertinent variable than u_μ when p_μ and m_μ are used as companion variables.

The next task is to eliminate the Lagrange multipliers from the equations of motion (17). As the first step, differentiate (17) with respect to x_ν and subtract from the result the same equation with μ and ν interchanged. Multiply then $\rho_0 u_\nu$ from the left-hand side, so that Lagrange multipliers φ and λ_μ can be removed by the help of (13), (15), and (19). Use is also made of the following identities:

$$\rho_0 u_\nu \frac{\partial}{\partial x_\nu}\left(\frac{1}{\alpha}P_{\mu\gamma}p_\gamma\right) - \rho_0 u_\nu \frac{\partial}{\partial x_\nu}(F_{\mu\gamma}p_\gamma) = \frac{\partial}{\partial x_\nu}\left(\frac{1}{\alpha}P_{\mu\gamma}P_{\nu\gamma} - F_{\mu\gamma}P_{\nu\gamma}\right) - \frac{\partial}{\partial x_\nu}\left[\left(F_{\mu\gamma} - \frac{1}{\alpha}P_{\mu\gamma}\right)\rho_0 u_\gamma p_\nu\right], \tag{23}$$

$$\rho_0 u_\nu \frac{\partial}{\partial x_\nu}\left(\frac{1}{i\beta}M_{\mu\gamma}m_\gamma\right) + \rho_0 u_\nu \frac{\partial}{\partial x_\nu}\left(\frac{1}{i}F^*_{\mu\gamma}m_\gamma\right) = \frac{\partial}{\partial x_\nu}\left(\frac{1}{\beta}M_{\mu\gamma}M_{\nu\gamma} + F^*_{\mu\gamma}M_{\nu\gamma}\right) - \frac{\partial}{\partial x_\nu}\left[i\left(F^*_{\mu\gamma} + \frac{1}{\beta}M_{\mu\gamma}\right)\rho_0 u_\gamma m_\nu\right], \tag{24}$$

$$\begin{aligned}\rho_0 u_\nu \frac{\partial}{\partial x_\mu}\left(\frac{1}{\alpha}P_{\nu\gamma}p_\gamma\right) - \rho_0 u_\nu \frac{\partial}{\partial x_\mu}(F_{\nu\gamma}p_\gamma) \\ = -\frac{1}{2}\frac{\partial F_{\nu\gamma}}{\partial x_\mu}P_{\nu\gamma} + \frac{1}{4}\frac{\partial}{\partial x_\mu}\left(\frac{1}{\alpha}P^2_{\gamma\delta}\right) + \frac{1}{4}\frac{\partial(1/\alpha)}{\partial x_\mu}P^2_{\gamma\delta} - \left(F_{\nu\gamma} - \frac{1}{\alpha}P_{\nu\gamma}\right)\rho_0 u_\nu \frac{\partial p_\gamma}{\partial x_\mu}\,,\end{aligned} \tag{25}$$

$$\begin{aligned}\rho_0 u_\nu \frac{\partial}{\partial x_\mu}\left(\frac{1}{i\beta}M_{\nu\gamma}m_\gamma\right) + \rho_0 u_\nu \frac{\partial}{\partial x_\mu}\left(\frac{1}{i}F^*_{\nu\gamma}m_\gamma\right) \\ = \frac{1}{2}\frac{\partial F^*_{\nu\gamma}}{\partial x_\mu}M_{\nu\gamma} + \frac{1}{4}\frac{\partial}{\partial x_\mu}\left(\frac{1}{\beta}M^2_{\gamma\delta}\right) + \frac{1}{4}\frac{\partial(1/\beta)}{\partial x_\mu}M^2_{\gamma\delta} - i\left(F^*_{\nu\gamma} + \frac{1}{\beta}M_{\nu\gamma}\right)\rho_0 u_\nu \frac{\partial m_\gamma}{\partial x_\mu}\,.\end{aligned} \tag{26}$$

As a result one obtains

$$\begin{aligned}&\frac{\partial}{\partial x_\nu}\Bigg\{\left[\rho_0 c^2 + \rho_0\epsilon_i + \rho_0^2\left(\frac{\partial\epsilon_i}{\partial\rho_0}\right)_s - \frac{\rho_0}{4}\left(\frac{\partial(1/\alpha)}{\partial\rho_0}\right)_s P^2_{\gamma\delta} + \frac{\rho_0}{4}\left(\frac{\partial(1/\beta)}{\partial\rho_0}\right)_s M^2_{\gamma\delta}\right]u_\mu u_\nu + \frac{1}{\alpha}P_{\mu\gamma}P_{\nu\gamma} - F_{\mu\gamma}P_{\nu\gamma} - \frac{1}{\beta}M_{\mu\gamma}M_{\nu\gamma} \\ &\quad - F^*_{\mu\gamma}M_{\nu\gamma} - \left(F_{\mu\gamma} - \frac{1}{\alpha}P_{\mu\gamma}\right)\rho_0 u_\gamma p_\nu + i\left(F^*_{\mu\gamma} + \frac{1}{\beta}M_{\mu\gamma}\right)\rho_0 u_\gamma m_\nu\Bigg\} \\ &\quad + \frac{\partial}{\partial x_\mu}\left[\rho_0^2\left(\frac{\partial\epsilon_i}{\partial\rho_0}\right)_s - \frac{\rho_0}{4}\left(\frac{\partial(1/\alpha)}{\partial\rho_0}\right)_s P^2_{\gamma\delta} + \frac{\rho_0}{4}\left(\frac{\partial(1/\beta)}{\partial\rho_0}\right)_s M^2_{\gamma\delta} - \frac{1}{4\alpha}P^2_{\gamma\delta} + \frac{1}{4\beta}M^2_{\gamma\delta}\right] \\ &\quad + \rho_0\frac{\partial\epsilon_i}{\partial x_\mu} - \rho_0\left(\frac{\partial\epsilon_i}{\partial\rho_0}\right)_s\frac{\partial\rho_0}{\partial x_\mu} + \frac{1}{4}\frac{\partial(1/\alpha)}{\partial x_\mu}P^2_{\gamma\delta} + \frac{1}{4}\frac{\partial(1/\beta)}{\partial x_\mu}M^2_{\gamma\delta} + \frac{1}{2}\frac{\partial F_{\nu\gamma}}{\partial x_\mu}P_{\nu\gamma} + \frac{1}{2}\frac{\partial F^*_{\nu\gamma}}{\partial x_\mu}M_{\nu\gamma} \\ &\quad + \left(F_{\nu\gamma} - \frac{1}{\alpha}P_{\nu\gamma}\right)\rho_0 u_\nu\frac{\partial p_\gamma}{\partial x_\mu} - i\left(F^*_{\nu\gamma} + \frac{1}{\beta}M_{\nu\gamma}\right)\rho_0 u_\nu\frac{\partial m_\gamma}{\partial x_\mu} - \rho_0 u_\nu\frac{\partial\zeta}{\partial x_\nu}\frac{\partial s}{\partial x_\mu} + \rho_0 u_\nu\frac{\partial\zeta}{\partial x_\mu}\frac{\partial s}{\partial x_\nu} - F_{\mu\nu}\rho_{E0}u_\nu = 0\,.\end{aligned} \tag{27}$$

