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Glauber approximation for the excitation of hydrogenlike ions by structureless charged particles. I*

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A Glauber approximation to the scattering amplitude for the direct inelastic scattering of structureless charged particles by ion targets is derived and applied to inelastic scattering by hydrogenlike ions with arbitrary nuclear charge. The $1s \rightarrow nlm$ Glauber scattering amplitudes are evaluated in closed form as simple sums of Meijer G functions. The asymptotic behavior of these amplitudes is examined for both large and small momentum transfers, and for the limit of infinitely large nuclear charge.

I. INTRODUCTION

The Glauber approximation[1] to the scattering amplitude has proved to be a useful and reasonably reliable predictor of intermediate- and high-energy *inelastic* scattering of structureless charged particles by neutral atoms.[2] For these inelastic collisions the Glauber predictions of both the integrated (over scattering angle) cross sections and the angular distributions for scattering at angles $\lesssim 90°$ are in remarkably good agreement with experiment when the incident particle speed v_i is greater than 1 a.u.[2,3] At scattering angles greater than 90°, the Glauber-predicted angular distributions are found to be somewhat less accurate when compared with very recent experiments.[3] On the other hand, it is well established[2,3] that the Glauber predictions of the *elastic* scattering in these same atomic collisions are unreliable; nevertheless, the utility of the Glauber approximation for inelastic collisions involving neutral targets cannot be gainsaid.

In the conventional Glauber approximation, the direct scattering-amplitude formula involves a straight line eikonallike path integral over the interaction potential V_i seen by the incident particle. For a direct collision between a charged particle and a neutral aggregate of charged particles, V_i is dipole at large distances; consequently, the Glauber amplitude formula is well defined, regardless of its physical significance. On the other hand, if both the incident particle and target are charged, then V_i is asymptotically Coulombic and the conventional Glauber amplitude formula is no longer mathematically well defined. Nevertheless, the success of the Glauber approximation in charged-particle–neutral-atom inelastic scattering has prompted a number of very recent attempts to seek a simple extension of the Glauber approximation appropriate for collisions between charged particles and ionic targets. These efforts include calculations by Narumi and Tsuji,[4] Ishihara and Chen,[5] and Thomas and Franco.[6] Although somewhat different methods are employed to derive the formula for this Coulomb-modified Glauber amplitude, the final amplitude formulas obtained are, not too surprisingly, identical. Differences do appear, however, when actual applications of this result are made to the excitation of hydrogenic ions by incident charged particles. In particular, Narumi and Tsuji, and Ishihara and Chen are able to compute these Glauber amplitudes only after reducing the Glauber multidimensional amplitude integrals to differing one-dimensional integral representations which require numerical integration. In contradistinction to those results, we have been able to reduce these Glauber amplitude integrals to closed form, as simple sums of Meijer G functions; thus we are able to compute these Glauber amplitudes with only slightly greater effort than that required to compute the corresponding closed form e^--H($1s$) Glauber amplitudes.[7] The purpose of this present paper is to detail the analysis leading to these new closed-form Glauber amplitude expressions for the excitation of ground-state hydrogenic ions by structureless charged particles. Numerical results for the Glauber-predicted excitation of He^+ and Li^{++} to the $n=2$ and $n=3$ levels by both incident electrons and protons shall be presented in a subsequent paper.

The contents of this paper may be summarized briefly as follows. The Coulomb-modified Glauber amplitude formula is derived in Sec. II. In Sec. III we describe the reduction to closed form of the Glauber amplitude integrals for general $1s \rightarrow nlm$ excitations of hydrogenlike ions by structureless charged particles. As useful examples of the general result, we give the explicit expressions for the $2s$ and $2p$ excitation amplitudes in Sec. IV. Finally, in Sec. V we evaluate the asymptotic forms for the Coulomb-modified Glauber amplitudes in the limits of large and small momentum transfers q. In addition we develop the asymptotic form of the appropriately scaled $1s\to 2s$ and $1s\to 2p$ Glauber amplitudes in the limit in which the nuclear charge of the hydrogenlike ion is allowed to approach infinity.

For convenience, we have deferred to Appendix A our discussion of an extremely useful generalization of an alternative integral representation for the integral,

$$\frac{1}{2\pi}\int_0^{2\pi} d\varphi\, e^{im\varphi}(1+s^2-2s\cos\varphi)^{i\eta},$$

developed originally by Thomas and Gerjuoy.[7] We employ the result of Ref. 7 to obtain an equivalent one-dimensional integral for the more general integral

$$\int d\Omega\, Y_l^{m*}(\hat{r})[1-(|\vec{b}-\vec{s}|/b)^{2i\eta}],$$

where Y_l^m is the conventional spherical harmonic and $\vec{s}$ is the projection of the three-dimensional vector $\vec{r}$ onto a plane containing the vector $\vec{b}$; η is purely real. This result is useful not only for our present purposes, but also in the general evaluation of the conventional Glauber amplitudes for induced transitions in multielectron atoms[2,8] and in the evaluation of the Glauber partial-wave ionization amplitudes for which it was originally developed.[9] Appendix B describes the generating functions which lead to the closed-form Glauber amplitudes for general $1s\to nlm$ transitions in atomic hydrogen. This result, which has not heretofore been generally published, is obtained from the results of Sec. III by the simple limiting process of setting the excess nuclear charge equal to zero.

II. DERIVATION OF THE COULOMB-MODIFIED GLAUBER FORMULA

The Glauber approximation to the scattering amplitude for a direct collision (excluding exchange or rearrangement, but including ionization) of a structureless particle of charge Z_ie with an atom which consequently undergoes a transition from an initial state i to a final state f is given by[1,2,8]

$$A(i\to f;\vec{q})=\frac{iK_i}{2\pi}\int e^{i\vec{q}\cdot\vec{b}}u_f^*(\vec{r})\Gamma(\vec{b},\vec{r})\times u_i(\vec{r})\,d^2b\,d\vec{r}, \tag{1a}$$

where

$$\Gamma(\vec{b},\vec{r})=1-\exp\left(-\frac{i}{\hbar v_i}\int_{-\infty}^{+\infty}dz'V_i(\vec{r}',\vec{r})\right). \tag{1b}$$

In Eqs. (1a) and (1b) $V_i(\vec{r}',\vec{r})$ is the interaction potential seen by the incident particle with coordinate $\vec{r}'$; $\vec{r}$ denotes the *collection* of internal coordinates required to specify the initial and final bound-state wave functions, u_i and u_f, of the target atom. The momentum transfer is given by $\vec{q}=\vec{K}_i-\vec{K}_f$ with $\hbar\vec{K}_i,\hbar\vec{K}_f=\mu\vec{v}_i,\mu\vec{v}_f$, where $\vec{v}_i$ and $\vec{v}_f$ are the initial and final relative velocities of the scattered particle in the center-of-mass system; μ is the reduced mass of the incident-particle–target-atom pair. It is by now well understood[2] that identifying Eqs. (1a) and (1b) as the direct scattering amplitude incorporates the subsumption that the z direction in Eq. (1b) is to be taken along a direction $\hat{\nu}$ perpendicular to $\vec{q}$ in the scattering plane. Thus in Eq. (1) $\vec{b}$ is the projection of $\vec{r}'$ onto the plane perpendicular to $\hat{\nu}$.

As long as the target atom (or molecule) is neutral, the potential V_i seen by the incident charged particle is proportional to $(r')^{-2}$ when r' is large so that the profile function $\Gamma(\vec{b},\vec{r})$ and Eq. (1a) are well defined. If the target system is an ion with nuclear charge Z_n and M bound electrons then, neglecting spin-dependent interactions,

$$V_i(\vec{r}',\vec{r})=\frac{Z_nZ_ie^2}{r'}-Z_ie^2\sum_{j=1}^{M}\frac{1}{|\vec{r}'-\vec{r}_j|} \tag{2a}$$

$$\equiv V_a(\vec{r}',\vec{r})+V_c(\vec{r}'), \tag{2b}$$

where

$$V_a(\vec{r}',\vec{r})\equiv\frac{MZ_ie^2}{r'}-Z_ie^2\sum_{j=1}^{M}\frac{1}{|\vec{r}'-\vec{r}_j|}, \tag{3a}$$

$$V_c(\vec{r}')=\frac{(Z_n-M)Z_ie^2}{r'}, \tag{3b}$$

and $\vec{r}_j$ is the coordinate of the jth bound electron. For the moment we have assumed the target to be a positive ion, i.e., $0<M<Z_n$. When Eq. (2b) is used in (1b), the path integral over V_a is well defined and convergent; on the other hand, the integral over the long-range Coulomb interaction V_c is not well defined so that the ensuing amplitude expression (1) is consequently ambiguous. Nevertheless, Eq. (1) still may be used to define a Glauber approximation for these charged-particle–ion-target collisions provided the residual Coulomb interaction V_c is treated as the limit of an appropriately chosen sequence of short-range potentials. The motivation for this course stems from a truly remarkable property of the Glauber potential scattering formula: aside from an overall constant phase factor, the Glauber-amplitude formula for either a cutoff Coulomb potential or Yukawa is, in the limit of zero screening, exact

not only in modulus, but also in phase.[1,2] Thus, in place of Eqs. (2) and (3), we write

$$V_i(\vec{r}', \vec{r}) = V_a(\vec{r}', \vec{r}) + V_c^\epsilon(\vec{r}') \tag{4a}$$

$$\equiv V_a(\vec{r}', \vec{r}) + \frac{(Z_n - M)Z_i e^2}{r'} e^{-\epsilon r'} . \tag{4b}$$

In the usual way[1,2] we now define the phase-shift functions χ_a and χ_c via

$$e^{i\chi_a} \equiv \exp\left(-\frac{i}{\hbar v_i}\int_{-\infty}^{\infty} dz'\, V_a(\vec{r}', \vec{r})\right)$$

$$= \prod_{j=1}^{M}\left(\frac{|\vec{b} - \vec{s}_j|}{b}\right)^{2i\eta} \tag{5a}$$

and

$$e^{i\chi_c(b)} \equiv \exp\left(-\frac{i}{\hbar v_i}\int_{-\infty}^{\infty} dz'\, V_c^\epsilon(\vec{r}')\right) = e^{2i\alpha_n K_0(\epsilon b)} , \tag{5b}$$

where

$$\eta = -Z_i e^2/\hbar v_i , \quad \alpha_n = -Z_i(Z_n - M)e^2/\hbar v_i ,$$

$\vec{s}_j$ is the projection of $\vec{r}_j$ onto the plane containing $\vec{q}$ and $\vec{b}$, and K_0 is the modified Bessel function.[10]

The Glauber amplitude for scattering by the ion, when the incident interaction is given by Eq. (4), is now well defined and given by

$$A_\epsilon(i \to f; \vec{q}) = \frac{iK_i}{2\pi}\int d^2b\, d\vec{r}\, e^{i\vec{q}\cdot\vec{b}} \times u_f^*(\vec{r})(1 - e^{i\chi_a}e^{i\chi_c})u_i(\vec{r}) . \tag{6}$$

In Eq. (6) the subscripted A_ϵ clearly denotes that we have constructed the scattering amplitude using the Yukawa V_c^ϵ of Eq. (4) rather than the exact Coulomb V_c of Eqs. (2) and (3). If the scattering is inelastic, $i \neq f$; then using the orthogonality of the wave functions u_f and u_i we may write

$$A_\epsilon(i \to f; \vec{q}) = \frac{iK_i}{2\pi}\int d^2b\, d\vec{r}\, e^{i\vec{q}\cdot\vec{b}} u_f^*(\vec{r}) \times (-e^{i\chi_a}e^{i\chi_c(b)})u_i(\vec{r}) \tag{7a}$$

$$= \frac{iK_i}{2\pi}\int d^2b\, d\vec{r}\, e^{i\vec{q}\cdot\vec{b}} e^{i\chi_c(b)} \times u_f^*(\vec{r})(1 - e^{i\chi_a})u_i(\vec{r}) . \tag{7b}$$

Equation (7b) follows from (7a) because $i \neq f$ and χ_c is independent of $\vec{r}$. The convenience of writing A_ϵ in the form (7b) will be made clear below.[11]

We now define the Coulomb-modified Glauber amplitude via the limiting relation

$$A(i \to f; \vec{q}) = \lim_{\epsilon\to 0} A_\epsilon(i \to f; \vec{q}) . \tag{8}$$

In principle, the limit of Eq. (8) should be taken only after performing the integrations indicated in Eq. (7); however, in actual practice this procedure is impractical. Therefore we employ the standard technique used to derive the exact Coulomb *potential* scattering amplitude in the Glauber approximation[1,2] and assume that the limit may be taken prior to performing the integrations. In other words, we replace $e^{i\chi_c}$ in Eq. (7b) by the $\lim e^{i\chi_c}$ as $\epsilon \to 0$. Now when ϵ is small we have from (5b) that[10]

$$\chi_c = 2\alpha_n K_0(\epsilon b) \sim -2\alpha_n[\gamma + \ln(\tfrac{1}{2}\epsilon b)] + O(\epsilon) ,$$

where γ is Euler's constant; consequently,

$$e^{i\chi_c} \sim \exp\{-2i\alpha_n[\ln(\epsilon/2) + \gamma]\}\, b^{-2i\alpha_n} . \tag{9}$$

Using Eq. (9) in (7b) and dropping inconsequential overall constant phase factors, we obtain our final formula for the Glauber amplitude for the inelastic scattering of a structureless charged particle by a positive ion, namely,

$$A(i \to f; \vec{q}) = \frac{iK_i}{2\pi}\int d^2b\, d\vec{r}\, e^{i\vec{q}\cdot\vec{b}} b^{-2i\alpha_n} \times u_f^*(\vec{r})(1 - e^{i\chi_a})\, u_i(\vec{r}) , \tag{10}$$

with

$$e^{i\chi_a} = \prod_{j=1}^{M}\left(\frac{|\vec{b} - \vec{s}_j|}{b}\right)^{2i\eta} ,$$

$$\eta = \frac{-Z_i e^2}{\hbar v_i}, \quad \alpha_n = \frac{-Z_i(Z_n - M)e^2}{\hbar v_i} ,$$

and M is the number of bound electrons in the ion.

