Coupled order-parameter treatment of the Dicke Hamiltonian*

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The Dicke Hamiltonian is linearized by expanding the field and atomic shift operators about disposable c-number parameters. When these parameters are chosen to be the expectation values of the corresponding operators with respect to the linearized Hamiltonian, the difference between the free energy of the original Hamiltonian and the free energy of the linearized Hamiltonian is minimized. This difference (per particle) vanishes in the thermodynamic limit; so the equilibrium statistical mechanics of the system is described by the linearized Hamiltonian. The disposable parameters are order parameters of the linearized Hamiltonian. The field and atom order parameters obey a system of coupled nonlinear equations characteristic of mean-field theories. These equations determine the critical temperature and the ordered-state behavior. The thermodynamic state of each subsystem is a statistical superposition of thermal noise and a coherent state produced by the complementary subsystem acting as a classical source. The classical driving terms are the order parameters. The order parameter and coherent-state parameter are equal for the field subsystem but not for the atomic subsystem. The ordered state is characterized by an enhanced condensation of the atomic subsystem into the state of maximum cooperation number r. This enhancement is attributed entirely to the Stark splitting of the atomic energy levels due to the classical driving field. A different but equivalent system of coupled nonlinear order-parameter equations is derived from the order-parameter equations of motion in thermodynamic equilibrium in the thermodynamic limit. This alternative form of the coupled self-consistent equations leads to a very simple method for locating the critical temperature and determining the ordered-state behavior of interacting systems. This method is illustrated by determining the gap equations for three model Hamiltonians. These describe (a) interaction of spin-j systems with a single mode of the radiation field, (b) interaction of two-level atoms with a finite number of modes of the radiation field, and (c) interaction of two-level atoms with one or two modes of the radiation field through double photon absorption and emission processes.

I. INTRODUCTION

In 1954 Dicke[1] proposed a model Hamiltonian which has since become an extremely useful tool in quantum optics for the study of the interaction between light and matter. Interest in this model has been renewed by the recent discovery by Hepp and Lieb[2] that this Hamiltonian predicts the existence of a second-order phase transition for a certain range of the coupling constants.

Wang and Hioe[3] considerably simplified the computational aspects of the Hepp and Lieb proof by introducing the Glauber[4] field coherent states. In addition, they determined the conditions for a phase transition in the multimode Dicke model for a finite number of modes. Hepp and Lieb[5] were able to put upper and lower bounds on the partition function for the infinite-mode Dicke model using atomic coherent states.[6,7] Their results reduced to those of Wang and Hioe in the finite-mode case.

The demonstration of the phase transition and and the location of the critical point were further simplified by Gibberd,[8] who applied a Bogoliubov transformation[9] to the field part of the Dicke Hamiltonian.

In the meantime the computational methods of Wang and Hioe have been used to discuss phase transitions and derive gap equations for "dressed" Dicke models. Carmichael, Gardiner, and Walls[10] showed that inclusion of the counterrotating terms merely serves to renormalize the coupling constant. Kudenko, Slivinsky, and Zaslavsky[11] studied the effect of Coulomb interactions on the location of the critical point. Thompson[12] studied the effects of phonon-assisted transitions on the critical temperature and found that for a certain range of the coupling constants a first-order phase transition was possible. Recently, Rzążewski, Wódkiewicz, and Żakowicz[13] showed that the phase transition disappears when the A^2 term is also included in the Dicke Hamiltonian. However, we believe[14] that the disappearance of the phase transition in their work stems entirely from the presence of the counterrotating terms in the interaction Hamiltonian.

The contributions stimulated by the original Hepp and Lieb result have been of two types, those that have simplified the original computation, and those that have considered more complicated models. In the present contribution we will simplify the computation even further by modifying the method introduced by Gibberd. In a subsequent contribution[15] we will consider more general model Hamiltonians.

The method to be introduced in this work is the following: The Dicke Hamiltonian is expanded

around the mean values of the shift operators $\sigma^{\pm}$, $a^{\dagger}$. In the thermodynamic limit the free energy per atom from the full and linearized Hamiltonians are equal. The existence of the phase transition and the location of the critical temperature can then be determined from the linearized Hamiltonian. The physical interpretation of the system state is then immediately determined from the density operator obtained from the linearized Hamiltonian.

The plan of this work is as follows: In Sec. II we carry out the dual order-parameter treatment of the Dicke Hamiltonian. In Sec. III we interpret the physical system state by comparing the density operators for the field and atomic subsystems with density operators representing statistical superpositions of thermal noise and coherent states. In Sec. IV we show that in the ordered phase there is an enhanced condensation into the atomic state of maximum cooperation. In Sec. V we compute generating functions for the energy moments of both the atomic and field subsystems. In Sec. VI a different but equivalent system of coupled nonlinear order-parameter equations is derived from the order-parameter equations of motion in thermodynamic equilibrium. These coupled equations are used in Sec. VII to locate the critical temperature and discuss the critical behavior for three variants of the Dicke Hamiltonian.