The remaining Lagrange multiplier ζ can be removed by making use of the relation

$$\rho_0 u_\nu\frac{\partial\zeta}{\partial x_\nu}\frac{\partial s}{\partial x_\mu} - \rho_0 u_\nu\frac{\partial\zeta}{\partial x_\mu}\frac{\partial s}{\partial x_\nu} = \left[\rho_0\left(\frac{\partial\epsilon_i}{\partial s}\right)_{\rho_0} - \frac{1}{4}\left(\frac{\partial(1/\alpha)}{\partial s}\right)_{\rho_0}P^2_{\gamma\delta} + \frac{1}{4}\left(\frac{\partial(1/\beta)}{\partial s}\right)_{\rho_0}M^2_{\gamma\delta}\right]\frac{\partial s}{\partial x_\mu}\,,$$

which follows from (14) and (18). As a result of compensation of a number of terms, the equations of motion can be written in the following form:

$$\frac{\partial}{\partial x_\nu}\Big[(\rho_0 c^2+\rho_0\epsilon_i)u_\mu u_\nu+(\pi_h-\tfrac14 K_\alpha P^2_{\gamma\delta}+\tfrac14 K_\beta M^2_{\gamma\delta})(u_\mu u_\nu+\delta_{\mu\nu})+\frac{1}{\alpha}(P_{\mu\gamma}P_{\nu\gamma}-\tfrac14 P^2_{\gamma\delta}\delta_{\mu\nu})$$
$$-\frac{1}{\beta}(M_{\mu\gamma}M_{\nu\gamma}-\tfrac14 M^2_{\gamma\delta}\delta_{\mu\nu})-F_{\mu\gamma}P_{\nu\gamma}-F^*_{\mu\gamma}M_{\nu\gamma}-\Big(F_{\mu\gamma}-\frac{1}{\alpha}P_{\mu\gamma}\Big)\rho_0 u_\gamma p_\nu+i\Big(F^*_{\mu\gamma}+\frac{1}{\beta}M_{\mu\gamma}\Big)\rho_0 u_\gamma m_\nu\Big]$$
$$=F_{\mu\nu}\rho_{E0}u_\nu-\frac12\frac{\partial F_{\gamma\delta}}{\partial x_\mu}P_{\gamma\delta}-\frac12\frac{\partial F^*_{\gamma\delta}}{\partial x_\mu}M_{\gamma\delta}+\Big(F_{\nu\gamma}-\frac{1}{\alpha}P_{\nu\gamma}\Big)\rho_0 u_\gamma\frac{\partial p_\nu}{\partial x_\mu}-i\Big(F^*_{\nu\gamma}+\frac{1}{\beta}M_{\nu\gamma}\Big)\rho_0 u_\gamma\frac{\partial m_\nu}{\partial x_\mu}, \tag{28}$$

where

$$K_\alpha=\rho_0\left(\frac{\partial(1/\alpha)}{\partial\rho_0}\right)_s,\quad K_\beta=\rho_0\left(\frac{\partial(1/\beta)}{\partial\rho_0}\right)_s,$$
$$\pi_h=\rho_0^2\left(\frac{\partial\epsilon_i}{\partial\rho_0}\right)_s.$$

Here, from thermodynamic considerations, π_h can be interpreted as the hydrostatic pressure in the fluid. Equation (28) is obviously the relativistic generalization of Euler's equation of hydrodynamics for simple electromagnetic fluids.

The other three basic equations, (20)–(22), will be now subjected to discussion. It is evident that Eqs. (20) and (21) state the constitutive laws for the material and Eq. (22) is the second pair of Maxwell's equations. Physical implications of these equations can be seen more easily if they are put into the three-dimensional form. In this connection the three-vectors $\vec{P}$ and $\vec{M}$ are defined by

$$P_i=\rho(p_i+ip_4 v_i/c), \tag{29}$$
$$M_i=\rho(m_i+im_4 v_i/c), \tag{30}$$

where the subscript i stands for 1, 2, or 3. The polarization and magnetization tensors can then be written in the form

$$P_{\mu\nu}=(1/c)(v_\mu P_\nu-v_\nu P_\mu),$$
$$M_{\mu\nu}=(1/ic)(v_\mu M_\nu-v_\nu M_\mu),$$

in terms of the three-vectors $\vec{v}$, $\vec{P}$, and $\vec{M}$. In the above equations the identities $v_4=ic$, $P_4=0$, and $M_4=0$ are implied. Note that v_μ, P_μ, and M_μ here are not four-vectors, in spite of their four-dimensional notation. Now Eqs. (20)–(22) can be transformed into a more familiar form,

$$\vec{D}+(\vec{v}/c)\times\vec{H}=(1+4\pi\alpha)[\vec{E}+(\vec{v}/c)\times\vec{B}],$$
$$\vec{B}-(\vec{v}/c)\times\vec{E}=(1-4\pi\beta)^{-1}[\vec{H}-(\vec{v}/c)\times\vec{D}], \tag{31}$$

and

$$\mathrm{curl}\vec{H}=\frac{1}{c}\frac{\partial\vec{D}}{\partial t}+\frac{4\pi}{c}\rho_E\vec{v},\quad \mathrm{div}\vec{D}=4\pi\rho_E. \tag{32}$$