Several remarks about Eq. (10) now are in order. Equation (10) differs from the conventional form of the Glauber amplitude for scattering by a neutral atom only by the additional b-dependent phase factor $b^{-2i\alpha_n}$. Thus standard techniques[2,8] may be employed to reduce (10) to a one-dimensional integral over b when the bound-state wave functions may be represented as products of one-electron orbitals. Since we are concerned only with inelastic scattering, the unit term in the profile function $\Gamma_a(\vec{b}, \vec{r}) = 1 - e^{i\chi_a}$ does not contribute to the amplitude (10); however, when we formally retain this term Γ_a has the convenient separation

$$\Gamma_a(\vec{b}, \vec{r}) = \sum_{j=1}^{M} \Gamma_j(\vec{b}, \vec{r}_j) - \sum_{j=1}^{M}\sum_{i=j+1}^{M} \Gamma_j(\vec{b}, \vec{r}_j)\Gamma_i(\vec{b}, \vec{r}_i) + \cdots , \tag{11}$$

with

$$\Gamma_j(\vec{b}, \vec{r}_j) = 1 - (|\vec{b} - \vec{s}_j|/b)^{2i\eta} ,$$

wherein the atomic scattering analog of the usual separation[1,2] of the Glauber amplitude into single scattering, double scattering, and higher-order scattering terms is made manifest. When Eq. (11) is used in the neutral-atom case to separate the Glauber amplitude the single scattering terms, involving only one $\Gamma_j(\vec{b}, \vec{r}_j)$, are found[8] to be the most difficult to compute numerically from a one-dimensional integral representation involving b. However, as Thomas and Chan[8] point out, these terms may be evaluated in closed form using the methods of Thomas and Gerjuoy.[7,2] Correspondingly, the use of Eq. (11) in (10) permits a similar separation of the so-called single scattering terms in the Coulomb-modified Glauber amplitude for a multielectron-ion target; now these single scattering terms may be evaluated in closed form using the analytic methods to be described in Sec. III.

In the derivation of Eq. (10) it was assumed that the target ion was positively charged. As long as only inelastic transitions are considered, the unit term in Γ_a does not contribute. Under these circumstances it can be shown easily that Eq. (10) also is the Coulomb-modified Glauber-amplitude formula when the target is a negative ion, i.e., when the number of bound electrons M is greater than the nuclear charge Z_n. Thus Eq. (10) may be employed to compute the Glauber predictions for excitation and detachment of negative ions like H^- and Cl^- by charged-particle impact.

In a quantum-mechanical treatment of the scattering of a charged particle by an ionic target, an exact treatment of the asymptotic final state of the system is one in which the outgoing scattered particle is represented by a Coulomb wave in the field of the excess ionic charge. In such a treatment the scattering amplitude $A(i \to f)$ is found to diverge as K_f^{-1} as the final momentum $\hbar K_f$ of the scattered particle approaches zero at threshold. This behavior of the scattering amplitude leads to the generally expected theoretical prediction that the integrated excitation cross section for a bound-bound transition will be finite and nonzero at threshold. Of course, this expectation is generally borne out by the results of experiments. On the other hand, in the foregoing Glauber treatment of this scattering problem the final state of the outgoing charged particle has been treated in the derivation of Eq. (10) [as in the derivation[2] of Eq. (1)] as a plane wave rather than as a Coulomb wave in the residual Coulomb field. Moreover, it should be clear from the structure of Eq. (10), wherein K_f appears only implicitly (via $\vec{q} = \vec{K}_i - \vec{K}_f$), that these Glauber predicted inelastic scattering amplitudes will be finite at an inelastic threshold since K_i and $\vec{q}$ will be nonzero and η and α_n are bounded. Thus the integrated cross sections obtained via Eq. (10) will not be finite and nonzero at an excitation threshold, as generally expected, but rather will go smoothly to zero. This result poses no serious limitation upon the utility of the Glauber predictions to be obtained from Eq. (10). The Glauber approximation is, on general theoretic grounds,[2] expected to be valid only for incident-particle speeds greater than the average orbital speed of the bound electron to be excited, and not at threshold as the extant Glauber applications to the excitation of neutral atoms demonstrate.[2]

Equation (10), or an equivalent rearrangement thereof, has been employed by Narumi and Tsuji[4] and Ishihara and Chen[5] to obtain Glauber predictions for $n = 2$ excitation of He^+ by incident electrons; in such applications of Eq. (10) to the excitation of hydrogenlike ions, there is only one bound electron ($M = 1$), of course, and

$$\Gamma_a(\vec{b}, \vec{r}) = 1 - (|\vec{b} - \vec{s}|/b)^{2i\eta}.$$

We remark that the $1s \to 2s$ and $1s \to 2p$ Glauber predictions shown by Narumi and Tsuji[4] are clearly finite and nonzero at threshold. In view of the previous paragraph, these results for the Glauber-predicted e^--$He^+(n=2)$ excitation are suspect near threshold.

Further limitations on the applicability and utility of the Coulomb-modified form of the Glauber approximation are imposed by a seemingly unphysical, overly restrictive, symmetry of the conventional Glauber formalism first discussed by Gau and Macek.[12] Recall that in the conventional Glauber approximation the profile function $\Gamma(\vec{b}, \vec{r})$ of Eq. (1b) is to be evaluated[2] by integrating along the direction $\hat{\nu}$ perpendicular to $\vec{q}$. Thus, as long as $V_i(\vec{r}', \vec{r})$ in Eq. (1b) [or V_a in Eqs. (4) and (5)] is a sum of pairwise purely Coulombic potentials, $\Gamma(\vec{b}, \vec{r})$ (or $\Gamma_a = 1 - e^{i\chi_a}$) will be symmetric under reflections in the plane perpendicular to $\hat{\nu}$. It is easily shown that this reflection symmetry leads directly to the Glauber selection rule[7] described in Sec. III; for $1s \to nlm$ transitions in hydrogenic targets the selection rule states that the Glauber amplitude will vanish identically when $l - m$ is odd and the target bound states are quantized along $\hat{\nu}$. The physical consequences of this selection rule, and therefore of the reflection symmetry of $\Gamma(\vec{b}, \vec{r})$, are twofold. Firstly, the limitations of the selection rule will be manifested in the Glauber predictions of wide-angle inelastic scattering. In, for example, $n = 2$ excitation of hydrogenic ions the selection rule (with quantization axis along $\hat{\nu}$) implies that the $m = 0$ contribution to the dominant $1s \to 2p$ excitation vanishes identically. As we show in Sec. V A, at large K_i and large momentum transfers $\vec{q}$ (i.e., large scattering angles) the absolute squares of the nonvanishing $1s \to 2p$ amplitudes are proportional to q^{-6} and not the q^{-4} characteristic of pure Coulomb scattering. The effect of this large-q behavior is to cause the conventional Glauber approximation to underestimate the observed absolute e^--H(1s) $n=2$ angular distribution at wide scattering angles[12] even at moderately high incident-electron energies (~200 eV), where the Glauber predictions at scattering angles $\lesssim 30°$ and the Glauber-predicted integrated cross sections are in quite good agreement with the data.[2,12] Second, and perhaps more significantly, Gau and Macek[12] have argued that the symmetry properties of $\Gamma(\vec{b}, \vec{r})$ preclude reliable Glauber predictions of some of the expected properties of line radiation emitted subsequent to collisional excitation and observed in coincidence with the inelastically scattered particle. In particular, Gau and Macek have shown for electron–hydrogen-atom collisions that the Glauber approximation—like the first Born approximation—predicts the emitted Lyman-α radiation produced by direct (nonexchange) excitation of the $2p$ levels and observed in coincidence with the scattered electron is purely linearly polarized perpendicular to $\hat{\nu}$, corresponding to the radiation of a single electric dipole lying along the momentum-transfer vector $\vec{q}$.[12,13] The arguments of Gau and Macek are easily generalized so that the same result is obtained for the Coulomb-modified Glauber prediction of $n=2$ excitation in hydrogenlike ions and for the conventional Glauber prediction of direct $2\,{}^1S \to 2\,{}^1P$ (and $3\,{}^1P$) excitation in neutral helium atoms. This latter result is of particular significance because measurements by Eminyan *et al.*[14] of electron-photon angular correlation in helium $2\,{}^1P$ (and $3\,{}^1P$) excitation cannot be explained solely in terms of the excitation of a single electric dipole lying along the momentum transfer $\vec{q}$.

Despite the limitations the foregoing considerations place upon the utility and applicability of the Glauber approximation, we again stress that the presently available evidence[2] indicates that the Glauber approximation is a reasonably reliable predictor of the angular distributions for intermediate- and high-energy charged particles *inelastically* scattered at angles $\lesssim 90°$ by neutral-atom targets and of the integrated cross sections for these same collisions. Similarly, the Glauber predictions of the polarization fraction of the radiation emitted in electron–neutral-atom collisions appear to be in good agreement with experiment.[2,12,13] In this context the very recent Glauber applications to e^--He$^+$ excitation[4,5,6] suggest that the Coulomb-modified form of Glauber approximation will be an equally reliable predictor of these same properties of the inelastic scattering of charged particles by ionic targets.

III. REDUCTION TO CLOSED FORM OF THE GLAUBER AMPLITUDES FOR HYDROGENLIKE IONS

In this section we consider the Coulomb-modified Glauber amplitudes for the excitation of hydrogenlike ions by charged-particle impact. We shall show, in particular, that these Glauber amplitudes for the general induced transition $1s \to nlm$ can be reduced to closed form; nlm are the usual quantum numbers specifying the bound states of a hydrogenlike atom, neglecting spin effects. For the special case under consideration Eq. (10) reduces to the simple form

$$A(1s\to nlm;\vec{q}) = \frac{iK_i}{2\pi}\int d^2b\, d\vec{r}\, e^{i\vec{q}\cdot\vec{b}}\, b^{-2i\alpha_n} \times u^*_{nlm}(\vec{r})\left[1-\left(\frac{|\vec{b}-\vec{s}|}{b}\right)^{2i\eta}\right]u_{1s}(\vec{r}). \qquad (12)$$

Now $\vec{r}$ is simply the coordinate of the bound electron relative to the nucleus and $\vec{s}$ is the projection of $\vec{r}$ onto the plane containing $\vec{q}$ and $\vec{b}$. Recall that the $\hat{z}$ direction in Eq. (1), and therefore in (12), is to be taken along a direction $\hat{\nu}$ perpendicular to the plane containing $\vec{q}$ and $\vec{b}$. Of course $\eta = -Z_i e^2/\hbar v_i$ and $\alpha_n = -Z_i(Z_n-1)e^2/\hbar v_i$.