II. DUAL ORDER-PARAMETER TREATMENT

The Dicke Hamiltonian for m modes of the radiation field interacting with N independent two-level atoms is[1]

$$\mathcal{H}=\sum_{s=1}^{m}\omega_s a_s^{\dagger}a_s+\epsilon\sum_{j=1}^{N}\tfrac{1}{2}\sigma_j^z + \frac{1}{\sqrt{N}}\sum_{j=1}^{N}\sum_{s=1}^{m}(\lambda_s^{*}a_s^{\dagger}\sigma_j^{-}+\lambda_s a_s\sigma_j^{+}). \tag{2.1}$$

The field mode operators $n_s=a_s^{\dagger}a_s$, $a_s^{\dagger}$, a_s, I obey independent harmonic-oscillator commutation relations,

$$[n_s,a_t^{\dagger}]=+a_t^{\dagger}\delta_{st},\quad [n_s,a_t]=-a_t\delta_{st},\quad [a_s^{\dagger},a_t]=-I\delta_{st}. \tag{2.2}$$

The atomic operators σ_j^z, σ_j^{+}, and σ_j^{-} obey independent SU(2) commutation relations:

$$[\sigma_j^z,\sigma_k^{+}]=+2\sigma_k^{+}\delta_{jk},\quad [\sigma_j^z,\sigma_k^{-}]=-2\sigma_k^{-}\delta_{jk},\quad [\sigma_j^{+},\sigma_k^{-}]=+\sigma_k^z\delta_{jk}. \tag{2.2a}$$

For the purposes of the present section we will assume that only one field mode is important ($m=1$), that $\omega_1=1$, and that $\lambda_1=\lambda$ is real.

The equilibrium statistical mechanics[2] of the system described by the Hamiltonian (2.1) is governed by the free energy F,

$$e^{-\beta F}=\mathrm{Tr}e^{-\beta\mathcal{H}}. \tag{2.3}$$

The exponential of the Dicke Hamiltonian is difficult to compute because of the bilinear coupling terms $a^{\dagger}\sigma_j^{-}$ and $a\sigma_j^{+}$.

It is therefore useful to rewrite (2.1) by making the following substitutions:

$$a=(a-\mu)+\mu,\quad \sigma_j^{-}=(\sigma_j^{-}-\nu_j)+\nu_j. \tag{2.4}$$

The Dicke Hamiltonian then assumes the form

$$\mathcal{H}=\mathcal{H}_L+\mathcal{H}_Q,\quad \mathcal{H}_L=H_0+H_1+H_2, \tag{2.5}$$

$$H_0=-\frac{\lambda}{\sqrt{N}}\left(\mu^{*}\sum_{j=1}^{N}\nu_j+\mu\sum_{j=1}^{N}\nu_j^{*}\right),$$

$$H_1=a^{\dagger}a+\frac{\lambda}{\sqrt{N}}\left(a^{\dagger}\sum_{j=1}^{N}\nu_j+a\sum_{j=1}^{N}\nu_j^{*}\right), \tag{2.6}$$

$$H_2=\epsilon\sum_{j=1}^{N}\tfrac{1}{2}\sigma_j^z+\frac{\lambda}{\sqrt{N}}\left(\mu^{*}\sum_{j=1}^{N}\sigma_j^{-}+\mu\sum_{j=1}^{N}\sigma_j^{+}\right), \quad =\sum_{j=1}^{N}h(j),$$

$$H_Q=\frac{\lambda}{\sqrt{N}}\left((a^{\dagger}-\mu^{*})\sum_{j=1}^{N}(\sigma_j^{-}-\nu_j) + (a-\mu)\sum_{j=1}^{N}(\sigma_j^{+}-\nu_j^{*})\right). \tag{2.7}$$

The Hamiltonian $\mathcal{H}_Q$ contains terms bilinear in the field and atomic operators, while $\mathcal{H}_L$ contains only linear terms. Moreover, $\mathcal{H}_L$ can be expressed as the sum of the three mutually commuting terms H_0, which is a c-number function of the parameters μ and ν_j, H_1, which depends only on field operators and the parameters ν_j, and H_2, which depends only on atomic operators and μ.