Here, a pair of three-vectors $\vec{D}$ and $\vec{H}$ are defined by

$$\vec{D}=\vec{E}+4\pi\vec{P}+4\pi[(\vec{v}/c)\times\vec{M}]$$
$$\vec{H}=\vec{B}-4\pi\vec{M}+4\pi[(\vec{v}/c\times\vec{P}]. \tag{33}$$

It is obvious that (32) is the second pair of Maxwell's equations and (31) the relativistic generalization of the simple constitutive law $\vec{D}=\epsilon\vec{E}$ and $\vec{B}=\mu\vec{H}$ assumed for stationary media. The rest-frame dielectric and magnetic permeabilities, ϵ and μ, should be related to the parameters α and β introduced in this paper by

$$\epsilon=1+4\pi\alpha,\quad \mu=(1-4\pi\beta)^{-1}.$$

Looking at Eqs. (31)–(33), one is naturally invited to interpret $\vec{D}$ as the electric displacement and $\vec{H}$ as the magnetic field intensity in polarizable and magnetizable media. Equation (31) is a well-known relation first derived by Minkowski.[16] It should be remarked that the Lorentz invariance of Maxwell's equations in such media depends on the assumption that the variables p_μ and m_μ, as well as the potentials $\vec{A}$ and $i\phi$, behave together as a four-vector. Another point to be noted in this connection is the fact that the four-vector p_μ provides the same three-vector $\vec{P}$ if transformed in such a way that $p_\mu\to p_\mu+iv_\mu\chi/c$, where χ is an arbitrary real, invariant function of space and time. As for m_μ, the situation is similar.

It has thus been shown that the postulated Lagrangian density, especially the interaction term (9), can reproduce the well-established electromagnetic properties of linearly polarizable and magnetizable fluids. This fact can be regarded as giving credibility to the proposed form of the Lagrangian density and, in consequence, to the hydrodynamic equations of motion derived therefrom. Thus it is concluded that Eq. (28), as well as Eqs. (20)–(22), has the proper form of the basic equations useful for the discussion of both mechanical and electromagnetic properties of nonviscous, compressible, and nondispersive perfect fluids when linear constitutive laws are assumed.

IV. ENERGY-MOMENTUM TENSOR

In Sec. III three sets of basic equations for simple electromagnetic fluids have been derived in a covariant way at every stage. They are the hydrodynamic equations of motion for the material, (28), the constitutive relations (20) and (21), and the field equations (22). These three sets of equations are clearly independent of each other in the sense that they have been obtained by making variations of the different groups of variables for the respective sets. Thus the energy-momentum conservation law for the material subsystem, which is derivable from the hydrodynamic equations of motion, and the corresponding law for the field subsystem, which is derivable from the field equations, can be established independently. The conservation of energy and momentum for the total system will be found to result as a logical consequence of these two laws for subsystems.

Owing to the constitutive laws (20) and (21) the last two terms on both sides of Eq. (28) can be dropped. It then turns out that

$$\frac{\partial T_{\mu\nu}^{(m)}}{\partial x_\nu} = F_{\mu\nu}\rho_{E0}u_\nu - \frac{1}{2}\frac{\partial F_{\gamma\delta}}{\partial x_\mu}P_{\gamma\delta} - \frac{1}{2}\frac{\partial F_{\gamma\delta}^*}{\partial x_\mu}M_{\gamma\delta}. \quad (34)$$

Here, the left-hand side of (28) is written as a four-dimensional divergence of a tensor which is defined by

$$\begin{aligned} T_{\mu\nu}^{(m)} = {} & (\rho_0 c^2 + \rho_0\epsilon_i)u_\mu u_\nu + \pi(u_\mu u_\nu + \delta_{\mu\nu}) \\ & + (1/\alpha)(P_{\mu\gamma}P_{\nu\gamma} - \tfrac{1}{4}P_{\gamma\delta}^2\delta_{\mu\nu}) \\ & - (1/\beta)(M_{\mu\gamma}M_{\nu\gamma} - \tfrac{1}{4}M_{\gamma\delta}^2\delta_{\mu\nu}) \\ & - F_{\mu\gamma}P_{\nu\gamma} - F_{\mu\gamma}^*M_{\nu\gamma}, \end{aligned} \quad (35)$$

where the total pressure with electro- and magnetostrictive effects involved is denoted by

$$\pi = \pi_h - \tfrac{1}{4}K_\alpha P_{\gamma\delta}^2 + \tfrac{1}{4}K_\beta M_{\gamma\delta}^2.$$

$T_{\mu\nu}^{(m)}$ may be interpreted as the material part of the energy-momentum tensor and Eq. (34) as the relation expressing the energy-momentum conservation law for the material subsystem. The right-hand side may be taken as the force exerted on the material by the field. Later, however, a more reasonable way of defining the material tensor will be proposed, based upon physical considerations.

As the next step, the conservation law for the field subsystem will be worked out. This is accomplished by starting with the field equations (22). First, change the subscripts μ, ν into ν, δ in this equation, and then multiply the result by $(1/4\pi)F_{\mu\nu}$. After some transformations, taking advantage of the defining equation for $F_{\gamma\delta}$, Eq. (6), one obtains

$$\frac{\partial T_{\mu\nu}^{(f)}}{\partial x_\nu} = -F_{\mu\nu}\rho_{E0}u_\nu + \frac{1}{2}\frac{\partial F_{\gamma\delta}}{\partial x_\mu}P_{\gamma\delta} + \frac{1}{2}\frac{\partial F_{\gamma\delta}}{\partial x_\mu}M_{\gamma\delta}^*, \quad (36)$$

where $T_{\mu\nu}^{(f)}$ is a tensor defined by

$$T_{\mu\nu}^{(f)} = (1/4\pi)(F_{\mu\gamma}F_{\nu\gamma} - \tfrac{1}{4}F_{\gamma\delta}^2\delta_{\mu\nu}) + F_{\mu\gamma}P_{\nu\gamma} + F_{\mu\gamma}M_{\nu\gamma}^*. \quad (37)$$