The bound-state wave functions appearing in Eq. (12) are known exactly; quantizing along the direction $\hat{\nu}\parallel\hat{z}$ they are given by[15]

$$u^*_{nlm}(\vec{r})u_{1s}(\vec{r}) = -\frac{2}{\sqrt{\pi}}\lambda_i^{3/2}\lambda_f^{3/2}\left(\frac{(n-l-1)!}{n[(n+l)!]^3}\right)^{1/2}(2\lambda_f r)^l \times L^{2l+1}_{n+l}(2\lambda_f r)e^{-(\lambda_i+\lambda_f)r}\,Y^{m*}_l(\hat{r}), \qquad (13)$$

with $\lambda_i = Z_n/a_0$, $\lambda_f = Z_n/na_0$, and $Y^m_l(\hat{r})$ is the conventional spherical harmonic as defined by Rose.[16] Rather than use the expansion of the Laguerre polynomial L^{2l+1}_{n+l} given by Schiff, we use a convenient alternative, namely,[7]

$$L^{2l+1}_{n+l}(x) = -\frac{[(n+l)!]^2}{(n-l-1)!(2l+1)!} \times {}_1F_1(-n+l+1;2l+2;x), \qquad (14)$$

where ${}_1F_1$ is the usual confluent hypergeometric function.[17] Thus we have

$$u^*_{nlm}u_{1s} = \frac{1}{\sqrt{\pi}}2^{l+1}\lambda_i^{3/2}\lambda_f^{3/2+l}\left(\frac{(n+l)!}{n(n-l-1)!}\right)^{1/2} \times \frac{1}{(2l+1)!}Y^{m*}_l(\hat{r}) \times \sum_{j=0}^{n-l-1}\frac{(-n+l+1)_j}{j!(2l+2)_j}(2\lambda_f)^j r^{j+l}e^{-\lambda r}, \qquad (15)$$

where $\lambda=\lambda_i+\lambda_f$ and $(a)_j$ is Pochhammer's symbol.[18] After inserting Eq. (15) into (12), we find that the amplitudes of (12) can be written as

$$A(1s\to nlm)=iK_i\,\mathfrak{A}_{nlm}\left\{\sum_{j=0}^{n-l-1}B_j\left(\frac{\partial}{\partial\lambda}\right)^{j+1}\mathcal{G}_{lm}(\lambda,\vec{q})\right\}\Bigg|_{\lambda=\lambda_i+\lambda_f}, \tag{16a}$$

where the generating function $\mathcal{G}_{lm}(\lambda,\vec{q})$ is given by

$$\mathcal{G}_{lm}(\lambda,\vec{q})=\frac{1}{2\pi}\int d^2b\,d\vec{r}\,e^{i\vec{q}\cdot\vec{b}}b^{-2i\alpha_n}r^{l-1}e^{-\lambda r}\times Y_l^{m*}(\hat{r})\left[1-\left(\frac{|\vec{b}-\vec{s}|}{b}\right)^{2i\eta}\right] \tag{16b}$$

and

$$\mathfrak{A}_{nlm}\equiv\frac{1}{\sqrt{\pi}}2^{l+1}\lambda_i^{3/2}\lambda_f^{3/2+l}\left(\frac{(n+l)!}{n(n-l-1)!}\right)^{1/2}\frac{1}{(2l+1)!}, \tag{16c}$$

$$B_j\equiv\frac{(-n+l+1)_j}{j!(2l+2)_j}(2\lambda_f)^j(-1)^{j+1}. \tag{16d}$$

We now proceed to show that the generating functions defined by Eq. (16b) can be evaluated in closed form. Unlike previous evaluations[7,8] of integrals like those in Eq. (16b), we introduce spherical coordinates for $\vec{r}$ with the z direction along $\hat{\nu}\perp\vec{q}$ and polar coordinates for $\vec{b}$. Thus

$$\mathcal{G}_{lm}(\lambda,\vec{q})=\int_0^\infty b\,db\,b^{-2i\alpha_n}\int_0^{2\pi}d\varphi_b\,e^{iqb\cos(\varphi_q-\varphi_b)}\int_0^\infty dr\,r^{l+1}e^{-\lambda r}\frac{1}{2\pi}\int d\Omega_r\,Y_l^{m*}(\hat{r})\left[1-\left(\frac{|\vec{b}-\vec{s}|}{b}\right)^{2i\eta}\right], \tag{17}$$

where φ_q and φ_b are the azimuthal angles of $\vec{q}$ and $\vec{b}$ in the plane perpendicular to $\hat{\nu}$, and $\vec{s}$ is specified by $s=r\sin\theta$ and the azimuthal angle φ.

The integration over solid angle Ω_r is described in Appendix A. Using Eq. (A13) of that appendix, we have

$$\mathcal{G}_{lm}(\lambda,\vec{q})=2(-1)^{(l-m)/2}\int_0^\infty b\,db\,b^{-2i\alpha_n}\int_0^{2\pi}d\varphi_b\,e^{iqb\cos(\varphi_q-\varphi_b)}Y_l^{m*}(\pi/2,\varphi_b)\int_0^\infty dr\,r^{l+1}e^{-\lambda r}\times\left\{\delta_{l,0}\delta_{m,0}+2^{2i\eta}\frac{\Gamma(1+i\eta)}{\Gamma(1-i\eta)}\int_0^\infty dt\,t^{-2i\eta}\frac{d}{dt}\left[J_m(t)\left(\frac{\pi b}{2rt}\right)^{1/2}J_{l+1/2}\left(\frac{rt}{b}\right)\right]\right\}. \tag{18}$$

In Eq. (18) we can exploit the elementary properties of spherical harmonics to extract the φ_q dependence of the integrals and obtain

$$\mathcal{G}_{lm}(\lambda,\vec{q})=2(-1)^{(l-m)/2}Y_l^{m*}(\pi/2,\varphi_q)\int_0^\infty b\,db\,b^{-2i\alpha_n}\int_0^{2\pi}d\varphi\,e^{iqb\cos\varphi+im\varphi}\int_0^\infty dr\,r^{l+1}e^{-\lambda r}\times\left\{\delta_{l,0}\delta_{m,0}+2^{2i\eta}\frac{\Gamma(1+i\eta)}{\Gamma(1-i\eta)}\int_0^\infty dt\,t^{-2i\eta}\frac{d}{dt}\left[J_m(t)\left(\frac{\pi b}{2rt}\right)^{1/2}J_{l+1/2}\left(\frac{rt}{b}\right)\right]\right\}. \tag{19}$$

Note that the spherical harmonic $Y_l^{m*}(\pi/2,\varphi_q)$ vanishes if $l-m$ is an odd positive or negative integer (see also Appendix A). Thus Eq. (19) incorporates in a succinct way the Glauber selection rule, namely, that the Glauber amplitudes of Eq. (12) vanish identically for $l-m$ odd when the bound-state quantization axis is along $\hat{\nu}\perp\vec{q}$. In Eq. (19), the integration over φ yields[19]

$$\int_0^{2\pi}d\varphi\,e^{iqb\cos\varphi+im\varphi}=2\pi i^m J_m(qb).$$

As for the r integration, the integral stemming from the first term in curly brackets is trivial; in the second term we assume interchanging orders of integration and differentiation is valid and find[20]

$$\int_0^\infty dr\,r^{l+1}e^{-\lambda r}\left(\frac{\pi b}{2rt}\right)^{1/2}J_{l+1/2}\left(\frac{rt}{b}\right)=2^l\Gamma(l+1)t^l b^{l+2}(t^2+\lambda^2b^2)^{-l-1}.$$

Consequently, Eq. (19) reduces to

$$\mathcal{G}_{lm}(\lambda,\vec{q}) = 4\pi i^l Y_l^{m*}(\pi/2,\varphi_q)\int_0^\infty db\, b^{-2i\alpha_n} J_m(qb) \times\left\{\lambda^{-2}\delta_{l,0}\delta_{m,0} + 2^{2i\eta+l}\frac{\Gamma(1+i\eta)}{\Gamma(1-i\eta)}\Gamma(l+1)b^{l+2}\int_0^\infty dt\, t^{-2i\eta}\frac{d}{dt}\left(\frac{t^l J_m(t)}{(t^2+\lambda^2 b^2)^{l+1}}\right)\right\}. \quad (20)$$

In view of the remarks following Eq. (A13) in Appendix A, we need to consider two separate cases of Eq. (20), namely, $l>0$ and $l=0$.

First the case $l>0$: When $l>0$ the first term in curly brackets in Eq. (20) does not appear and the second term may be integrated once by parts on t. Since[21] $J_{-m}(x)=(-1)^m J_m(x)$, $J_m(x)J_m(y)=J_{|m|}(x)J_{|m|}(y)$ and

$$\mathcal{G}_{lm}(\lambda,\vec{q}) = 4\pi i^l Y_l^{m*}(\pi/2,\varphi_q)(2i\eta)2^{2i\eta+l}\frac{\Gamma(1+i\eta)}{\Gamma(1-i\eta)}\Gamma(l+1)\int_0^\infty db\, b^{-2i\alpha_n+l+3}J_{|m|}(qb) \times\int_0^\infty dt\, t^{-2i\eta-1+l}J_{|m|}(t)\,(t^2+\lambda^2 b^2)^{-l-1}. \quad (21)$$

We now observe that the rational function $(t^2+\lambda^2b^2)^{-l-1}$ is a special case of the hypergeometric function; in particular,[22]

$$(t^2+\lambda^2b^2)^{-l-1} = t^{-2l-2}\left(1+\frac{\lambda^2b^2}{t^2}\right)^{-l-1} = t^{-2l-2}\,{}_2F_1\left(l+1, c; c; -\frac{\lambda^2b^2}{t^2}\right), \quad (22)$$

where c is an arbitrary complex number. Since $l+1$ is not a negative integer, the ${}_2F_1$ in Eq. (22) has the Mellin-Barnes integral representation[23]

$$(t^2+\lambda^2b^2)^{-l-1} = t^{-2l-2}\frac{1}{\Gamma(l+1)}\frac{1}{2\pi i}\int_{-\delta-i\infty}^{-\delta+i\infty} ds\,\Gamma(l+1+s)\Gamma(-s)\left(\frac{\lambda^2b^2}{t^2}\right)^s, \quad (23)$$

provided $|\arg(\lambda^2b^2/t^2)|<\pi$. The path of integration in (23) is a straight line parallel to the imaginary s axis; δ is real and satisfies $0<\delta<l+1$. We now use Eq. (23) in (21). Furthermore, we assume that there exists a range of δ, bounded by the limits $0<\delta<l+1$, for which both the t and b integrations in (21) are well defined and convergent for each and every value of s on the contour, so that we may interchange the order of integration over s with the integrations over b and t. With this assumption we obtain

$$\mathcal{G}_{lm}(\lambda,\vec{q}) = 4\pi i^l Y_l^{m*}(\pi/2,\varphi_q)i\eta 2^{2i\eta+l+1}\frac{\Gamma(1+i\eta)}{\Gamma(1-i\eta)}\frac{1}{2\pi i} \times\int_{-\delta-i\infty}^{-\delta+i\infty} ds\,\Gamma(l+1+s)\Gamma(-s)\lambda^{2s}\int_0^\infty db\, b^{-2i\alpha_n+l+3+2s}J_{|m|}(qb)\int_0^\infty dt\, t^{-2i\eta-3-l-2s}J_{|m|}(t), \quad (24)$$

wherein the b and t integrations have been separated. In Eq. (24), the b and t integrals each are of the form[20]

$$\int_0^\infty dx\, x^{\mu-1}J_\nu(ax) = 2^{\mu-1}a^{-\mu}\frac{\Gamma(\frac{1}{2}(\nu+\mu))}{\Gamma(1+\frac{1}{2}(\nu-\mu))} \quad (25)$$

subject to the restriction $-\mathrm{Re}\,\nu<\mathrm{Re}\,\mu<\frac{3}{2}$. Thus in Eq. (24) the b integration will converge if

$$-|m|<\mathrm{Re}(-2i\alpha_n+l+4+2s)<\tfrac{3}{2}, \quad (26a)$$

i.e.,

$$\tfrac{1}{2}l+\tfrac{5}{4}<\delta<\tfrac{1}{2}(l+|m|)+2. \quad (26b)$$

Similarly the t integration will converge provided

$$-|m|<\mathrm{Re}(-2i\eta-2-l-2s)<\tfrac{3}{2}, \quad (27a)$$

or

$$\tfrac{1}{2}(l-|m|)+1<\delta<\tfrac{1}{2}l+\tfrac{7}{4}. \quad (27b)$$

Recalling that $0\le|m|\le l$, we see that $\frac{1}{2}(l-|m|)+1<\frac{1}{2}l+\frac{5}{4}$ and that $\frac{1}{2}l+\frac{7}{4}<\frac{1}{2}(l+|m|)+2$. Therefore, the inequalities (26b) and (27b) combine to yield the condition

$$\tfrac{1}{2}l+\tfrac{5}{4}<\delta<\tfrac{1}{2}l+\tfrac{7}{4} \quad (28)$$

for convergence of *both* the t and b integrations. But $l>0$ and δ also must satisfy $0<\delta<l+1$ in order for (23) to be valid. Since $l+1>\frac{1}{2}l+\frac{5}{4}$ for all $l>0$, we must have

$$\tfrac{1}{2}l+\tfrac{5}{4}<\delta<\min(l+1, \tfrac{1}{2}l+\tfrac{7}{4}) \quad (29)$$

in order for the b and t integrations in (24) to be well defined and convergent for each and every value of s on the s contour when (23) holds. Thus if (29) holds, Eq. (24) is equivalent to Eq. (21).