Since H_0, H_1, and H_2 commute, F_L is the sum of the free energies associated with each of these terms,

$$F_L=F_0+F_1+F_2, \tag{2.8}$$

$$F_0=-\frac{\lambda}{\sqrt{N}}\sum_{j=1}^{N}(\mu^{*}\nu_j+\mu\nu_j^{*}),$$

$$F_1=-\frac{\lambda^2}{N}\left|\sum_{j=1}^{N}\nu_j\right|^2+\frac{1}{\beta}\ln(1-e^{-\beta}), \tag{2.9}$$

$$F_2=-(1/\beta)N\ln 2\cosh\beta[(\tfrac{1}{2}\epsilon)^2+|\lambda\mu|^2/N]^{1/2}.$$

So far the values of the disposable parameters

μ, and ν_j have not been specified. Since the equilibrium statistical mechanics of (2.1) is governed by F but only $F_L(\mu, \nu_j)$ can be computed, we must choose the parameters μ, and ν_j to make the two free energies equal in the thermodynamic limit,

$$\lim_{N\to\infty} \frac{F - F_L(\mu, \nu_j)}{N} \to 0. \tag{2.10}$$

This can be done by comparing the ratios

$$\frac{e^{-\beta F}}{e^{-\beta F_L}} = \frac{\mathrm{Tr} e^{-\beta(\mathcal{H}_L + \mathcal{H}_Q)}}{\mathrm{Tr} e^{-\beta \mathcal{H}_L}}$$

$$\sim 1 + \int \langle \mathcal{H}_Q \rangle + \int\int \langle \mathcal{H}_Q^2 \rangle + \cdots . \tag{2.11}$$

The perturbation expansion is given explicitly in the Appendix. If the expansion on the right-hand side of (2.11) coverges then (2.10) is valid.

The values of the parameters μ, and ν_j for which (2.11) converges can be determined by investigating the term $\langle \mathcal{H}_Q^2 \rangle$,

$$\langle \mathcal{H}_Q^2 \rangle = \sum_j \sum_K \frac{\lambda^2}{N} [\langle (a^\dagger - \mu^*)(a - \mu) \rangle_1 \langle (\sigma_j^- - \nu_j)(\sigma_k^+ - \nu_k^*) \rangle_2$$

$$+ \langle (a - \mu)(a^\dagger - \mu^*) \rangle_1 \langle (\sigma_k^+ - \nu_k^*)(\sigma_j^- - \nu_j) \rangle_2]. \tag{2.12}$$

The subscripts 1 and 2 indicate that the expectation values are taken with respect to H_1 and H_2, respectively, and the integrals have been suppressed. It is clear that the choice of the parameters μ, ν_j which minimizes the expectation value of the fluctuations in (2.12) is

$$\mu = \langle a \rangle_1, \quad \nu_j = \langle \sigma_j^- \rangle_2. \tag{2.13}$$

The choice (2.13) causes all odd terms in (2.11) to vanish and all even terms to be minimized. Therefore if the expansion in (2.11) converges the choice (2.13) will guarantee convergence.

The arguments leading to (2.13) are heuristic in nature and could have been anticipated from (2.4). We remark that a rigorous variational argument[16] leads to the same result.[15]

The result (2.13) shows clearly that the (variational) parameters μ and ν_j have a physical interpretation as order parameters for the field and atomic subsystem, respectively.

The order parameters μ and ν_j may be determined as follows: $\mu = \langle a \rangle_1$, where the expectation value is taken with respect to $H_1(\nu)$, which is itself an explicit function of the unknown order parameters ν_j. Similarly, $\nu_j = \nu_k$ is a function of μ. Direct computation of the expectation values leads to a pair of coupled nonlinear equations for μ and ν,

$$\mu = -\lambda \sqrt{N} \nu,$$

$$\nu_j = \nu_k = \nu = \frac{-\lambda(\mu/\sqrt{N})}{2\theta} \tanh\beta\theta, \tag{2.14}$$

$$\theta^2 = (\tfrac{1}{2}\epsilon)^2 + \lambda^2 |\mu/\sqrt{N}|^2.$$

It is convenient to replace the extensive order parameter μ by the intensive order parameter $\mu' = \mu/\sqrt{N}$.

These equations can be solved for μ' and ν using the standard methods of mean-field theory.[17] For example, ν can be eliminated to obtain

$$\mu' = (\lambda^2 \mu'/2\theta) \tanh\beta [(\tfrac{1}{2}\epsilon)^2 + |\lambda\mu'|^2]^{1/2}. \tag{2.15}$$

It is clear that $\mu' = 0$, $\nu = 0$ is always a solution of (2.15) and (2.14). This solution corresponds to the disordered state. If a nontrivial solution $\mu' \neq 0$, $\nu \neq 0$ exists, then μ' is determined implicitly from (2.15). At $T = 0$,

$$|\mu'|^2 = (\epsilon/2\lambda)^2 [(\lambda^2/\epsilon)^2 - 1];$$

thus an ordered state is possible only if $\lambda^2/\epsilon > 1$. The order parameter μ' decreases with increasing temperature and approaches zero as the critical temperature is approached from below. Therefore the critical temperature is determined from the "gap equation"

$$1 = (\lambda^2/\epsilon) \tanh \tfrac{1}{2} \beta_c \epsilon. \tag{2.16}$$

The behavior of the order parameter μ as a function of temperature on both the disordered and the ordered branch is shown in Fig. 1. The behavior of the free energy F/N,