Equation (36) expresses the energy-momentum conservation law for the field subsystem, and $T_{\mu\nu}^{(f)}$ may be interpreted as the field part of the energy-momentum tensor. Note that the first term on the right-hand side of (37) is the well-known electromagnetic energy-momentum tensor in free space. If one defines the tensor for the total system by the sum

$$T_{\mu\nu} = T_{\mu\nu}^{(m)} + T_{\mu\nu}^{(f)}, \quad (38)$$

then (34) and (36) can be summed to give

$$\frac{\partial T_{\mu\nu}}{\partial x_\nu} = -\frac{1}{2}\frac{\partial F_{\gamma\delta}^*}{\partial x_\mu}M_{\gamma\delta} + \frac{1}{2}\frac{\partial F_{\gamma\delta}}{\partial x_\mu}M_{\gamma\delta}^* = 0, \quad (39)$$

where the second equation is evident. Conservation of the total energy and momentum has thus been established.

There is another important theorem in which the total tensor is required to be symmetric in order that the total angular momentum conserve. The only quantity which has a nonsymmetric form in the sum (38) is the one expressed in the form $-F_{\mu\gamma}^*M_{\nu\gamma} + F_{\nu\gamma}M_{\mu\gamma}^*$. Each constituent term here is nonsymmetric, but symmetry of the total expression can be seen from a trivial identity $-F_{\mu\gamma}^*M_{\nu\gamma} = F_{\nu\gamma}M_{\mu\gamma}^*$ validated for $\mu \neq \nu$. Thus the tensor $T_{\mu\nu}$ defined by (38) is well qualified as the energy-momentum tensor for the total system.

The proper identification of the energy-momentum tensor for the material or the field subsystem has been a controversial issue for years. Those defined by (35) and (37) will be found indeed to be not qualified for physical reasons to be discussed shortly. The total tensor, in fact, may be split into two parts in any other way if the corresponding formal modifications are made in the equations expressing the conservation laws for subsystems. Such alteration in the definition of subsystem tensors brings forth nothing new mathematically, but brings about changes in physical interpretations of mathematical expressions. This is because the elements of the energy-momentum tensor should be given the same physical interpretations no matter how they are explicitly defined. The controversy over the identification of the field part of the momentum density reflects this situation. Upon these considerations one is led to the idea that introduction of some physical criteria is necessary to make a reasonable choice out of all possible ways of splitting the total tensor into the material and the field parts. The total tensor can obviously be expressed by a sum in which each constituent term is a four-tensor. To classify these terms the following criteria are proposed: (i) The part which has the same form as the pure material tensor is assigned to the material part; (ii) the part which has the same form as the field tensor for free space is assigned to the field part; and (iii) of the terms that cannot be classified by (i) and (ii), those which have nonvanishing space-space or time-space elements in the case of the material velocity $\vec{v}$ set equal to zero must be assigned to the field part. According to (i) the term involving the pressure π should be assigned to the material part. No further explanation will be necessary as to the first two criteria. The third criterion has been introduced to take account of the following situation: The space-space elements of the energy-momentum tensor can be interpreted as either the stresses or the momentum flux densities in the system. Since the only mechanical stress extant in nonviscous fluids is a pressure and the pressure-dependent part has already been attributed to the material part, the space-space elements of the terms to be classified by (iii) should be understood to be some kind of momentum flux density. Now, the flow of the momentum supported by the material is possible only when the fluid particles are in motion. Thus in the criterion (iii) terms having nonvanishing space-space elements for $\vec{v}=0$ are assigned to the field part. Similar arguments are applied to the energy flow and terms having nonvanishing time-space elements for $\vec{v}=0$ should also be assigned to the field part.

An attempt will now be made to define the material and field parts of the energy-momentum tensor with the help of the above criteria. For this purpose it will be convenient to rewrite the total tensor in the following form:

$$T_{\mu\nu}=t_{\mu\nu}^{(m)}+t_{\mu\nu}^{(f)}+t_{\mu\nu}^{(P)}+t_{\mu\nu}^{(M)}+D_{\mu\nu}+\bar{D}_{\mu\nu}, \tag{40}$$

where

$$t_{\mu\nu}^{(m)}=(\rho_0c^2+\rho_0\epsilon_i)u_\mu u_\nu+\pi(u_\mu u_\nu+\delta_{\mu\nu}), \tag{41}$$

$$t_{\mu\nu}^{(f)}=(1/4\pi)(F_{\mu\gamma}F_{\nu\gamma}-\tfrac{1}{4}F_{\gamma\delta}^2\delta_{\mu\nu}), \tag{42}$$

$$t_{\mu\nu}^{(P)}=(1/\alpha)(P_{\mu\gamma}P_{\nu\gamma}-\tfrac{1}{4}P_{\gamma\delta}^2\delta_{\mu\nu}), \tag{43}$$

$$t_{\mu\nu}^{(M)}=(1/\beta)(M_{\mu\gamma}M_{\nu\gamma}-\tfrac{1}{4}M_{\gamma\delta}^2\delta_{\mu\nu}), \tag{44}$$

$$D_{\mu\nu}=-(1/\beta)M_{\mu\gamma}M_{\nu\gamma}-F_{\mu\gamma}^*M_{\nu\gamma}, \tag{45}$$

$$\bar{D}_{\mu\nu}=-(1/\beta)M_{\mu\gamma}M_{\nu\gamma}+F_{\mu\gamma}M_{\nu\gamma}^*+(1/2\beta)M_{\gamma\delta}^2\delta_{\mu\nu}. \tag{46}$$