Having determined the required condition (29) on δ, we may proceed to evaluate the b and t integrals in (24) via (25). It is easily seen that Eq. (24) reduces to

$$\mathcal{g}_{lm}(\lambda, \vec{q}) = 4\pi i^l Y_l^{m*}(\pi/2, \varphi_q) i\eta 2^{-2i\alpha_n+l+1} q^{2i\alpha_n-l-4} \frac{\Gamma(1+i\eta)}{\Gamma(1-i\eta)}$$

$$\times \frac{1}{2\pi i} \int_{-\delta-i\infty}^{-\delta+i\infty} ds \frac{\Gamma(l+1+s)\Gamma(s+2+\frac{1}{2}(l+|m|)-i\alpha_n)\Gamma(-s)\Gamma(\frac{1}{2}(|m|-l)-1-i\eta-s)}{\Gamma(\frac{1}{2}(|m|-l)-1+i\alpha_n-s)\Gamma(s+2+\frac{1}{2}(l+|m|)+i\eta)} \left(\frac{\lambda^2}{q^2}\right)^s \tag{30}$$

provided $l>0$ and (29) holds. Note that the condition (29) on δ is sufficiently strong to ensure that the poles of $\Gamma(l+1+s)$ and $\Gamma(s+2+\frac{1}{2}(l+|m|)-i\alpha_n)$ lie to the left of the contour, and that the poles of $\Gamma(-s)$ and $\Gamma(\frac{1}{2}(|m|-l)-1-i\eta-s)$ lie to the right. Indeed, the weaker condition

$$1+\tfrac{1}{2}(l-|m|)<\delta<\min(l+1, 2+\tfrac{1}{2}(l+|m|)) \tag{31}$$

is sufficient to guarantee the separation of these poles. We now change variables in (30) via $t = s+1+\frac{1}{2}(l-|m|)+i\eta$ so that

$$\mathcal{g}_{lm}(\lambda, \vec{q}) = 4\pi i^l Y_l^{m*}(\pi/2, \varphi_q) i\eta 2^{l+1-2i\alpha_n} q^{2i\alpha_n-l-4} \frac{\Gamma(1+i\eta)}{\Gamma(1-i\eta)}$$

$$\times \frac{1}{2\pi i} \int_{-\epsilon-i\infty}^{-\epsilon+i\infty} dt \frac{\Gamma(t-i\eta+\frac{1}{2}(l+|m|))\Gamma(t+1+|m|-i\eta-i\alpha_n)}{\Gamma(i\eta+i\alpha_n-t)} \frac{\Gamma(1+\frac{1}{2}(l-|m|)+i\eta-t)\Gamma(-t)}{\Gamma(t+1+|m|)} \left(\frac{\lambda^2}{q^2}\right)^{t-i\eta-(l-|m|)/2-1}, \tag{32a}$$

where the inequality (31) implies that

$$0<\epsilon<\min(\tfrac{1}{2}(l+|m|), 1+|m|) . \tag{32b}$$

The integral on the right side of (32a) can be identified directly as the standard Mellin-Barnes representation of a Meijer G function; in particular, we note that the function $G_{33}^{22}(x)$ can be defined via[24]

$$G_{33}^{22}\left(x \,\middle|\, {a,b,c \atop d,e,f}\right) = \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} ds \frac{\Gamma(1+s-a)\Gamma(1+s-b)\Gamma(d-s)\Gamma(e-s)}{\Gamma(c-s)\Gamma(1+s-f)} x^s \tag{33}$$

provided $|\arg x|<\pi$. In Eq. (33) the integration path runs from $-i\infty$ to $+i\infty$ so that the poles of $\Gamma(1+s-a)$ and $\Gamma(1+s-b)$ lie to the left of the path, while the poles of $\Gamma(d-s)$ and $\Gamma(e-s)$ lie to the right of the path; in other words a, b, d, and e are restricted so that none of the poles of $\Gamma(1+s-a)$ and $\Gamma(1+s-b)$ may coincide with the poles of $\Gamma(d-s)$ and $\Gamma(e-s)$.

Since (32b) holds, we may apply (33) directly to (32a) and find, when $l>0$, that Eq. (20) reduces to

$$\mathcal{g}_{lm}(\lambda, \vec{q}) = 4\pi i^l Y_l^{m*}(\pi/2, \varphi_q) i\eta 2^{l+1-2i\alpha_n} \frac{\Gamma(1+i\eta)}{\Gamma(1-i\eta)} \lambda^{-2i\eta-l+|m|-2}$$

$$\times q^{2i\alpha_n+2i\eta-|m|-2} G_{33}^{22}\left(\frac{\lambda^2}{q^2} \,\middle|\, {1+i\eta-(l+|m|)/2,\, i\eta+i\alpha_n-|m|,\, i\eta+i\alpha_n \atop 0,\, 1+i\eta+(l-|m|)/2,\, -|m|}\right) \tag{34}$$

provided $|\arg(\lambda^2/q^2)|<\pi$. The G function in (34) will be discussed below. Note that $l\pm|m|$ must be an even integer in (34) since $Y_L^{m*}(\pi/2, \varphi_q)$ vanishes otherwise.

We now consider the special case of Eq. (20) when $l=0$. As we remark in Appendix A following Eq. (A13), when $l=0$, integrating Eq. (20) once by parts on the variable t is no longer valid; the indicated differentiation must be performed. When $l=0$, the terms enclosed by curly brackets in (20) reduce to

$$\{\cdots\} = \lambda^{-2} + 2^{2i\eta} \frac{\Gamma(1+i\eta)}{\Gamma(1-i\eta)} b^2 \int_0^\infty dt\, t^{-2i\eta} \frac{d}{dt}\left(\frac{J_0(t)}{t^2+\lambda^2 b^2}\right) \tag{35a}$$

$$= \lambda^{-2} \left\{ 1 + 2^{2i\eta} \frac{\Gamma(1+i\eta)}{\Gamma(1-i\eta)} \times \int_0^\infty dt\, t^{-2i\eta} \frac{d}{dt}\left(J_0(t) \frac{\lambda^2 b^2}{t^2+\lambda^2 b^2}\right) \right\} . \tag{35b}$$

But

$$\lambda^2 b^2(t^2+\lambda^2 b^2)^{-1} = 1 - t^2(t^2+\lambda^2 b^2)^{-1}, \quad (36)$$

so that

$$\{\cdots\} = \lambda^{-2}\left\{1 - 2^{2i\eta}\frac{\Gamma(1+i\eta)}{\Gamma(1-i\eta)}\int_0^\infty dt\, t^{-2i\eta}J_1(t) - 2^{2i\eta}\frac{\Gamma(1+i\eta)}{\Gamma(1-i\eta)} \times \int_0^\infty dt\, t^{-2i\eta}\frac{d}{dt}\left(\frac{t^2 J_0(t)}{t^2+\lambda^2 b^2}\right)\right\}, \quad (37)$$

where we have used Eq. (36) in (35b), together with the relation[25] $dJ_0(t)/dt = -J_1(t)$, to obtain Eq. (37). The b-independent integral on the right side of (37) may be evaluated via (25); it is easily seen that

$$\int_0^\infty dt\, t^{-2i\eta}J_1(t) = 2^{-2i\eta}\frac{\Gamma(1-i\eta)}{\Gamma(1+i\eta)},$$

and the unit term in (37) is canceled exactly. The remaining term in (37) may be integrated once by parts. Thus when $l=0$, the term in curly brackets in Eq. (20) becomes

$$\{\cdots\} = -i\eta\lambda^{-2}2^{2i\eta+1}\frac{\Gamma(1+i\eta)}{\Gamma(1-i\eta)} \times \int_0^\infty dt\, t^{-2i\eta+1}J_0(t)(t^2+\lambda^2 b^2)^{-1}. \quad (38)$$

Equation (38) now may be used directly in Eq. (20); the subsequent reduction of $\mathcal{G}_{00}(\lambda,\vec{q})$ to closed form employs the same methods used to reduce Eq. (21) to the form (32) when $l>0$. Indeed we find that

$$\mathcal{G}_{00}(\lambda,\vec{q}) = -4\pi Y_0^{0*}(\pi/2,\varphi_q)i\eta 2^{1-2i\alpha_n}\frac{\Gamma(1+i\eta)}{\Gamma(1-i\eta)}\lambda^{-2i\eta-2}q^{2i\alpha_n+2i\eta-2} \times \frac{1}{2\pi i}\int_{-\epsilon-i\infty}^{-\epsilon+i\infty} dt\,\frac{\Gamma(t+1-i\eta)\Gamma(t+1-i\alpha_n-i\eta)\Gamma(i\eta-t)\Gamma(-t)}{\Gamma(i\eta+i\alpha_n-t)\Gamma(1+t)}\left(\frac{\lambda^2}{q^2}\right)^t, \quad (39)$$

where $0<\epsilon<1$ and the straight-line integration path separates the poles of $\Gamma(t+1-i\eta)$ and $\Gamma(t+1-i\eta-i\alpha_n)$ from those of $\Gamma(-t)$ and $\Gamma(i\eta-t)$. Finally, we again employ (33) and obtain

$$\mathcal{G}_{00}(\lambda,\vec{q}) = -4\pi Y_0^{0*}(\pi/2,\varphi_q)i\eta 2^{1-2i\alpha_n}\frac{\Gamma(1+i\eta)}{\Gamma(1-i\eta)}\lambda^{-2i\eta-2}q^{2i\alpha_n+2i\eta-2}\,G_{33}^{22}\left(\frac{\lambda^2}{q^2}\middle|\begin{matrix}i\eta,\,i\eta+i\alpha_n,\,i\eta+i\alpha_n\\0,\,i\eta,\,0\end{matrix}\right) \quad (40)$$

provided $|\arg(\lambda^2/q^2)|<\pi$.

It should be noted that Eq. (40) cannot be obtained from (34) simply by setting $l=0$ and $m=0$: the G function so obtained from (34) is different from that in (40). This difference is a manifestation of the presence of the term $\lambda^{-2}\delta_{l,0}\delta_{m,0}$ in (20) when $l=0$ and $m=0$. On the other hand the functions G_{33}^{22} do satisfy the relation[24]

$$(1+a-b)G_{33}^{22}(x\,|\,{}^{a,\,i\eta+i\alpha_n,\,i\eta+i\alpha_n}_{0,\,b,\,0}) = G_{33}^{22}(x\,|\,{}^{a-1,\,i\eta+i\alpha_n,\,i\eta+i\alpha_n}_{0,\,b,\,0}) + G_{33}^{22}(x\,|\,{}^{a,\,i\eta+i\alpha_n,\,i\eta+i\alpha_n}_{0,\,b+1,\,0}). \quad (41)$$

When $a=1+i\eta$ and $b=i\eta$, the G functions on the right side of (41) are just those appearing in (40) and in (34) when $l=0$. But $1-a+b$ would then vanish. If one naively assumes that the left-hand side of (41) therefore vanishes, one is led to the erroneous conclusion that for $l=0$ (34) is the same as (40). On the contrary, it can be shown that the left-hand side of (41) does not vanish when $a=1+i\eta$ and $b=i\eta$. From the standard expansions of the G function in terms of convergent generalized hypergeometric functions[24] (about which more shall be said later), one can show that

$$(1-a+b)G_{33}^{22}(x\,|\,{}^{a,\,i\eta+i\alpha_n,\,i\eta+i\alpha_n}_{0,\,b,\,0})\Big|_{\substack{a=1+i\eta\\ b=i\eta}} = \frac{\Gamma(-i\eta)\Gamma(1-i\alpha_n)}{\Gamma(1+i\eta)\Gamma(i\alpha_n)}x^{i\eta}; \quad (42)$$

in other words,

$$G_{33}^{22}(x\,|\,{}^{1+i\eta,\,i\eta+i\alpha_n,\,i\eta+i\alpha_n}_{0,\,1+i\eta,\,0}) = -G_{33}^{22}(x\,|\,{}^{i\eta,\,i\eta+i\alpha_n,\,i\eta+i\alpha_n}_{0,\,i\eta,\,0}) + \frac{\Gamma(-i\eta)\Gamma(1-i\alpha_n)}{\Gamma(1+i\eta)\Gamma(i\alpha_n)}x^{i\eta}. \quad (43)$$

Of course, the relation (43) may be proved directly from the integral representation (33). However, the foregoing remarks serve to emphasize sufficiently strongly the fundamental differences between the $\mathcal{G}_{00}$ obtained by improper usage of Eq. (34) with $l=0$, and the correct result (40).

One may legitimately ask what would be the consequences of using (34), rather than (40), in Eq. (16a) to compute the Coulomb-modified Glauber amplitudes for the excitation of s states. Equation (43) with $x=\lambda^2/q^2$ makes clear that the $\mathcal{G}_{00}$ obtained from (34) would separate into the sum of two terms, one of which would be proportional to λ^{-2}, with the other term given by (40). As long as we are considering the excitation of s states in a hydrogenlike ion (i.e., $n>1$), for which the product of the wave functions $u_f^* u_i$ is given exactly by (13), wherein $\lambda_i = Z_n/a_0$ and $\lambda_f = Z_n/na_0$ ($\lambda = \lambda_i + \lambda_f$), one can show that the differential operator of Eq. (16) acting upon λ^{-2} yields a term

$$\sum_{j=0}^{n-1} \frac{(-n+1)_j}{j!\,(2)_j}(2\lambda_f)^j\left(-\frac{\partial}{\partial\lambda}\right)^{j+1}\lambda^{-2} \equiv 0 \tag{44}$$

only if $\lambda_f/\lambda_i \equiv 1/n$. Equation (44) may be proved directly by performing the differentiation and summing the series using the standard transformation formulas[26] for the hypergeometric functions ${}_2F_1$. On the other hand we may simply observe that the λ^{-2} term of Eq. (20) stems directly from the unit term in square brackets in (16b); thus aside from multiplicative factors, λ^{-2} is the generating function which yields, via the left side of (44), the integral $\int d\vec{r}\, u_{ns}^*(\vec{r}) u_{1s}(\vec{r})$ in Eq. (12). But u_{ns} and u_{1s} are *exactly* orthogonal; hence (44) must be true. Despite the fact that the λ^{-2} term implicitly contained in (34) cannot finally contribute to the $1s \to ns$ inelastic scattering amplitude, there is no guarantee that the direct use of (34) in (16a) will not lead to spurious results when the amplitudes are computed numerically. Thus (40) should be used for these transitions to avoid this possible source of numerical error.