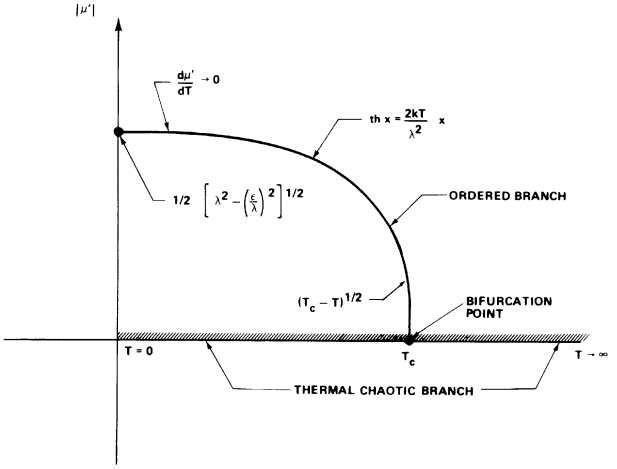

FIG. 1. Behavior of the order parameter μ' as a function of temperature shown both above and below the critical temperature T_c. In the ordered state the order parameter $|\mu'|^2$ is determined implicitly through the self-consistent equations characteristic of mean-field theories, $\tanh x = (2kT/\lambda^2)x$, where $x = \beta[(\frac{1}{2}\epsilon)^2 + |\lambda\mu'|^2]^{1/2}$.

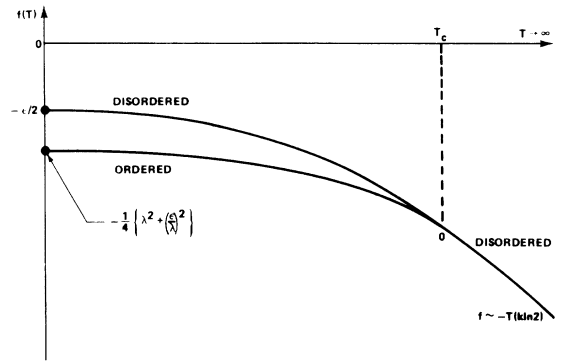

FIG. 2. Free energy per particle $F/N = f(\beta\mu', \mu'^*) = +|\mu'|^2 - (1/\beta)\ln 2\cosh\beta[(\frac{1}{2}\epsilon)^2 + |\lambda\mu'|^2]^{1/2}$ plotted in both the disordered and the ordered phases. The ordered phase is stable below T_c and the phase transition is second order.

$$F/N = f(\mu, \mu') = |\mu'|^2 - (1/\beta)\ln 2\cosh\beta[(\tfrac{1}{2}\epsilon)^2 + |\lambda\mu'|^2]^{1/2}, \tag{2.17}$$

as a function of the temperature $T = 1/K\beta$ is shown on both the disordered and the ordered branch in Fig. 2.

III. PHYSICAL INTERPRETATION OF THE ORDERED STATES

In the thermodynamic limit the equilibrium statistical mechanics of the systems described by (2.1) and its linearlized version $\mathcal{H}_L$, (2.6), are equivalent. The density operator obtained from $\mathcal{H}_L$ factors into the product of density operators, one describing the field subsystem only, and the other describing the atomic subsystem only. The influence of each subsystem on the other is contained in the order parameters ν and μ appearing in H_1 and H_2. As a result, we can discuss the physical state of the field and the atomic subsystem separately.

A. Density operator for the field

The density operator for a single field mode of energy $\hbar\omega$ in thermal equilibrium is[4]

$$\rho(\beta, \hbar\omega) \sim \exp(-\beta\hbar\omega a^\dagger a). \tag{3.1}$$

The density operator for a pure coherent state $|\alpha\rangle$ is

$$\rho(|\alpha\rangle) = |\alpha\rangle\langle\alpha| = U(\alpha)|0\rangle\langle 0|U^\dagger(\alpha), \tag{3.2}$$

$$U(\alpha) = \exp(\alpha a^\dagger - \alpha^* a). \tag{3.3}$$

The operator $|0\rangle\langle 0|$ is the density operator $\rho(\text{vac})$ for the field vacuum state. From (3.2), the operator $U(\alpha)\rho(\text{vac})U^\dagger(\alpha)$ describes a system originally in thermodynamic equilibrium at $T = 0$ which is driven by a classical current. The density operator describing a system initially in thermodynamic equilibrium at temperature T, which is driven by the same classical current, is

$$\rho(\beta, \hbar\omega; |\alpha\rangle) \sim U(\alpha)\exp(-\beta\hbar\omega a^\dagger a)U^\dagger(\alpha) \sim \exp[-\beta\hbar\omega(a^\dagger a - \alpha a^\dagger - \alpha^* a)]. \tag{3.4}$$

Comparison of the density operator $\rho_1 \sim e^{-\beta H_1(\nu)}$ for the field subsystem with (3.4) leads immediately to the conclusion that the physical state of the field subsystem can be represented by a statistical superposition of thermal noise characterized by a temperature $T = 1/K\beta$, $\hbar\omega = 1$, and a coherent state characterized by the coherent-state parameter $\alpha = -\lambda\sqrt{N}\nu$. From the coupled order-parameter equation (2.14) we conclude that the order parameter μ and the coherent-state parameter α for the field subsystem are equal.