Here, $t_{\mu\nu}^{(m)}$ should be assigned to the material part according to criterion (i) and $t_{\mu\nu}^{(f)}$ to the field part according to (ii). The next two terms, $t_{\mu\nu}^{(P)}$ and $t_{\mu\nu}^{(M)}$, both have nonvanishing space-space and time-space elements for $\vec{v}=0$ and therefore should be assigned to the field part. As may easily be found, the tensor $D_{\mu\nu}$ has vanishing space-space and time-space elements and $\bar{D}_{\mu\nu}$ has vanishing space-space but nonvanishing time-space elements, for $\vec{v}=0$. Thus $D_{\mu\nu}$ may be assigned to either the material or the field part, but $\bar{D}_{\mu\nu}$ must be assigned to the field part. Upon these considerations, one possible way of dividing the total tensor in two parts is obtained by defining

$$T_{\mu\nu}^{(m)A}=t_{\mu\nu}^{(m)}, \tag{47}$$

$$T_{\mu\nu}^{(f)A}=t_{\mu\nu}^{(f)}+t_{\mu\nu}^{(P)}+t_{\mu\nu}^{(M)}+D_{\mu\nu}+\bar{D}_{\mu\nu}. \tag{48}$$

Here, the superscript A is used to suggest that the field part $T_{\mu\nu}^{(f)A}$ is the generalization of the field tensor assumed by Abraham for stationary media, as will be shown in Sec. V. The conservation laws for individual subsystems are now expressed in the form

$$\frac{\partial T_{\mu\nu}^{(m)A}}{\partial x_\nu}=-\frac{\partial T_{\mu\nu}^{(f)A}}{\partial x_\nu}$$
$$=F_{\mu\nu}\rho_{E0}u_\nu-\frac{\partial C_{\mu\nu}}{\partial x_\nu}-\frac{\partial D_{\mu\nu}}{\partial x_\nu}-\frac{1}{2}\frac{\partial F_{\gamma\delta}}{\partial x_\mu}P_{\gamma\delta}+\frac{1}{4}\frac{\partial}{\partial x_\mu}\left(\frac{1}{\alpha}P_{\gamma\delta}^2\right)-\frac{1}{2}\frac{\partial F_{\gamma\delta}}{\partial x_\mu}M_{\gamma\delta}^*-\frac{1}{4}\frac{\partial}{\partial x_\mu}\left(\frac{1}{\beta}M_{\gamma\delta}^2\right), \tag{49}$$

where $C_{\mu\nu}$ is the electric counterpart of $D_{\mu\nu}$ defined by

$$C_{\mu\nu}=(1/\alpha)P_{\mu\gamma}P_{\nu\gamma}-F_{\mu\gamma}P_{\nu\gamma}. \tag{50}$$

Note that the $C_{\mu\nu}$ have vanishing space-space and time-space elements for $\vec{v}=0$ (see Sec. V).

An alternative way of defining the tensors for subsystems, conforming to the criteria, is the following:

$$T_{\mu\nu}^{(m)M}=t_{\mu\nu}^{(m)}+C_{\mu\nu}+D_{\mu\nu}, \tag{51}$$

$$T_{\mu\nu}^{(f)M}=t_{\mu\nu}^{(f)}+t_{\mu\nu}^{(P)}+t_{\mu\nu}^{(M)}-C_{\mu\nu}+\bar{D}_{\mu\nu}. \tag{52}$$

Here, the superscript M is used to suggest that $T_{\mu\nu}^{(f)M}$ is the generalization of the field tensor assumed by Minkowski (see Sec. V). The conservation laws are

$$\frac{\partial T_{\mu\nu}^{(m)M}}{\partial x_\nu}=-\frac{\partial T_{\mu\nu}^{(f)M}}{\partial x_\nu}$$
$$=F_{\mu\nu}\rho_{E0}u_\nu-\frac{1}{2}\frac{\partial F_{\gamma\delta}}{\partial x_\mu}P_{\gamma\delta}+\frac{1}{4}\frac{\partial}{\partial x_\mu}\left(\frac{1}{\alpha}P_{\gamma\delta}^2\right)-\frac{1}{2}\frac{\partial F_{\gamma\delta}}{\partial x_\mu}M_{\gamma\delta}^*-\frac{1}{4}\frac{\partial}{\partial x_\mu}\left(\frac{1}{\beta}M_{\gamma\delta}^2\right). \tag{53}$$

These two alternative ways of decomposing the total tensor into subsystem parts are cited here on account of their historical importance and their usefulness in the discussion of the radiation pressure problem, which will be the subject of Sec. V. There are, of course, many other methods of decomposition conforming to the above-mentioned three criteria. It is noted that all of them, including Abraham's and Minkowski's forms, have the same space-space and time-space elements of subsystem tensors for $\vec{v}=0$ and therefore predict the same density of the electromagnetic energy-momentum flux in nonrelativistic limits. The subsystem tensors labeled A are characterized by their symmetry, but this property by no means endorses their privileged legitimacy. This is because what is required from fundamental principles is the symmetry of the total tensor, not the symmetry of subsystem tensors.

V. DISCUSSION ON RADIATION PRESSURE IN MEDIA

The topic discussed in this section can be investigated on the basis of nonrelativistic theory, since the velocity of the fluid particles can be assumed to be well below the speed of light in all feasible experiments on radiation pressures. For the sake of clarity, the three-dimensional description will be employed throughout this section.

It is easy to show that the tensors $C_{\mu\nu}$, $D_{\mu\nu}$, and $\overline{D}_{\mu\nu}$ all have a remarkably simple form in nonrelativistic limits,

$$C_{\mu\nu}=\begin{bmatrix}0&0&0&-i(\vec{P}\times\vec{B})_1\\0&0&0&-i(\vec{P}\times\vec{B})_2\\0&0&0&-i(\vec{P}\times\vec{B})_3\\0&0&0&0\end{bmatrix},$$

$$D_{\mu\nu}=\begin{bmatrix}0&0&0&-i(\vec{E}\times\vec{M})_1\\0&0&0&-i(\vec{E}\times\vec{M})_2\\0&0&0&-i(\vec{E}\times\vec{M})_3\\0&0&0&0\end{bmatrix}, \qquad (54)$$

$$\overline{D}_{\mu\nu}=\begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\-i(\vec{E}\times\vec{M})_1&-i(\vec{E}\times\vec{M})_2&-i(\vec{E}\times\vec{M})_3&0\end{bmatrix}.$$