To summarize the results obtained in this present section, we have shown, for a hydrogenlike ionic target, with nuclear charge $Z_n e$, that the Coulomb-modified Glauber amplitudes for the direct excitation of the transition $1s \to nlm$ by a structureless particle of charge $Z_i e$ can be reduced to closed form. With quantization axis along $\hat{\nu} \perp \vec{q}$, these amplitudes have the form

$$A(1s \to nlm; \vec{q}) = iK_i \frac{1}{\sqrt{\pi}} 2^{l+1} \lambda_i^{3/2} \lambda_f^{3/2+l} \left(\frac{(n+l)!}{n(n-l-1)!}\right)^{1/2} \frac{1}{(2l+1)!}$$

$$\times \sum_{j=0}^{n-l-1} \frac{(-n+l+1)_j}{j!\,(2l+2)_j}(2\lambda_f)^j \left(-\frac{\partial}{\partial\lambda}\right)^{j+1} \mathcal{G}_{lm}(\lambda, \vec{q})\bigg|_{\lambda=\lambda_i+\lambda_f}, \tag{45a}$$

where, for $l \neq 0$,

$$\mathcal{G}_{lm}(\lambda, \vec{q}) = i\eta 4\pi i^l Y_l^{m*}(\pi/2, \varphi_q) 2^{l+1-2i\alpha_n} \frac{\Gamma(1+i\eta)}{\Gamma(1-i\eta)} \lambda^{-2i\eta-l+|m|-2}$$

$$\times q^{2i\alpha_n+2i\eta-|m|-2} G_{33}^{22}\left(\frac{\lambda^2}{q^2}\,\middle|\,\begin{matrix} 1+i\eta-(l+|m|)/2,\, i\eta+i\alpha_n-|m|,\, i\eta+i\alpha_n \\ 0,\, 1+i\eta+(l-|m|)/2,\, -|m| \end{matrix}\right); \tag{45b}$$

if $l=0$

$$\mathcal{G}_{00}(\lambda, \vec{q}) = -i\eta 4\pi Y_0^{0*}(\pi/2, \varphi_q) 2^{1-2i\alpha_n} \frac{\Gamma(1+i\eta)}{\Gamma(1-i\eta)} \lambda^{-2i\eta-2}\, q^{2i\alpha_n+2i\eta-2} G_{33}^{22}\left(\frac{\lambda^2}{q^2}\,\middle|\,\begin{matrix} i\eta,\, i\eta+i\alpha_n,\, i\eta+i\alpha_n \\ 0,\, i\eta,\, 0 \end{matrix}\right), \tag{45c}$$

with

$$\eta = -Z_i e^2/\hbar v_i, \quad \alpha_n = -Z_i(Z_n-1)e^2/\hbar v_i, \quad \lambda_i = Z_n/a_0, \quad \lambda_f = Z_n/na_0,$$

and G_{33}^{22} is the Meijer G function. We again emphasize that the Glauber selection rule for these transitions is contained within the spherical harmonic $Y_l^m(\pi/2, \varphi_q)$, which vanishes identically if $l \pm m$ is an *odd* integer.

Although the Meijer G functions in Eq. (45) appear formidable, they are actually comparatively trivial to compute. In fact the functions G_{33}^{22} have standard absolutely convergent expansions[24] in terms of the generalized hypergeometric functions ${}_3F_2$; these expansions are obtained directly from the integral representations (32) or (40). As long as $\alpha_n \neq 0$, the poles of the integrand in (32) or (40) are simple. Thus when $\lambda^2/q^2 < 1$ we close the contour at infinity in the right half plane and obtain, via the residue theorem, a power-series expansion for G_{33}^{22} in powers of λ^2/q^2. When $\lambda^2/q^2 > 1$, the contour is closed on the left, which yields an expansion in powers of q^2/λ^2. Now the G functions in (45) have the general form

$$G_{33}^{22}\left(x\,\middle|\,\begin{matrix} i\eta-p,\, i\eta+i\alpha_n-|m|,\, i\eta+i\alpha_n \\ 0,\, i\eta+r,\, -|m| \end{matrix}\right),$$

where p and r are integers such that $p, r \geq 0$; of course $|m| \geq 0$. It is found[24] that when $|x| < 1$

$$G_{33}^{22}\left(x \,\middle|\, {}^{i\eta-p,\; i\eta+i\alpha_n-|m|,\; i\eta+i\alpha_n}_{0,\; i\eta+r,\; -|m|}\right)$$

$$= \frac{\Gamma(i\eta+r)\Gamma(1+p-i\eta)\Gamma(1+|m|-i\eta-i\alpha_n)}{\Gamma(1+|m|)\Gamma(i\eta+i\alpha_n)}\, {}_3F_2(1+p-i\eta,\, 1+|m|-i\eta-i\alpha_n,\, 1-i\eta-i\alpha_n;\, 1-r-i\eta,\, 1+|m|;\, -x)$$

$$+ \frac{\Gamma(-i\eta-r)\Gamma(1+p+r)\Gamma(1+r+|m|-i\alpha_n)}{\Gamma(1+r+|m|+i\eta)\Gamma(i\alpha_n-r)}$$

$$\times x^{i\eta+r}\, {}_3F_2(1+r+p,\, 1+r+|m|-i\alpha_n,\, 1+r-i\alpha_n;\, 1+r+i\eta,\, 1+r+|m|+i\eta;\, -x)\,; \tag{46a}$$

when $|x|>1$

$$G_{33}^{22}\left(x \,\middle|\, {}^{i\eta-p,\; i\eta+i\alpha_n-|m|,\; i\eta+i\alpha_n}_{0,\; i\eta+r,\; -|m|}\right)$$

$$= \frac{\Gamma(|m|-p-i\alpha_n)\Gamma(1+p-i\eta)\Gamma(1+p+r)}{\Gamma(1+p+i\alpha_n)\Gamma(|m|-p+i\eta)}$$

$$\times x^{i\eta-p-1}\, {}_3F_2(1+p-i\eta,\, 1+p+r,\, 1+p-|m|-i\eta;\, 1+p-|m|+i\alpha_n,\, 1+p+i\alpha_n;\, -1/x)$$

$$+ \frac{\Gamma(p-|m|+i\alpha_n)\Gamma(1+|m|-i\eta-i\alpha_n)\Gamma(1+|m|+r-i\alpha_n)}{\Gamma(1+|m|)\Gamma(i\eta+i\alpha_n)}$$

$$\times x^{i\eta+i\alpha_n-|m|-1}\, {}_3F_2(1+|m|-i\eta-i\alpha_n,\, 1+|m|+r-i\alpha_n,\, 1-i\eta-i\alpha_n;\, 1+|m|-p-i\alpha_n,\, 1+|m|;\, -1/x)\,. \tag{46b}$$

The functions ${}_3F_2$ appearing in Eq. (46) are absolutely convergent in the specified range of x provided α_n and η are both nonzero.[27] Of course, the amplitudes themselves are constructed from the function $\mathcal{G}_{lm}$ by differentiation according to (45a). The G functions may be differentiated via the relation[24]

$$x\frac{d}{dx}G_{33}^{22}\left(x \,\middle|\, {}^{i\eta-p,\; i\eta+i\alpha_n-|m|,\; i\eta+i\alpha_n}_{0,\; i\eta+r,\; -|m|}\right)$$

$$= G_{33}^{22}\left(x \,\middle|\, {}^{i\eta-p-1,\; i\eta+i\alpha_n-|m|,\; i\eta+i\alpha_n}_{0,\; i\eta+r,\; -|m|}\right) + (i\eta-p-1)G_{33}^{22}\left(x \,\middle|\, {}^{i\eta-p,\; i\eta+i\alpha_n-|m|,\; i\eta+i\alpha_n}_{0,\; i\eta+r,\; -|m|}\right). \tag{47}$$

Since (45a) involves only a finite number of differentiations, the amplitudes $A(1s\to nlm;\vec{q})$ may be expressed using (47) as a finite sum of G functions, each of which may be computed via Eq. (46). Although this procedure always may be carried out in principle, in actual practice it may prove excessively tedious to do so, especially if $n-l\gtrsim 5$. Alternatively, we may take useful advantage of the absolute convergence of each of the series in Eq. (46); the differentiation required by (45a) may be performed by explicitly differentiating the series expansions obtained by using Eq. (46) in (45b) or (45c). Even though the resulting series will no longer be readily identifiable as simple functions like ${}_3F_2$ or G_{33}^{22}, these series will be absolutely convergent and of simple enough structure so that numerical evaluation of the amplitude is straightforward and rapid.

IV. SPECIAL CASES

The Coulomb-modified Glauber amplitudes for the two transitions $1s\to 2s$ and $1s\to 2p$ have been discussed by Narumi and Tsuji[4] and Ishihara and Chen.[5] As mentioned previously, these authors are only able to compute these Glauber amplitudes from one-dimensional integral representations which must be evaluated numerically. From Eq. (45) these amplitudes have comparatively simple closed-form expressions which are useful examples of the general result.

Consider first the $1s\to 2s$ transition amplitude for which $n=2$ and $l=m=0$. We have from (45a)

$$A(1s\to 2s;\vec{q}) = -iK_i\frac{2}{\sqrt{\pi}}\lambda_i^{3/2}\lambda_f^{3/2} \times\left\{\left(1+\lambda_f\frac{\partial}{\partial\lambda}\right)\frac{\partial}{\partial\lambda}\mathcal{G}_{00}(\lambda,\vec{q})\right\}\bigg|_{\lambda=\lambda_i+\lambda_f}, \tag{48a}$$

with $\mathcal{G}_{00}(\lambda,\vec{q})$ given by (45c). Performing the indicated differentiation in (48a) via (47), we make use of the fact that $\lambda_i = Z_n/a_0$, $\lambda_f=\frac{1}{2}\lambda_i$, and $\lambda=\frac{3}{2}\lambda_i$ to collect terms and obtain

$$A(1s\to 2s;\vec{q})=-K_i\eta 2^4\left(\frac{\sqrt{2}}{3}\right)^3 2^{-2i\alpha_n}\frac{\Gamma(1+i\eta)}{\Gamma(1-i\eta)}q^{2i\alpha_n+2i\eta-2}\lambda^{-2i\eta}$$
$$\times\left\{\tfrac{4}{3}G_{33}^{22}\left(\frac{\lambda^2}{q^2}\middle|{i\eta,\,i\eta+i\alpha_n,\,i\eta+i\alpha_n \atop 0,\,i\eta,\,0}\right)-\tfrac{8}{3}G_{33}^{22}\left(\frac{\lambda^2}{q^2}\middle|{i\eta-1,\,i\eta+i\alpha_n,\,i\eta+i\alpha_n \atop 0,\,i\eta,\,0}\right)+\tfrac{2}{3}G_{33}^{22}\left(\frac{\lambda^2}{q^2}\middle|{i\eta-2,\,i\eta+i\alpha_n,\,i\eta+i\alpha_n \atop 0,\,i\eta,\,0}\right)\right\}\tag{48b}$$

for arbitrary $Z_n>1$.

For the $1s\to 2p$ transition we now have

$$A(1s\to 2p_m;\vec{q})=-iK_i\frac{1}{\sqrt{\pi}}\lambda_i^{3/2}\lambda_f^{5/2}\frac{2}{\sqrt{3}}\left\{\frac{\partial}{\partial\lambda}\mathcal{G}_{1m}(\lambda,\vec{q})\right\}\Bigg|_{\lambda=\lambda_i+\lambda_f},\tag{49a}$$

where $\mathcal{G}_{1m}$ is given by (45b). Now $Y_1^{0*}(\pi/2,\varphi_q)\equiv 0$; thus the $m=0$ transition amplitude vanishes and we are left with

$$\mathcal{G}_{1,\pm 1}(\lambda,\vec{q})=\mp i\eta(3\pi)^{1/2}\frac{i}{\sqrt{2}}e^{\mp i\varphi_q}2^{3-2i\alpha_n}\frac{\Gamma(1+i\eta)}{\Gamma(1-i\eta)}q^{2i\alpha_n+2i\eta-3}\lambda^{-2i\eta-2}G_{33}^{22}\left(\frac{\lambda^2}{q^2}\middle|{i\eta,\,i\eta+i\alpha_n-1,\,i\eta+i\alpha_n \atop 0,\,1+i\eta,\,-1}\right).\tag{49b}$$

Again we use (47) to perform the differentiation required in Eq. (49a). Now, however, the result is the sum of two G functions which may be combined via the generalization[24] of Eq. (41). We find that $1s\to 2p$ amplitudes are given by

$$A(1s\to 2p_0;\vec{q})\equiv 0\tag{50a}$$

and

$$A(1s\to 2p_{\pm 1};\vec{q})=\pm\frac{i}{\sqrt{2}}e^{\mp i\varphi_q}K_i\eta\left(\frac{\sqrt{2}}{3}\right)^3\lambda_f 2^{5-2i\alpha_n}\times\frac{\Gamma(1+i\eta)}{\Gamma(1-i\eta)}\lambda^{-2i\eta}q^{2i\alpha_n+2i\eta-3}\times G_{33}^{22}\left(\frac{\lambda^2}{q^2}\middle|{i\eta,\,i\eta+i\alpha_n-1,\,i\eta+i\alpha_n \atop 0,\,2+i\eta,\,-1}\right),\tag{50b}$$

where $\lambda_f=Z_n/2a_0$ and $\lambda=3Z_n/2a_0$ and the z axis is along $\hat{v}\perp\vec{q}$.