B. Density operator for the atomic subsystem

The density operator for a single two-level atom with energy level separation E in thermal equilibrium is

$$\rho(\beta, E) \sim e^{-\beta E\sigma^z/2}. \tag{3.5}$$

The density operator for a pure atomic coherent state $|\theta\phi\rangle$ is[7]

$$\rho(|\theta\phi\rangle) = |\theta\phi\rangle\langle\theta\phi| = U(\theta, \phi)|-\tfrac{1}{2}\rangle\langle-\tfrac{1}{2}|U^\dagger(\theta, \phi), \tag{3.6}$$

$$U(\theta, \phi) = \exp[\tfrac{1}{2}(\theta e^{-i\phi}\sigma^+ - \theta e^{i\phi}\sigma^-)]. \tag{3.7}$$

The density operator describing a statistical superposition of thermal noise and atomic coherent state is

$$\rho(\beta, E, |\theta\phi\rangle) \sim U(\theta, \phi)e^{-\beta E\sigma^z/2}U^\dagger(\theta, \phi) \sim \exp\{-\beta E[\tfrac{1}{2}\sigma^z\cos\theta - \tfrac{1}{2}(e^{-i\varphi}\sigma^+ - e^{i\varphi}\sigma^-)\sin\theta]\}. \tag{3.8}$$

Comparison of the density operator $\rho_2 \sim e^{-\beta H_2(\mu)}$ from (2.6) for the atomic subsystem with (3.8) leads immediately to the conclusion that the physical state of the atomic subsystem can be represented by a statistical superposition of thermal noise characterized by a temperature $T = 1/K\beta$ and energy $E = 2[(\frac{1}{2}\epsilon)^2 + |\lambda\mu'|^2]^{1/2}$ and a coherent state characterized by Bloch angles θ and φ, where

$$\cos\theta = \epsilon/E, \qquad (\sin\theta)e^{-i\varphi} = \nu/[(\epsilon/2\lambda^2)^2 + |\nu|^2]^{1/2}. \tag{3.9}$$

For the atomic subsystem the order parameter ν

and the coherent-state parameter $\frac{1}{2}\theta e^{-i\phi}$ are uniquely related to each other but are not equal, while for the field subsystem the order parameter μ and coherent-state parameter α are equal.

The results of this section make it clear why the coupled order-parameter treatment (2.4)–(2.13) reduces to a mean-field theory. In the thermodynamic limit the Hamiltonian (2.1) can be replaced by $\mathcal{H}_L$, Eq. (2.5). Since the components of $\mathcal{H}_L$ commute, the corresponding density operator factors

$$\rho \sim e^{-\beta\mathcal{H}_L} \to e^{-\beta H_1} e^{-\beta H_2} \sim \rho_f \otimes \rho_a, \tag{3.10}$$

$$\rho_f \sim \exp\left[-\beta\left(a^\dagger a + \frac{\lambda}{\sqrt{N}} a^\dagger \sum_{j=1}^{N} \langle \sigma_j^- \rangle + \frac{\lambda}{\sqrt{N}} a \sum_{j=1}^{N} \langle \sigma_j^+ \rangle\right)\right], \tag{3.11}$$

$$\rho_a \sim \prod_{j=1}^{N} \exp\left[-\beta\left(\tfrac{1}{2}\epsilon\sigma_j^z + \frac{\lambda}{\sqrt{N}} \langle a^\dagger \rangle \sigma_j^- + \frac{\lambda}{\sqrt{N}} \langle a \rangle \sigma_j^+\right)\right]. \tag{3.12}$$

Equation (3.11) describes a field driven by a classical current ($\langle \sigma_j^\pm \rangle$), and (3.12) describes an atomic system driven by a classical field ($\langle a \rangle, \langle a^\dagger \rangle$). Thus each subsystem is treated as a quantum-mechanical system driven by its dual classical counterpart. The effective classical driving fields (μ, μ^*) and currents (ν_j, ν_j^*) are related to each other in a self-consistent way by (2.14). The onset of order in each subsystem induces an order in its complementary subsystem.

In the ordered phase the effective classical driving field μ produces an increase in the energy level separation between the ground and excited atomic levels. This Stark splitting $E = [\epsilon^2 + 4\lambda^2 |\mu'|^2]^{1/2}$ ranges from $E = \epsilon$ at the critical temperature to $E = \lambda^2$ at $T = 0$.