In the first place, the problem will be discussed on the basis of the generalized Minkowski concept. The material and field tensors are considerably simplified in the limit $\vec{v}/c\to 0$. Thus from (51) and (52) one obtains

$$T^{(m)M}_{\mu\nu}=\begin{pmatrix}\rho\vec{v}\vec{v}+\pi\overleftrightarrow{I} & i\rho c\vec{v}-i(\vec{P}\times\vec{B})-i(\vec{E}\times\vec{M})\\ i\rho c\vec{v} & -\rho c^2-\rho\epsilon_i\end{pmatrix}, \qquad (55)$$

$$T^{(f)M}_{\mu\nu}=\frac{1}{4\pi}\begin{pmatrix}-\vec{E}\vec{D}-\vec{H}\vec{B}+\frac{1}{2}(\vec{E}\cdot\vec{D}+\vec{H}\cdot\vec{B})\overleftrightarrow{I} & i(\vec{D}\times\vec{B})\\ i(\vec{E}\times\vec{H}) & -\frac{1}{2}(\vec{E}\cdot\vec{D}+\vec{H}\cdot\vec{B})\end{pmatrix}, \qquad (56)$$

where $\overleftrightarrow{I}$ stands for the three-dimensional unit dyadic. It is evident that the right-hand side of (56) is identical to Minkowski's field tensor, as suggested before. The conservation laws (53) now reduce to

$$\frac{\partial T^{(m)M}_{i\nu}}{\partial x_\nu}=-\frac{\partial T^{(f)M}_{i\nu}}{\partial x_\nu}=\rho_E E_i+\frac{1}{c}(\rho_E\vec{v}\times\vec{B})_i-\frac{1}{4}\frac{\partial(1/\alpha)}{\partial x_i}\vec{P}^2+\frac{1}{4}\frac{\partial(1/\beta)}{\partial x_i}\vec{M}^2, \qquad (57)$$

$$\frac{\partial T^{(m)M}_{4\nu}}{\partial x_\nu}=-\frac{\partial T^{(f)M}_{4\nu}}{\partial x_\nu}=i\rho_E\vec{v}\cdot\vec{E}+\frac{i}{4c}\frac{\partial(1/\alpha)}{\partial t}\vec{P}^2-\frac{i}{4c}\frac{\partial(1/\beta)}{\partial t}\vec{M}^2. \qquad (58)$$

For the discussion of radiation pressure we can assume that free charges are absent and the effect of space-time dependence of α and β can be ignored. This remarkably simplifies the basic equations (57) and (58) and enables one to write them in the form

$$\frac{\partial T^{(m)M}_{\mu\nu}}{\partial x_\nu}=0, \qquad (59)$$

$$\frac{\partial T^{(f)M}_{\mu\nu}}{\partial x_\nu}=0. \qquad (60)$$

It should be noted that they are not tensor equations, in spite of their appearance. They are correct only in the rest frame. According to (59) and (60), in the generalized Minkowski version energy and momentum conserve within the material and the field subsystems separately. This leads to a natural conclusion that the radiation pressure on opaque bodies immersed in electromagnetic fluids is given by Minkowski's field momentum density multiplied by the velocity of light in matter, i.e., $(\vec{D}\times\vec{B})/4\pi(\epsilon\mu)^{1/2}$. This is the result well confirmed by the experiments on refractive liquids.[7]

Next, the same problem will be considered in the second alternative version originally proposed by Abraham. The material and field tensors now have the form

$$T^{(m)A}_{\mu\nu}=\begin{pmatrix}\rho\vec{v}\,\vec{v}+\pi\overleftrightarrow{I} & i\rho c\vec{v}\\ i\rho c\vec{v} & -\rho c^2-\rho\epsilon_i\end{pmatrix}, \tag{61}$$

$$T^{(f)A}_{\mu\nu}=\frac{1}{4\pi}\begin{pmatrix}-\vec{E}\,\vec{D}-\vec{H}\vec{B}+\frac{1}{2}(\vec{E}\cdot\vec{D}+\vec{H}\cdot\vec{B})\overleftrightarrow{I} & i(\vec{E}\times\vec{H})\\ i(\vec{E}\times\vec{H}) & -\frac{1}{2}(\vec{E}\cdot\vec{D}+\vec{H}\cdot\vec{B})\end{pmatrix}, \tag{62}$$

as easily derived from (47) and (48). The conservation laws are then

$$\frac{\partial T^{(m)A}_{i\nu}}{\partial x_\nu}=\frac{1}{c}\frac{\partial}{\partial t}(\vec{P}\times\vec{B}+\vec{E}\times\vec{M})_i\,,\quad \frac{\partial T^{(m)A}_{4\nu}}{\partial x_\nu}=0, \tag{63}$$

$$\frac{\partial T^{(f)A}_{i\nu}}{\partial x_\nu}=-\frac{1}{c}\frac{\partial}{\partial t}(\vec{P}\times\vec{B}+\vec{E}\times\vec{M}),\quad \frac{\partial T^{(f)A}_{4\nu}}{\partial x_\nu}=0. \tag{64}$$

Note that momentum here does not conserve separately within the respective subsystems, as in the case of Minkowski. This obviously means that the density of the electromagnetic momentum flux cannot be given by the product of the density of the momentum, $(\vec{E}\times\vec{H})/4\pi c$, and the velocity of light, $c'=c/(\epsilon\mu)^{1/2}$. Comparison of (62) with (56) shows clearly that $T^{(f)A}_{\mu\nu}$ has the same space-space and time-space elements as $T^{(f)M}_{\mu\nu}$, differing only as to the space-time elements. Thus the expression for the density of the electromagnetic momentum flux can be expected to be the same for both versions, although the density of the momentum is not. Similar situations are true for any other versions in which subsystem tensors are taken so as to conform the three criteria discussed before. This means that agreement between theory and experiment on the radiation pressure of the type under discussion can be obtained if one adopts any form of subsystem tensors conforming to the above criteria. It does not seem necessary to answer definitely the question as to what is the proper form of the field momentum density. What is essential is the density of the momentum flux rather than the momentum density.