It should be noted, that each of the G functions appearing in (48b) and (50b) may be computed via Eq. (46).

V. ASYMPTOTIC FORMS

In this section we discuss the asymptotic behavior of the Coulomb-modified Glauber amplitudes $A(1s\to nlm;\vec{q})$ in various limits of both physical and general theoretic interest. First we shall detail the behavior of $A(1s\to nlm)$ for large and small values of the momentum transfer q; these limits are of practical interest since they determine the general shape of the Glauber-predicted angular distribution for the scattered particle in these inelastic collisions. We then shall examine these amplitudes in the limit of large nuclear charge Z_n, i.e., as $Z_n\to\infty$. Throughout this section we usually shall require $\alpha_n\neq 0$.

A. q dependence

In this subsection we consider the behavior of the amplitudes $A(1s\to nlm)$ of Eqs. (45) in the limits of large and small momentum transfers q. Since q is defined by $\vec{q}=\vec{K}_i-\vec{K}_f$, $q^2=K_i^2+K_f^2-2K_iK_f\cos\theta$, where θ is the center-of-mass scattering angle. For a bound-bound transition $1s\to nlm$, energy conservation in the center-of-mass system requires that

$$\hbar^2K_i^2/2\mu+\epsilon_i=\hbar^2K_f^2/2\mu+\epsilon_f,$$

where ϵ_i and ϵ_f are, respectively, the initial and final bound-state energies of the target atom. Thus, for fixed K_i and fixed final excited state f with $\epsilon_f>\epsilon_i$, K_f is fixed and $K_f<K_i$. Consequently, at each K_i the physical momentum transfer obeys

$$K_i-K_f\leq q\leq K_i+K_f,\tag{51}$$

where $q=K_i-K_f$ corresponds to forward scattering ($\theta=0$) and $q=K_i+K_f$ corresponds to backward scattering ($\theta=180°$). The asymptotic behavior of the scattering amplitude at large and small values of q is of practical interest when these limiting values of q are contained within the physical range of q specified by (51). Since we do not consider elastic scattering (for which $K_i=K_f$), $q>0$; however, K_i-K_f can be small if K_i is sufficiently large. Correspondingly, K_i+K_f is large, when K_i is large. Thus at large K_i, the asymptotic behavior of the amplitude at small q determines the scattering at small angles near the forward direction. Similarly, the large-q asymptotic behavior of the amplitude determines the scattering at wide scattering angles.

Since the amplitudes $A(1s \to nlm)$ are obtained from the generating functions $\mathcal{G}_{lm}(\lambda, \vec{q})$ of Eqs. (45b)- and (45c) by differentiation with respect to λ, the q dependence of the amplitudes, at fixed K_i and K_f, is determined not only by the q dependence, but also by the λ dependence of the functions $\mathcal{G}_{lm}$. When η and α_n are fixed and nonzero, the G functions in Eqs. (45b) and (45c) have absolutely convergent expansions given by Eq. (46a), when $\lambda^2/q^2 < 1$, and by Eq. (46b), when $\lambda^2/q^2 > 1$. Thus, the procedure for determining the asymptotic q dependence of amplitudes is straightforward. When q is large, $\lambda^2/q^2 \ll 1$; therefore, to obtain the asymptotic behavior of the G functions, we expand each of the hypergeometric functions ${}_3F_2$ in Eq. (46a) in powers of λ^2/q^2, retaining for each ${}_3F_2$ only the leading-order terms. When this expansion for the G function is used in Eq. (45b) [or (45c)], $\mathcal{G}_{lm}$ reduces to a sum of two terms, each of which is a product of powers of λ and powers of q^{-1}; e.g., when q is large

$$\mathcal{G}_{00}(\lambda, \vec{q}) \sim A_1 \lambda^{-2i\eta-2} q^{2i\eta+2i\alpha_n-2} + A_2 \lambda^{-2} q^{2i\alpha_n-2},$$

where A_1 and A_2 are independent of λ and q. The asymptotic expression for $\mathcal{G}_{lm}$ must then be differentiated with respect to λ according to Eq. (45a) and the leading q dependence extracted. [Thus the large q dependence of the $1s \to ns$ transitions is determined by differentiating $\mathcal{G}_{00}$ via Eq. (45a); for these transitions the term proportional to A_2 does not contribute since Eq. (44) holds.] In this way it can be shown easily that for fixed K_i and *large* q

$$A(1s \to nlm) \propto q^{2i\alpha_n + 2i\eta - |m| - 2} \tag{52a}$$

and

$$|A(1s \to nlm)|^2 \propto q^{-2|m|-4}. \tag{52b}$$

Consequently, the only transitions for which at large q $|A(i \to f)|^2$ behaves like pure Coulomb scattering are those for which the magnetic quantum number of the excited state is zero. But the scattering amplitude vanishes identically when $l - |m|$ is an odd integer. Thus while the absolute squares of the ns and nd $(m=0)$ amplitudes are proportional to q^{-4} at large q, the only surviving np transition amplitudes are those for $m = \pm 1$ for which $|A|^2 \propto q^{-6}$. Since Eq. (46a) also is valid when $\alpha_n = 0$ [whereas Eq. (46b) is no longer valid[27]], the foregoing conclusions hold when $\alpha_n = 0$, i.e., even when the target is atomic hydrogen.

The procedure for determining the asymptotic behavior of these amplitudes for small momentum transfers q is similar to that described above; now, however, Eq. (46b) is used to expand the G functions in Eqs. (45b) and (45c) in powers of q^2/λ^2. Again each of hypergeometric functions appearing in Eq. (46b) is expanded, retaining only the lowest-order terms in each ${}_3F_2$. The resulting small-q asymptotic form for $\mathcal{G}_{lm}$ is then differentiated according to Eq. (45a). In contradistinction to the large-q behavior of the amplitudes described by the relations (52), the small-q behavior is found to depend explicitly upon the orbital quantum number l, as well as m. Three separate cases now are required to describe the behavior of the amplitude:

(i) If $l - |m| < 2$, but $l \neq 0$, as $q \to 0$ at fixed K_i

$$|A(1s \to nlm)|^2 \propto q^{2l-4}. \tag{53a}$$

(ii) If $l - |m| = 2$, or $l = 0$,

$$|A(1s \to nlm)|^2 \propto q^{2|m|} f(q^2), \tag{53b}$$

where $f(q^2)$ is bounded but does not approach a well defined limit as $q \to 0$.

(iii) Finally, if $l - |m| > 2$, then as $q \to 0$

$$|A(1s \to nlm)| \propto q^{2|m|}. \tag{53c}$$

Again these relations are supplemented by the Glauber selection rule $l - |m|$ even. We stress that the relation (53b) does not apply to the elastic scattering Glauber amplitude which is *not* given by Eq. (12). Moreover, these results do not hold, in general, when $\alpha_n = 0$ because (46b) does not then hold;[27] indeed, it has been shown[2,7] that the $1s \to ns$ Glauber amplitudes for an atomic-hydrogen target diverge as $ln(q)$ in the limit of small q. On the other hand, the relation (53a) implies that the nonzero $1s \to np$ Glauber predictions for arbitrary $Z_n > 1$ diverge as q^{-2} as $q \to 0$, as do the corresponding e^--H(1s) $1s \to np$ Glauber amplitudes.[7]

B. The limit $Z_n \to \infty$

We now consider the limiting form of the Coulomb-modified Glauber amplitudes specified by Eq. (45) as the nuclear charge Z_n of the target atom is allowed to become infinitely large. We shall not discuss the limiting form of the general $1s \to nlm$ amplitudes; in what follows we shall only consider the large-Z_n behavior of the $1s \to 2s$, $2p$ amplitudes described in Sec. IV. For these transitions the initial and final nonrelativistic bound-state energies of the ion are given by $\epsilon_i = -Z_n^2 e^2/2a_0$ and $\epsilon_f = -Z_n^2 e^2/8a_0$; thus energy conservation in the center-of-mass system implies that

$$\frac{\hbar^2}{2\mu} K_i^2 - \frac{Z_n^2 e^2}{2a_0} = \frac{\hbar^2}{2\mu} K_f^2 - \frac{Z_n^2 e^2}{8a_0}. \tag{54}$$

Since the physical reduced mass μ remains finite as $Z_n\to\infty$, Equation (54) implies that the threshold value of K_i for inducing an inelastic transition to the $n=2$ level is proportional to Z_n, so that $K_i\to\infty$ as $Z_n\to\infty$. Because $\lambda_i=Z_n/a_0$ and $\lambda_f=\lambda_i/n$, we now introduce scaled quantities via $K_i=Z_n\overline{K}_i$, $K_f=Z_n\overline{K}_f$ and, holding the center-of-mass scattering angle fixed, $\overline{q}^2=Z_n^{-2}q^2=\overline{K}_i^2+\overline{K}_f^2-2\overline{K}_i\overline{K}_f\cos\theta$. Similarly $\lambda_i=Z_n\overline{\lambda}_i$ and $\lambda=\lambda_i+\lambda_f=Z_n\overline{\lambda}$. Moreover,

$$\eta=-\frac{Z_ie^2}{\hbar v_i}=-\frac{\mu Z_ie^2}{\hbar^2K_i}\equiv\frac{1}{Z_n}\overline{\eta}$$

and

$$\alpha_n+\eta=-\frac{\mu Z_iZ_ne^2}{\hbar^2K_i}=\overline{\eta}\,.$$

We emphasize that all barred quantities remain finite as $Z_n\to\infty$.

Since $\lambda^2/q^2=\overline{\lambda}^2/\overline{q}^2$ and $\partial/\partial\lambda=(1/Z_n)(\partial/\partial\overline{\lambda})$, it can be easily shown from Eq. (45) that in general

$$A(1s\to nlm;\vec{q})=Z_n^{2i\overline{\eta}(1-1/Z_n)-2}\overline{A}(1s\to nlm;\vec{\overline{q}})\,, \tag{55}$$

where the scaled amplitude $\overline{A}(1s\to nlm)$ remains finite and, in general, nonzero as $Z_n\to\infty$. In particular, from (48a) and (50b) we have

$$\overline{A}(1s\to 2s;\vec{\overline{q}})=-\overline{K}_i\overline{\eta}2^{3-2i\overline{\eta}(1-1/Z_n)}(\overline{\lambda}_i\overline{\lambda}_f)^{3/2}\frac{\Gamma(1+i\overline{\eta}/Z_n)}{\Gamma(1-i\overline{\eta}/Z_n)}\overline{q}^{2i\overline{\eta}-2}$$
$$\times\left\{\left(1+\overline{\lambda}_f\frac{\partial}{\partial\overline{\lambda}}\right)\frac{\partial}{\partial\overline{\lambda}}\left[\overline{\lambda}^{-2i\overline{\eta}/Z_n-2}G_{33}^{22}\left(\frac{\overline{\lambda}^2}{\overline{q}^2}\,\middle|\,{i\overline{\eta}/Z_n,\,i\overline{\eta},\,i\overline{\eta}\atop 0,\,i\overline{\eta}/Z_n,\,0}\right)\right]\right\}\Bigg|_{\overline{\lambda}=3/2a_0} \tag{56}$$

and

$$A(1s\to 2p_\pm;\vec{\overline{q}})=\pm\frac{i}{\sqrt{2}}e^{\mp i\varphi_q}\overline{K}_i\overline{\eta}\left(\frac{\sqrt{2}}{3}\right)^3a_0^{-1}2^{4-2i\overline{\eta}(1-1/Z_n)}\frac{\Gamma(1+i\overline{\eta}/Z_n)}{\Gamma(1-i\overline{\eta}/Z_n)}$$
$$\times\overline{q}^{2i\overline{\eta}-3}\overline{\lambda}^{-2i\overline{\eta}/Z_n}G_{33}^{22}\left(\frac{\overline{\lambda}^2}{\overline{q}^2}\,\middle|\,{i\overline{\eta}/Z_n,\,i\overline{\eta}-1,\,i\overline{\eta}\atop 0,\,2+i\overline{\eta}/Z_n,\,-1}\right)\Bigg|_{\overline{\lambda}=3/2a_0}; \tag{57}$$

recall that $A(1s\to 2p_0)=0$ and the quantization axis is along $\hat{\nu}\perp\vec{q}$. To determine the limiting behavior of $A(1s\to 2s,2p)$ as $Z_n\to\infty$ we need specifically to consider the limiting forms of the G functions appearing in Eqs. (56) and (57); the limits of the remaining terms in these equations are clearly evident since all barred quantities remain finite as $Z_n\to\infty$. The behavior of the G functions is most easily established by directly examining the Mellin-Barnes integrals of Eq. (33) which represent these functions.