IV. ENHANCED COOPERATION CONDENSATION

Since the atomic subsystem is described by N independent SU(2) algebras, it is useful to introduce the total angular momentum operator

$$J^2 = J_z^2 + \tfrac{1}{2}(J_+ J_- + J_- J_+),$$

where

$$J_z = \sum_{j=1}^{N} \tfrac{1}{2}\sigma_j^z \quad \text{and} \quad J_\pm = \sum_{j=1}^{N} \sigma_j^\pm.$$

Then the cooperation number r defined by

$$r(r+1) = \langle J^2 \rangle = \mathrm{Tr}\rho_2 J^2 \tag{4.1}$$

describes the extent to which the individual atomic Bloch vectors are aligned. In the disordered state r increases monotonically as the temperature decreases and assumes the maximum value $r = \frac{1}{2}N$ at $T = 0$, since all atoms are in the ground state $m = -\frac{1}{2}$. It is not immediately obvious, however, that $r \to \frac{1}{2}N$ as $T \to 0$ in the ordered state.

The cooperation number as a function of temperature in both the disordered and the ordered states is given by

$$r(r+1) = \langle J^2 \rangle = \tfrac{3}{4}N + \tfrac{1}{4}N(N-1)(\tanh\tfrac{1}{2}\beta E)^2. \tag{4.2}$$

In the limit of high temperature ($\beta \to 0$), the atomic system behaves like N independent spin-$\frac{1}{2}$ systems. In the limit of low temperatures ($\tanh\frac{1}{2}\beta E \to 1$), the atomic system behaves like a single giant spin with $J = \frac{1}{2}N$. Below the critical temperature $\frac{1}{2}E = [(\frac{1}{2}\epsilon)^2 + |\lambda\mu'|^2]^{1/2}$ in the ordered state and $E = \epsilon$ in the disordered state. As a result, at fixed temperature $r(\mathrm{ord}) \geq r(\mathrm{dis})$. The onset of order below the critical temperature leads to an increase in the cooperation number r. As a result we can say that the ordered state is characterized by an enhanced condensation into the state of maximum cooperation number.

From (3.8) it is clear that the density operator for the atomic system in the ordered state can be diagonalized by a unitary transformation $U^{-1}(\theta, \phi)$. This unitary transformation rotates the south pole of the Bloch sphere into a new position characterized by the atomic coherent-state parameters $\frac{1}{2}\theta e^{-i\phi}$. In this new representation the density operator has the diagonal form

$$\rho_2 \sim \prod_{j=1}^{N} \exp\left(\frac{-\beta E \sigma_j^z}{2}\right).$$

It is apparent that the enhancement in the cooperation number r stems entirely from the Stark splitting of the atomic levels produced by the effective classical driving field μ' as discussed at the end of Sec. III.

V. GENERATING FUNCTIONS FOR OPERATOR MOMENTS

Generating functions for the moments of the diagonal operators $a^\dagger a, \sigma_j^z$, and the off-diagonal operators $a^\dagger$, a, $\sigma_j^\pm$, can be computed by standard techniques involving Baker-Campbell-Hausdorff formulas.[18] We will compute the generating functions for the diagonal operators in this section.

The expectation value

$$\langle e^{\gamma a^\dagger a} \rangle = \frac{\mathrm{Tr} e^{\gamma a^\dagger a} \exp[-\beta(a^\dagger a + \alpha a^\dagger + \alpha^* a)]}{\mathrm{Tr}(\gamma \to 0)} \tag{5.1}$$

is[18]

$$\langle e^{\gamma a^\dagger a} \rangle = \frac{1}{1 - (e^\gamma - 1)\langle n \rangle} \exp\left(\frac{|\alpha|^2}{\langle n \rangle} \frac{(e^\gamma - 1)\langle n \rangle}{1 - (e^\gamma - 1)\langle n \rangle}\right), \tag{5.2}$$

where $\langle n\rangle = (e^{\beta}-1)^{-1}$. Further application of a standard Laguerre polynomial identity,[19] valid for $|e^{\gamma}-1|<1$, leads to

$$\langle e^{\gamma a^{\dagger}a}\rangle = \sum_{k=0}^{\infty} L_k\left(-\frac{|\alpha|^2}{\langle n\rangle}\right)(e^{\gamma}-1)^k\langle n\rangle^k. \tag{5.3}$$

The first two moments obtained from this generating function are

$$\langle a^{\dagger}a\rangle = \langle n\rangle + |\alpha|^2 = \mathfrak{N}+\mathfrak{S},$$
$$\langle (a^{\dagger}a)^2\rangle = (2\mathfrak{N}^2+4\mathfrak{N}\mathfrak{S}+\mathfrak{S}^2)+(\mathfrak{N}+\mathfrak{S}),$$

where $\mathfrak{N}$ is the chaotic or thermal contribution and $\mathfrak{S}$ is the ordered contribution. The variance

$$\langle (\Delta a^{\dagger}a)^2\rangle = \mathfrak{S}+\mathfrak{N}(1+\mathfrak{N})+2\mathfrak{S}\mathfrak{N} \tag{5.4}$$

is the sum of variances expected from a coherent source only, from a thermal source only, and the interference term arising from the presence of both sources.[20]