In this connection, the total momentum carried by a wave packet of light seems worth investigation. The impulse G imparted to an opaque body embedded in a fluid when the packet impinges on its surface is just given by this total momentum. As it equals the density of the electromagnetic momentum integrated over the whole volume of the packet, it might be thought to be different for differently assumed momentum densities. However, it will be shown that the packet momentum is determined by the density of the momentum flux, as in the case already discussed, rather than the density of the momentum. For simplicity, imagine a wave packet of cylindrical shape of cross-sectional area S and length L traveling along the direction of the cylinder axis. The time required for the packet to pass a particular plane perpendicular to the direction of propagation is denoted by $\tau=L/c'$. The total momentum carried by the packet can be measured by the total amount of momentum which passes the plane for this time duration. If one denotes the density of the electromagnetic momentum flux by Γ, then the impulse under consideration is given by

$$G=\Gamma S\tau=\Gamma SL/c'. \tag{65}$$

The average radiation force exerted on the body is obviously given by ΓS, and is thus determined by the density of the momentum flux rather than by the density of the momentum. This is the same conclusion as derived before.

Some authors[9,10] argue that the pressure observed in Jones and Richards's experiments on refractive liquids is not entirely the true electromagnetic radiation pressure. According to them, it must consist of two parts, namely, the true radiation pressure $(\vec{E}\times\vec{H})/4\pi(\epsilon\mu)^{1/2}$ and the mechanical pressure $(\vec{D}\times\vec{B}-\vec{E}\times\vec{H})/4\pi(\epsilon\mu)^{1/2}$. The preceding analyses in this paper, however, have revealed that the true radiation pressure is given by $(\vec{D}\times\vec{B})/4\pi(\epsilon\mu)^{1/2}$, thus suggesting no need for such a mechanical pressure to be taken into account in interpreting observed data. A brief account of theoretical estimation of the mechanical contribution will be given below in further justification of this statement. There are apparently two possible types of mechanical contribution deserving of investigation. One is the electro- and magnetostrictive excess pressures, $\frac{1}{4}\alpha^2K_\alpha E^2$ and $-\frac{1}{4}\beta^2K_\beta B^2$ in the nonrelativistic case, and the other is the acoustic radiation pressure of the density waves arising from the forced oscillations of fluid particles under the influence of the light fields.

First, consider the electro- and magnetostrictive excess pressures. For ordinary liquids they cannot be regarded as small in comparison with the excess radiation pressures actually observed in such liquids.[17] In the case of steady-beam experiments, however, the time-average effect of the electro- and magnetostrictive excess pressures will be cancelled by the elastic hydrostatic pressure and will not affect the radiation pressure measurements, while it may be important in transient phenomena. The two existing experiments[7,8] are concerned with pulsed beams, and whether they are steady-state or transient experiments depends on the width a and the duration τ of the pulse employed. Let the velocity of sound in liquids be denoted by v_s; then a steady-state situation can be assumed when the condition $a \ll v_s \tau$ is satisfied. Robinson,[18] in his recent review article, has fully discussed this issue and has concluded that the steady-state conditions are applicable in both existing experiments. Thus it will be justified to confine the arguments in this section to steady-state phenomena, so that the direct contribution from the electro- and magnetostrictive forces may be ignored.

The other type of mechanical contribution will now be studied. Imagine a monochromatic plane wave of light traveling along the z axis with the direction of polarization of the electric field parallel to the x axis. Then, the only nonvanishing field components are

$$E_x = E_0 \cos[k(z - c't)],$$
$$H_y = (\epsilon/\mu)^{1/2} E_0 \cos[k(z - c't)], \tag{66}$$

where k is the wave number. From (63) one obtains for the electromagnetic force density in the present case the following expression:

$$-\frac{\mu}{c}(\alpha+\beta)\frac{\partial}{\partial t}(E_x H_y) - \frac{\alpha^2 K_\alpha}{2}\frac{\partial}{\partial z}E_x^2 + \frac{\mu^2\beta^2}{2}K_\beta\frac{\partial}{\partial z}H_y^2,$$

where the second and third terms are the electrostrictive and magnetostrictive forces, respectively. If the fluid is electrically nonpolar and magnetically nonpermeable ($\mu = 1$), then the third term vanishes and the second term can be shown, by applying the Clausius-Mossotti relation, to be considerably smaller than the first term. It is tempting then to assume that the electro- and magnetostrictive forces play no important role in the phenomena under discussion for most common liquids. Thus the influence of these forces will be discarded in the following treatment.

The force exerted by the fields is clearly parallel to the z direction. This indicates that the forced motion of the fluid particles caused by these fields is also restricted to this direction. Obviously, it is sufficient for the present purpose to study the case in which only this type of motion exists, and therefore v_x and v_y can be set equal to zero. The equations of motion in (59) or (63) then read

$$\frac{\partial}{\partial t}(\rho v) + \frac{\partial \pi_h}{\partial z} = -\frac{\mu}{c}(\alpha+\beta)\frac{\partial}{\partial t}(E_x H_y), \tag{67}$$

where v is used for v_z and π is replaced by π_h. If the equilibrium values of ρ and π_h are denoted by $\bar{\rho}$ and $\bar{\pi}$ and the acoustic radiation pressure by p, then

$$\rho = \bar{\rho}(1+\kappa p), \quad \pi_h = \bar{\pi} + p, \tag{68}$$

where κ is the compressibility of the fluid. Here, $\bar{\rho}$, $\bar{\pi}$, κ, and the specific entropy s are all assumed to be constant. Let the displacement of the fluid particles be denoted by η; then $v = \partial\eta/\partial t$. Since the displacement occurs only in the z direction, from the second equation in (68) one obtains

$$\frac{\partial p}{\partial z} = -\frac{1}{\kappa}\frac{\partial \eta}{\partial z}. \tag{69}$$

Now, differentiation of (67) with respect to z and substitution of (66), (68), and (69) into the result yields

$$\bar{\rho}\frac{\partial^2\eta}{\partial t^2} - \frac{1}{\kappa}\frac{\partial^2\eta}{\partial z^2} = -(\alpha+\beta)kE_0^2 \sin[2k(z - c't)], \tag{70}$$

where ρ is replaced approximately by $\bar{\rho}$ in the first term on the left hand side. The displacement associated with the acoustic waves excited by the fields can be given by the following particular solution of (70):

$$\eta = \frac{(\alpha+\beta)E_0^2}{k(\bar{\rho}c'^2 - 1/\kappa)}\sin[2k(z - c't)].$$