We consider first the limit of the G function in (56): from (33)

$$\lim_{Z_n\to\infty}G_{33}^{22}\left(x\,\middle|\,{i\overline{\eta}/Z_n,\,i\overline{\eta},\,i\overline{\eta}\atop 0,\,i\overline{\eta}/Z_n,\,0}\right)=\frac{1}{2\pi i}\int_{-\epsilon-i\infty}^{-\epsilon+i\infty}ds\,\frac{\Gamma(1+s)\Gamma(1+s-i\overline{\eta})\Gamma(-s)\Gamma(-s)}{\Gamma(i\overline{\eta}-s)\Gamma(1+s)}x^s \tag{58a}$$

$$=x^{-1+i\overline{\eta}}\frac{1}{2\pi i}\int_{-\delta-i\infty}^{-\delta+i\infty}dw\,\frac{\Gamma(w+1-i\overline{\eta})\Gamma(w+1-i\overline{\eta})\Gamma(-w)}{\Gamma(1+w)}x^{-w}\,, \tag{58b}$$

where (58b) is obtained from (58a) by the change of variable $-w=1+s-i\overline{\eta}$; since $0<\epsilon<1$ in (58a), $0<\delta<1$ in (58b). Now the contour integral in (58b) is simply the Mellin-Barnes integral representation[23] of the standard hypergeometric function ${}_2F_1$, i.e.,

$$\frac{\Gamma(a)\Gamma(b)}{\Gamma(c)}\,{}_2F_1(a,b;c;z)=\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty}ds\,\frac{\Gamma(s+a)\Gamma(s+b)\Gamma(-s)}{\Gamma(s+c)}(-z)^s\,, \tag{59}$$

where $|\arg(-z)|<\pi$ and the contour separates the poles of $\Gamma(s+a)$ and $\Gamma(s+b)$ from those of $\Gamma(-s)$. Thus

$$\lim_{Z_n\to\infty}G_{33}^{22}\left(\frac{\overline{\lambda}^2}{\overline{q}^2}\,\middle|\,{i\overline{\eta}/Z_n,\,i\overline{\eta},\,i\overline{\eta}\atop 0,\,i\overline{\eta}/Z_n,\,0}\right)=\left(\frac{\overline{q}^2}{\overline{\lambda}^2}\right)^{1-i\overline{\eta}}[\Gamma(1-i\overline{\eta})]^2\,{}_2F_1\left(1-i\overline{\eta},1-i\overline{\eta};1;-\frac{\overline{q}^2}{\overline{\lambda}^2}\right)$$

and

$$\lim_{Z_n\to\infty}\overline{A}(1s\to 2s;\vec{\overline{q}})=-\overline{K}_i\overline{\eta}2^{3-i\overline{\eta}}(\overline{\lambda}_i\overline{\lambda}_f)^{3/2}\Gamma(1-i\overline{\eta})\Gamma(1-i\overline{\eta})$$
$$\times\left\{\left(1+\overline{\lambda}_f\frac{\partial}{\partial\overline{\lambda}}\right)\frac{\partial}{\partial\overline{\lambda}}\left[\overline{\lambda}^{2i\overline{\eta}-4}\,{}_2F_1\left(1-i\overline{\eta},1-i\overline{\eta};1;-\frac{\overline{q}^2}{\overline{\lambda}^2}\right)\right]\right\}\Bigg|_{\overline{\lambda}=3/2a_0}. \tag{60}$$

The standard differentiation formula[26]

$$\frac{d}{dx}\,{}_2F_1(a,b;c;x)=\frac{ab}{c}\,{}_2F_1(a+1,b+1;c+1;x)$$

together with $\bar{\lambda}/\bar{\lambda}_f=3$ may be used to evaluate Eq. (60). We find that

$$\begin{aligned}\lim_{Z_n\to\infty}\bar{A}(1s\to 2s;\vec{\bar{q}}) = -\bar{K}_i\bar{\eta}2^{4-2i\bar{\eta}}\tfrac{1}{3}\left(\frac{\sqrt{2}}{3}\right)^3\Gamma(1-i\bar{\eta})\Gamma(1-i\bar{\eta})\bar{\lambda}^{2i\bar{\eta}-2}\\ \times\Bigg\{2(i\bar{\eta}-1)(i\bar{\eta}-2)\,{}_2F_1\left(1-i\bar{\eta},1-i\bar{\eta};1;-\frac{\bar{q}^2}{\bar{\lambda}^2}\right)\\ +4(i\bar{\eta}-2)(1-i\bar{\eta})^2\frac{\bar{q}^2}{\bar{\lambda}^2}\,{}_2F_1\left(2-i\bar{\eta},2-i\bar{\eta};2;-\frac{\bar{q}^2}{\bar{\lambda}^2}\right)\\ +(1-i\bar{\eta})^2(2-i\bar{\eta})^2\left(\frac{\bar{q}}{\bar{\lambda}}\right)^4{}_2F_1\left(3-i\bar{\eta},3-i\bar{\eta};3;-\frac{\bar{q}^2}{\bar{\lambda}^2}\right)\Bigg\}\Bigg|_{\bar{\lambda}=3/2a_0}.\end{aligned} \tag{61}$$

We now consider the G function in (57): again from (33) we have

$$\lim_{Z_n\to\infty}G_{33}^{22}\left(x\,\Big|\,{}^{i\bar{\eta}/Z_n,\,i\bar{\eta}-1,\,i\bar{\eta}}_{0,\,2+i\bar{\eta}/Z_n,\,-1}\right)=\frac{1}{2\pi i}\int_{-\epsilon-i\infty}^{-\epsilon+i\infty}ds\,\frac{\Gamma(1+s)\Gamma(2+s-i\bar{\eta})\Gamma(-s)\Gamma(2-s)}{\Gamma(i\bar{\eta}-s)\Gamma(2+s)}x^s, \tag{62}$$

where $0<\epsilon<1$. But

$$\frac{\Gamma(1+s)\Gamma(-s)}{\Gamma(2+s)}=\frac{\Gamma(-s)}{1+s}=-\Gamma(-s-1).$$

Thus letting $-w=2+s-i\bar{\eta}$,

$$\lim_{Z_n\to\infty}G_{33}^{22}\left(x\,\Big|\,{}^{i\bar{\eta}/Z_n,\,i\bar{\eta}-1,\,i\bar{\eta}}_{0,\,2+i\bar{\eta}/Z_n,\,-1}\right)=-x^{-2+i\bar{\eta}}\frac{1}{2\pi i}\int_{-\delta-i\infty}^{-\delta+i\infty}dw\,\frac{\Gamma(1+w-i\bar{\eta})\Gamma(4+w-i\bar{\eta})}{\Gamma(2+w)}\Gamma(-w)x^{-w}, \tag{63}$$

where $1<\delta<2$. Although the integrand in (63) has the same form as (59), the integral in (63) cannot be directly identified as a ${}_2F_1$ because a pole of $\Gamma(1+w-i\bar{\eta})$ at $w_0=-1+i\bar{\eta}$ lies to the right of the contour. Let C_1 denote the straight-line path in (63) and C_0 a closed counterclockwise contour solely about the simple pole of $\Gamma(1+w-i\bar{\eta})$ at $w_0=-1+i\bar{\eta}$. Then

$$\frac{1}{2\pi i}\int_{C_1}dw\cdots=\frac{1}{2\pi i}\int_{C_2}dw\cdots-\frac{1}{2\pi i}\oint_{C_0}dw\cdots,$$

where C_2 is a straight-line path from $-\epsilon-i\infty$ to $-\epsilon+i\infty$ with $0<\epsilon<1$. The integral along C_2 now may be identified as a ${}_2F_1$ via (59) while the integral around C_0 may be evaluated simply via the residue theorem. Thus

$$\begin{aligned}\lim_{Z_n\to\infty}G_{33}^{22}\left(x\,\Big|\,{}^{i\bar{\eta}/Z_n,\,i\bar{\eta}-1,\,i\bar{\eta}}_{0,\,2+i\bar{\eta}/Z_n,\,-1}\right)=-x^{-2+i\bar{\eta}}\Bigg\{\frac{\Gamma(4-i\bar{\eta})\Gamma(1-i\bar{\eta})}{\Gamma(2)}\,{}_2F_1\left(1-i\bar{\eta},4-i\bar{\eta};2;-\frac{1}{x}\right)\\ -\frac{1}{2\pi i}\oint_{C_0}dw\,\frac{\Gamma(1+w-i\bar{\eta})\Gamma(4+w-i\bar{\eta})\Gamma(-w)}{\Gamma(2+w)}x^{-w}\Bigg\}\end{aligned} \tag{64a}$$

$$=\frac{\Gamma(1-i\bar{\eta})}{\Gamma(1+i\bar{\eta})}\frac{2}{x}-x^{-2+i\bar{\eta}}\Gamma(1-i\bar{\eta})\Gamma(4-i\bar{\eta})\,{}_2F_1\left(1-i\bar{\eta},4-i\bar{\eta};2;-\frac{1}{x}\right). \tag{64b}$$

Using (64b) in (57) we finally obtain

$$\begin{aligned}\lim_{Z_n\to\infty}\bar{A}(1s\to 2p_{\pm1};\vec{\bar{q}})=\pm\frac{i}{\sqrt{2}}e^{\mp i\varphi_q}\bar{K}_i\bar{\eta}\left(\frac{\sqrt{2}}{3}\right)^3a_0^{-1}2^{4-2i\bar{\eta}}\,\bar{\lambda}^{-2}\bar{q}^{2i\bar{\eta}-1}\Gamma(1-i\bar{\eta})\\ \times\left\{\frac{2}{\Gamma(1+i\bar{\eta})}-\left(\frac{\bar{q}^2}{\bar{\lambda}^2}\right)^{1-i\bar{\eta}}\Gamma(4-i\bar{\eta})\,{}_2F_1\left(1-i\bar{\eta},4-i\bar{\eta};2;-\frac{\bar{q}^2}{\bar{\lambda}^2}\right)\right\}\Bigg|_{\bar{\lambda}=3/2a_0}.\end{aligned} \tag{65}$$

Equations (61) and (65) together with Eq. (55) complete the specification of the $Z_n \to \infty$ limiting behavior of the nonvanishing $1s \to 2s$ and $1s \to 2p$ Coulomb-modified Glauber amplitudes. Similar methods may be employed to determine the more general result for arbitrary $1s \to nlm$ excitations.

APPENDIX A: USEFUL INTEGRAL REPRESENTATION

In this Appendix we describe a generalization of an integral representation developed by Thomas and Gerjuoy[7] in their reduction of e^--H Glauber amplitudes to closed form. Thomas and Gerjuoy showed that

$$\frac{1}{2\pi}\int_0^{2\pi} d\varphi\, e^{im\varphi}(1+s^2-2s\cos\varphi)^{i\eta} = -2^{2i\eta}\frac{\Gamma(1+i\eta)}{\Gamma(1-i\eta)}\int_0^{\infty} dt\, t^{-2i\eta}\frac{d}{dt}[J_{|m|}(t)J_{|m|}(st)] \tag{A1}$$

for $s>0$, η real, and arbitrary (positive or negative) integers m. The functions $\Gamma(z)$ and $J_\nu(z)$ are, respectively, the Gamma function and Bessel function of the first kind. The generalization of (A1) we consider here is the integral

$$I=\frac{1}{2\pi}\int d\Omega_r\, Y_l^{m*}(\hat{r})[1-(|\vec{b}-\vec{s}|/b)^{2i\eta}], \tag{A2}$$

where $\vec{b}$ is an arbitrary vector in the x-y plane, $\vec{s}$ is the projection of a three-dimensional vector $\vec{r}$ onto that plane, and η is real. The reduction, presented here, of the integral (A2) to an equivalent one-dimensional integral representation of the type (A1) was first described by Thomas.[9]

We employ the Rose[16] convention for the spherical harmonic Y_l^m so that

$$Y_l^m(\hat{r})=\left(\frac{2l+1}{4\pi}\right)^{1/2}\left(\frac{(l-m)!}{(l+m)!}\right)^{1/2} e^{im\varphi}P_l^m(\cos\theta). \tag{A3}$$

Note that the generalized Legendre polynomials P_l^m of Eq. (A3) are defined consistent with the conventional definition[28] of the Legendre functions. With (A3), Eq. (A2) becomes

$$I=\left(\frac{2l+1}{4\pi}\right)^{1/2}\left(\frac{(l-m)!}{(l+m)!}\right)^{1/2}\frac{1}{2\pi}\int_0^{\pi}\sin\theta d\theta P_l^m(\cos\theta)\int_0^{2\pi}d\varphi e^{-im\varphi}\left[1-\left(1+\frac{s^2}{b^2}-\frac{2s}{b}\cos(\varphi_b-\varphi)\right)^{i\eta}\right], \tag{A4}$$

where φ_b and φ are the azimuthal angles of $\vec{b}$ and $\vec{s}$ (i.e., $\vec{r}$) in the x-y plane. We make use of the periodic properties of the trigonometric functions to remove the φ_b dependence from inside the integrals; after integrating over φ and using (A1) for $s/b>0$ we have

$$I=2\left(\frac{2l+1}{4\pi}\right)^{1/2}\left(\frac{(l-m)!}{(l+m)!}\right)^{1/2} e^{-im\varphi_b}\left\{\delta_{l,0}\delta_{m,0}+2^{2i\eta-1}\frac{\Gamma(1+i\eta)}{\Gamma(1-i\eta)}\int_0^{\pi}\sin\theta\, d\theta\, P_l^m(\cos\theta)\times\int_0^{\infty}dt\, t^{-2i\eta}\frac{d}{dt}\left[J_m(t)J_m\left(\frac{s}{b}t\right)\right]\right\}. \tag{A5}$$