The generating function for the diagonal operator $\frac{1}{2}\sigma_j^z$ can be computed similarly,

$$\langle e^{\gamma\sigma^z/2}\rangle = \frac{\mathrm{Tr}e^{\gamma\sigma^z/2}\exp[-\beta(\frac{1}{2}\epsilon\sigma^z+\lambda\mu'\sigma^{+}+\lambda\mu'^{*}\sigma^{-})]}{\mathrm{Tr}(\gamma\to 0)}$$
$$= \cosh\tfrac{1}{2}\gamma - (\tfrac{1}{2}\epsilon/\theta)\tanh\beta\theta\sinh\tfrac{1}{2}\gamma, \tag{5.5}$$

where

$$\theta = [(\tfrac{1}{2}\epsilon)^2+|\lambda\mu'|^2]^{1/2}.$$

The first two moments and the variance of $\frac{1}{2}\sigma^z$ are

$$\langle\tfrac{1}{2}\sigma^z\rangle = -\tfrac{1}{2}(\tfrac{1}{2}\epsilon/\theta)\tanh\beta\theta, \quad \langle(\tfrac{1}{2}\sigma^z)^2\rangle = \tfrac{1}{4},$$
$$\langle(\Delta\tfrac{1}{2}\sigma^z)^2\rangle = \tfrac{1}{4}[1-(\epsilon/2\theta)^2\tanh^2\beta\theta]. \tag{5.6}$$

It should be remarked that in the disordered phase $\langle a^{\dagger}a\rangle = \langle n\rangle = (e^{\beta}-1)^{-1}$ and $\langle\frac{1}{2}\sigma^z\rangle = -\frac{1}{2}\tanh\frac{1}{2}\beta\epsilon$ go to 0 and $-\frac{1}{2}$, respectively, as $T\to 0$. In the ordered phase $\langle a^{\dagger}a\rangle \to (e^{\beta}-1)^{-1}+N[(\frac{1}{2}\lambda)^2-(\epsilon/2\lambda)^2]$ and $\langle\frac{1}{2}\sigma^z\rangle \to -(\epsilon/2\lambda^2)$ by (2.15) and (5.6).

VI. ALTERNATIVE FORM FOR THE COUPLED EQUATIONS

The interpretation of the disposable parameters μ and ν_j, (2.4), as order parameters, which is forced on us by (2.13), suggests an alternative derivation of the coupled order parameter equations.[21] This involves computing the equation of motion for the order parameters $\langle a\rangle$, and $\langle\sigma_j^{-}\rangle$,

$$\frac{d}{dt}\langle a\rangle = \frac{i}{\hbar}\left\langle\left[a^{\dagger}a+\epsilon\sum_{j=1}^{N}\tfrac{1}{2}\sigma_j^z + \frac{\lambda}{\sqrt{N}}\left(a^{\dagger}\sum_{j=1}^{N}\sigma_j^{-}+a\sum_{j=1}^{N}\sigma_j^{+}\right), a\right]\right\rangle. \tag{6.1}$$

The assumptions of steady-state conditions, $d\langle a\rangle/dt = 0$, and the thermodynamic limit, $\langle[a^{\dagger}\sigma_j^{-}a]\rangle = \langle[a^{\dagger}a]\rangle_1\langle\sigma_j^{-}\rangle_2$, lead directly to an expression for $\langle a\rangle$. A dual calculation leads to an analogous expression for $\langle\sigma_j^{-}\rangle$. The coupled nonlinear equations obtained in this way are

$$\mu = -\frac{\lambda}{\sqrt{N}}\sum_{j=1}^{N}\nu_j, \quad \epsilon\nu_j = \frac{2\lambda}{\sqrt{N}}\langle\tfrac{1}{2}\sigma_j^z\rangle_2\mu. \tag{6.2}$$

Although the coupled order-parameter equations (6.2) appear different from the coupled equations (2.14), the two sets may be shown to be equivalent[15] using Schur's formula (see the Appendix).

For purposes of discussing the existence of phase transitions and locating critical temperatures, coupled nonlinear order-parameter equations derived by the methods of this section are more useful than the nonlinear order-parameter equations derived by the methods of Sec. II. In the neighborhood of a critical temperature, the order parameters μ' and ν_j appearing in (6.2) may be taken as infinitesimals of the first order. The expectation values of the commutators may then be taken with respect to the disordered state. For example, $\langle\frac{1}{2}\sigma_j^z\rangle_2$ is simply a Brillouin function for a spin-$\frac{1}{2}$ system. Using the result that $\frac{1}{2}B_{1/2}(\beta\epsilon) = \frac{1}{2}\tanh\frac{1}{2}\beta\epsilon$ in (6.2) leads directly to the gap equation (2.16).