This represents a sound wave traveling along the same direction and with the same velocity as the wave packet of light. The acoustic radiation pressure can be evaluated from the equation

$$p = \bar{\rho}\frac{\partial\eta}{\partial t}c' = \bar{\rho}(1+\kappa p)\frac{\partial\eta}{\partial t}c'.$$

This equation can be solved for p by means of successive approximations to give

$$\langle p\rangle_{\rm av} = [(\alpha+\beta)E_0^2]^2/\bar{\rho}v_s^2,$$

where $v_s = (\bar{\rho}\kappa)^{-1/2}$ is the velocity of sound in the fluid. The result shows that such an acoustic pressure is smaller than the electromagnetic radiation pressure by the factor (in orders of magnitude) $E_0^2/\bar{\rho} v_s^2$. Note that this factor has an exceedingly small value for the light intensities available in practical experiments. Thus the mechanical pressure associated with the propagation of light waves cannot be expected to contribute a measurable part of the observed radiation pressure.

Hitherto in this section the subject of discussion has been the light forces exerted on objects immersed in fluids. Now, consider a force exerted on a free liquid surface by an impinging light pulse. Recent experiments of Ashkin and Dziedzic[8] showed that this force was a tension. It has been argued theoretically on occasion that Abraham's tensor predicts a pressure and Minkowski's tensor a tension. This conclusion has been obtained by comparing the momenta of the light packet before and after the impact. As has already been shown, the momentum in the liquid is given by $\Gamma S L/c'$ [Eq. (65)] and that in the air by $\Gamma_0 S L_0/c$, where Γ_0 and L_0 are the density of the field momentum flux and the length of the packet, respectively, in the air. Since the length of the packet and the velocity of light are both proportional to the reciprocal refractive index n^{-1} $[n=(\epsilon\mu)^{1/2}]$, the ratio is the same in both media. With due account of the reflectivity R of the liquid surface, the total impulse imparted to the unit area of the air-liquid interface is given by

$$G=\Gamma_0\tau(1+R)-\Gamma\tau(1-R), \qquad (71)$$

where L_0/c or L/c' is replaced by τ, the time required for the packet to pass the air-liquid interface. It has already been shown that Γ is given by the product of Minkowski's momentum density and the velocity of light in media. Thus for the same intensity of light one gets $\Gamma=n\Gamma_0$. With the use of this fact and the well-known relation $R=(n-1)^2/(n+1)^2$ one obtains for the average force exerted on the air-liquid interface by the impinging light pulse

$$f=-2\Gamma_0(n-1)/(n+1).$$

This force is clearly a tension. It is remarked that the result is the same for all alternative assumptions on the field part of the energy-momentum tensor conforming to the three criteria discussed before, since they all predict the same expression for Γ.

Finally, it is worth referring to the argument,[19,20] based on a hypothetical "experiment," that only Abraham's symmetric form of the electromagnetic energy-momentum tensor satisfies the momentum conservation and center-of-mass theorems simultaneously. As such an experiment is concerned with a wave packet, arguments similar to those just given may apply also in the present case and reveal that the ruling factor here is again the density of the momentum flux, not the density of the momentum, and that in consequence any alternative form of the field tensor which is admissible in the above sense can be consistent with both theorems.

Summarizing this section, one may say that although the exact form of the material and the field part of the energy-momentum tensor cannot be determined in a unique way, theory can provide a definite and satisfactory answer to the problem of the radiation pressure in matter.

[1]H. Minkowski, Nachr. Ges. Wiss. Gött. 53 (1908); Math. Ann. 68, 472 (1910).

[2]M. Abraham, Rend. Circ. Mat. Palermo 28, 1 (1908); 30, 33 (1910).

[3]M. V. Laue, *Relativitätstheorie*, 6th ed. (Vieweg, Braunshweig, 1956).

[4]C. Møller, *The Theory of Relativity* (Oxford U.P., Oxford, 1952), Sec. 75.

[5]L. D. Landau and E. M. Lifshitz, *Electrodynamics of Continuous Media* (Pergamon, London, 1960), p. 242.

[6]W. Pauli, *Theory of Relativity* (Pergamon, London, 1958).

[7]R. V. Jones and J. C. S. Richards, Proc. R. Soc. A 221, 480 (1954).

[8]A. Ashkin and J. M. Dziedzic, Phys. Rev. Lett. 30, 139 (1973).

[9]H. A. Haus, Physica (Utr.) 43, 77 (1969).

[10]J. P. Gordon, Phys. Rev. A 8, 14 (1973).

[11]M. G. Burt and R. Peierls, Proc. R. Soc. A 333, 149 (1973).

[12]L. D. Landau and E. H. Lifshitz, *The Classical Theory of Fields*, 2nd ed. (Pergamon, London, 1962), p. 77.

[13]J. Serrin, in *Handbuch der Physik*, edited by S. Flügge (Springer, Berlin, 1959), Vol. III/1.

[14]P. Penfield, Jr., Phys. Fluids 9, 1184 (1966).

[15]P. Penfield, Jr., and H. A. Haus, Phys. Fluids 9, 1195 (1966).

[16]See, for example, Ref. 5, p. 245.

[17]For instance, if the fluid is a simple nonpolar liquid, then the electrostrictive excess pressure $\frac{1}{4}\alpha^2 K_\alpha E^2$ can be reduced to the well-known Helmholtz formula $(\epsilon-1)(\epsilon+2)E^2/24\pi$ with the aid of the Clausius-Mossotti equation $(\epsilon-1)/(\epsilon+2)=(\text{const})\rho_0$. The excess pressures evaluated with this formula can be found to be of the same order of magnitude as the change in radiation pressure actually observed in such liquids.

[18]F. N. H. Robinson, Phys. Rep. C 16, 313 (1975).

[19]N. L. Balasz, Phys. Rev. 91, 408 (1953).

[20]W. Shockley, Proc. Natl. Acad. Sci. USA 60, 807 (1968).