In Eq. (A5) we have removed the absolute value signs on the order of the Bessel functions, because[21] for integer m, $J_{-m}(z)=(-1)^m J_m(z)$, so that $J_{|m|}(x)J_{|m|}(y)=J_m(x)J_m(y)$. We now note that $s=r\sin\theta$ and assume we may interchange the order of integration over θ with the integrations and differentiation with respect to t. Thus we need only consider the remaining integral

$$I'=\int_0^{\pi}\sin\theta d\theta P_l^m(\cos\theta)J_m\left(\frac{rt}{b}\sin\theta\right). \tag{A6}$$

If $l\geq m\geq 0$ in (A6) we may use the relation[29]

$$P_l^m(\cos\theta)=(-1)^m\frac{\Gamma(2m+1)}{\Gamma(m+1)}2^{-m}(\sin\theta)^m C_{l-m}^{m+1/2}(\cos\theta), \tag{A7}$$

where C_n^λ is the Gegenbauer polynomial. Therefore, for $m\geq 0$

$$I'=(-1)^m\frac{\Gamma(2m+1)}{\Gamma(m+1)}2^{-m}\times\int_0^{\pi}d\theta(\sin\theta)^{m+1}C_{l-m}^{m+1/2}(\cos\theta)J_m\left(\frac{rt}{b}\sin\theta\right). \tag{A8}$$

But the integral over θ in (A8) is the special case of a known result,[30] which yields

$$I'=(-1)^m\frac{\Gamma(2m+1)}{\Gamma(m+1)}2^{-m}(-1)^{(l-m)/2}\left(\frac{2\pi b}{rt}\right)^{1/2}\times C_{l-m}^{m+1/2}(0)J_{l+1/2}\left(\frac{rt}{b}\right)\quad\text{if } l-m \text{ is even} \tag{A9a}$$

$$=0\quad\text{if } l-m \text{ is odd}. \tag{A9b}$$

Since[31] $P_l^m(0)=0$ if $l-m$ is odd, we may again use (A7) to write Eq. (9) in the compact form

$$I' = (-1)^{(l-m)/2} P_l^m(0) \left(\frac{2\pi b}{rt}\right)^{1/2} J_{l+1/2}\left(\frac{rt}{b}\right), \quad \text{(A10)}$$

where $m \geq 0$. If in Eq. (A6) $m < 0$, then $m = -|m|$ and[32]

$$P_l^m(\cos\theta) = P_l^{-|m|}(\cos\theta) = \frac{\Gamma(l+1-|m|)}{\Gamma(l+1+|m|)} (-1)^{|m|} P_l^{|m|}(\cos\theta); \quad \text{(A11)}$$

again[21] $J_m(x) = J_{-|m|}(x) = (-1)^{|m|} J_{|m|}(x)$. Therefore when $m < 0$

$$I' = (-1)^{|m|} \frac{\Gamma(l+1-|m|)}{\Gamma(l+1+|m|)} (-1)^{|m|} \times \int_0^\pi \sin\theta \, d\theta \, P_l^{|m|}(\cos\theta) J_{|m|}\left(\frac{rt}{b} \sin\theta\right).$$

Now apply (A10) wherein $m \geq 0$, to obtain

$$I' = (-1)^{(l+|m|)/2} P_l^m(0) \left(\frac{2\pi b}{rt}\right)^{1/2} J_{l+1/2}\left(\frac{rt}{b}\right),$$

after again employing (A11). Since $m<0$, $(-1)^{(l+|m|)/2} = (-1)^{(l-m)/2}$. Therefore, for *all m* satisfying $-l \leq m \leq l$,

$$\int_0^\pi \sin\theta \, d\theta \, P_l^m(\cos\theta) J_m\left(\frac{rt}{b}\sin\theta\right) = (-1)^{(l-m)/2} P_l^m(0) \left(\frac{2\pi b}{rt}\right)^{1/2} J_{l+1/2}\left(\frac{rt}{b}\right). \quad \text{(A12)}$$

With the foregoing result, Eq. (A5) reduces to

$$I = 2\left(\frac{2l+1}{4\pi}\right)^{1/2} \left(\frac{(l-m)!}{(l+m)!}\right)^{1/2} e^{-im\varphi_b} \times \left\{\delta_{l,0}\delta_{m,0} + (-1)^{(l-m)/2} P_l^m(0) 2^{2i\eta-1} \frac{\Gamma(1+i\eta)}{\Gamma(1-i\eta)} \int_0^\infty dt \, t^{-2i\eta} \frac{d}{dt}\left[J_m(t)\left(\frac{2\pi b}{rt}\right)^{1/2} J_{l+1/2}\left(\frac{rt}{b}\right)\right]\right\}.$$

Finally, we may pull the $P_l^m(0)$ outside the curly brackets since $P_0^0(0) = 1$ and use Eq. (A3) to simplify the result. We obtain for arbitrary integers l and m $(-l \leq m \leq l)$

$$\frac{1}{2\pi} \int d\Omega_r \, Y_l^{m*}(\hat{r}) \left[1 - \left(\frac{|\vec{b}-\vec{s}|}{b}\right)^{2i\eta}\right] = 2(-1)^{(l-m)/2} Y_l^{m*}(\pi/2, \varphi_b) \times \left\{\delta_{l,0}\delta_{m,0} + 2^{2i\eta} \frac{\Gamma(1+i\eta)}{\Gamma(1-i\eta)} \int_0^\infty dt \, t^{-2i\eta} \frac{d}{dt}\left[J_m(t)\left(\frac{\pi b}{2rt}\right)^{1/2} J_{l+1/2}\left(\frac{rt}{b}\right)\right]\right\}. \quad \text{(A13)}$$

We stress that $Y_l^m(\pi/2, \varphi_b) \equiv 0$ if $l-m$ is an odd (positive or negative) integer. When $l>0$, (A13) may be simplified further: the first term in the curly brackets vanishes, whereas the second term may be integrated once by parts. When $l=0$ $(m=0)$ the partial integration is no longer valid for real η, and the indicated differentiation with respect to t must be carried out directly. The failure of partial integration in this case reflects the fact that in (A13) the integral over t contains a term which ultimately leads to the exact cancellation of terms stemming from the first term in the curly brackets in (A13) when (A13) is integrated over r as in Eq. (17) of Sec. III.

The utility of the result (A13) is twofold. Unlike previous treatments[2,7,8] of integrals like (A2), the foregoing reduction of (A2) to the form (A13) has employed spherical coordinates for $\vec{r}$, rather than cylindrical coordinates, thereby ensuring the full exploitation of the properties of the functions $Y_l^m(\hat{r})$. When cylindrical coordinates are used, the detailed properties of the radial wave functions multiplying $Y_l^{m*}(\hat{r})$ [as in Eq. (17)] become important and the integrals can be evaluated only after explicitly expanding the P_l^m in a power series. The result (A13) is clearly more compact and versatile than any such power-series expansion. Second, (A13) preserves the explicit r dependence of the integrals in Eq. (1) or (10), so that analytic forms for the radial functions need not be required; these wave functions now may be numerically tabulated (as, e.g., the close-coupling wave functions for an ejected electron in the field of its parent ion) and the integral over r performed numerically after first explicitly integrating (A13) over t.

APPENDIX B: GENERATING FUNCTION $\mathcal{G}_{lm}^H(\lambda, \vec{q})$ FOR TRANSITIONS IN ATOMIC HYDROGEN

Here we consider the generating function for the Glauber $1s \to nlm$ amplitudes for excitation of atomic hydrogen. Although the general Glauber transition amplitudes have been considered elsewhere by Thomas and Gerjuoy, these results have not been generally published heretofore (see footnote 14 of Ref. 7). Moreover, these previous results are especially awkward to employ in actual practice.

The Glauber amplitudes for $1s \to nlm$ transitions in atomic hydrogen still are specified by Eq. (45) after setting $Z_n = 1$ and $\alpha_n = 0$. Now, however, expressing the generating functions $\mathcal{G}_{lm}$ in terms of Meijer G functions G_{33}^{22} is no longer convenient nor practical. Indeed Thomas and Gerjuoy[7] have shown that these $1s \to ns$ and $1s \to np$ Glauber amplitudes may be expressed in terms of hypergeometric functions ${}_2F_1$ rather than the functions G_{33}^{22} of Eq. (45). Since Thomas and Gerjuoy have already examined the $1s \to ns$ transitions in general, we consider here only those transitions for which the final angular momentum quantum number $l > 0$. Moreover, since (45a) holds we shall consider only the special case of the generating function $\mathcal{G}_{lm}^H$ for these transitions. The amplitudes themselves are to be constructed in the usual way (see Ref. 7) from the generating function.

For $l > 0$, the generating function $\mathcal{G}_{lm}^H$ is given by Eq. (32); setting $\alpha_n = 0$ in (32) we have

$$\mathcal{G}_{lm}^H(\lambda, \vec{q}) = 4\pi i^l Y_l^{m*}(\pi/2, \varphi_q) i\eta 2^{l+1} q^{-l-4} \frac{\Gamma(1+i\eta)}{\Gamma(1-i\eta)} \frac{1}{2\pi i}$$
$$\times \int_{-\epsilon - i\infty}^{-\epsilon + i\infty} dt \left(\frac{\lambda^2}{q^2}\right)^{t - i\eta - (l-|m|)/2 - 1} \frac{\Gamma(t - i\eta + \frac{1}{2}(l+|m|))\Gamma(t+1+|m|-i\eta)\Gamma(1+\frac{1}{2}(l-|m|)+i\eta - t)\Gamma(-t)}{\Gamma(i\eta - t)\Gamma(t+1+|m|)}, \tag{B1}$$

where $0 < \epsilon < \min(\frac{1}{2}(l+|m|), 1+|m|)$. Since $l - |m|$ is even [otherwise $Y_l^m(\pi/2, \varphi_q) = 0$] we may write

$$\frac{\Gamma(1+\frac{1}{2}(l-|m|)+i\eta - t)}{\Gamma(i\eta - t)} = (i\eta - t)_{1+(l-|m|)/2},$$

where $(a)_n$ is the Pochhammer symbol.[18] But

$$\left(\frac{\partial}{\partial x}\right)^j x^{-a} = (-1)^j (a)_j x^{-a-j};$$

therefore

$$(i\eta - t)_{1+(l-|m|)/2}\, z^{t - i\eta - (l-|m|)/2 - 1} = (-1)^{1+(l-|m|)/2} \left(\frac{\partial}{\partial z}\right)^{1+(l-|m|)/2} z^{t - i\eta}.$$

Thus

$$\mathcal{G}_{lm}^H(\lambda, \vec{q}) = i\eta 4\pi i^l Y_l^{m*}(\pi/2, \varphi_q) 2^{l+1} q^{-l-4} \frac{\Gamma(1+i\eta)}{\Gamma(1-i\eta)} (-1)^{1+(l-|m|)/2}$$
$$\times \left(\frac{\partial}{\partial z}\right)^{1+(l-|m|)/2} \left\{\frac{z^{-i\eta}}{2\pi i} \int_{-\epsilon - i\infty}^{-\epsilon + i\infty} dt \frac{\Gamma(t - i\eta + \frac{1}{2}(l+|m|))\Gamma(t+1+|m|-i\eta)\Gamma(-t)}{\Gamma(t+1+|m|)} z^t\right\}, \tag{B2}$$

where $z = \lambda^2/q^2$. Because $l > 0$ and $0 < \epsilon < \min(\frac{1}{2}(l+|m|), 1+|m|)$ the integral in (B2) may be immediately identified as a hypergeometric function via Eq. (65). Therefore, with $z = \lambda^2/q^2$,

$$\mathcal{G}_{lm}^H(\lambda, \vec{q}) = i\eta 4\pi i^l Y_l^{m*}(\pi/2, \varphi_q) 2^{l+1} q^{-l-4} \frac{\Gamma(1+i\eta)}{\Gamma(1-i\eta)} \frac{\Gamma(\frac{1}{2}(l+|m|) - i\eta)\Gamma(1+|m|-i\eta)}{\Gamma(1+|m|)}$$
$$\times \left(-\frac{\partial}{\partial z}\right)^{1+(l-|m|)/2} \left[z^{-i\eta} {}_2F_1\left(\frac{l+|m|}{2} - i\eta, 1+|m|-i\eta; 1+|m|; -z\right)\right]. \tag{B3}$$

Although the differentiations specified by (B3) together with those required by (45a) still need to be performed, in actual practice all these differentiations may be performed algebraically by fully exploiting the numerous differentiation formulas and recursion relations which the hypergeometric functions satisfy. Furthermore, we wish to point out that Eq. (B3) remains valid when $l = 0$, as can be shown explicitly. The validity of (B3) when $l = 0$ is a reflection of the fact that the second term on the right-hand side of Eq. (43) vanishes identically when $\alpha_n = 0$ so that (45b) and (45c) are then equivalent.

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