Coupled equations of the form (6.2) are more useful than coupled equations of the form (2.14) for locating critical temperatures because they are already in linear form in the neighborhood of the critical temperature. Moreover, Eqs. (6.2) provide both local and global information about the nature of the order-disorder phase transition. It may be verified from (6.2) that $\mu, \nu_j \sim |T-T_c|^{1/2}$ in the neighborhood of T_c. From (2.17) it is easily verified that the phase transition is second order.

It is also possible to make global statements of a less precise nature about the phase transition. If the order parameters behave like $(T_c-T)^{1/2}$ in the neighborhood of T_c, then the phase transition will be second order at T_c, but if they behave like $(T-T_c)^{1/2}$ in the neighborhood of the largest critical temperature, then there will be[15] a first-order phase transition at a temperature $T_1 > T_c$ in addition to the second-order transition at T_c. This latter behavior appears for a certain range of the coupling constants in a model Hamiltonian discussed by Thompson.[12]

The results in this section dealing with the location of critical temperatures are a special case of a more general theorem called the fluctuation-transformation theorem, which will be discussed elsewhere.[15]

VII. APPLICATIONS TO OTHER MODEL HAMILTONIANS

The methods outlined in Sec. VI will be illustrated by application to three simple modifications of the Dicke Hamiltonian. These models describe (a) interaction of N identical spin-j systems with a single mode of the radiation field, (b) interaction of N identical two-level atoms with m different modes of the radiation field (m finite), and (c) interaction of N identical two-level atoms with the radiation field through double photon absorption and emission processes. In each of these models the gap equation for the critical temperature will be derived. In addition, for model (b) we will show that the atomic system order parameter behaves exactly as in the Dicke model, provided the coupling constant is appropriately chosen.

A. Spin-j systems

The model Hamiltonian describing the interaction of N identical spin-j systems with a single mode of the electromagnetic field is obtained from (2.1) by replacing the spin operators $\sigma_j^{\pm}$, $\frac{1}{2}\sigma_j^z$ by the angular momentum operators $J_j^{\pm}$, J_j^z. With the definitions

$$\langle a\rangle = \mu = \mu'\sqrt{N}, \quad \langle J_k^-\rangle = \nu_k = \nu, \tag{7.1}$$

the following coupled nonlinear equations are obtained using the procedures leading to (6.2):

$$\mu' = -\frac{\lambda}{N}\sum_{k=1}^{N}\nu_k, \quad \epsilon\nu_k = 2\lambda\langle J_k^z\rangle\,\mu'. \tag{7.2}$$

Linearization of (7.2) around the critical temperatures leads to the gap equation for the critical temperature $T_c = 1/\beta_c K$,

$$(\lambda^2/\epsilon)(2j)B_j(\beta_c\epsilon) = 1, \tag{7.3}$$

where $B_j(x)$ is the Brillouin function. This reduces to the gap equation (2.16) for $j = \frac{1}{2}$. Comparison of (7.3) with (2.16) reveals that it is possible for an ensemble of "atoms" with $2j+1$ equally spaced energy levels to undergo a phase transition for smaller values of the coupling constant λ than that required for the corresponding system of two-level atoms.

B. Multimode systems

The Hamiltonian describing the interaction of N identical two-level atoms with m modes (m finite) of the radiation field is given by (2.1). Making the identifications

$$\langle a_s\rangle = \mu_s, \quad \langle \sigma_k^-\rangle = \nu_k = \nu \tag{7.4}$$

and using the equations of motion method leads to the following coupled nonlinear order-parameter equations:

$$\omega_s\mu_s = -\frac{\lambda_s^*}{\sqrt{N}}\sum_{s=1}^{N}\nu_k, \quad \epsilon\nu_k = \frac{\langle\sigma_k^z\rangle}{\sqrt{N}}\sum_{k=1}^{m}\lambda_s\mu_s. \tag{7.5}$$

Linearization about β_c leads immediately to the gap equation

$$1 = \frac{1}{\epsilon}\left(\sum_{s=1}^{m}\frac{|\lambda_s|^2}{\omega_s}\right)\tanh\tfrac{1}{2}\beta_c\epsilon. \tag{7.6}$$

This is identical to the single-mode gap equation (2.16) under the identification

$$\lambda^2 = \sum_{s=1}^{m}\frac{|\lambda_s|^2}{\omega_s}. \tag{7.7}$$

In order to show that the ordered-state behavior in the multimode and single-mode models is the same, it is useful to eliminate the order parameters μ_s from (7.5). This leads to the relation

$$\epsilon\nu = \nu\left(\sum_{s=1}^{m}\frac{|\lambda_s|^2}{\omega_s}\right)\frac{\epsilon}{2\theta}\tanh\beta\theta,$$
$$\theta = \left[\left(\frac{\epsilon}{2}\right)^2 + \left|\nu\sum_{s=1}^{m}\frac{\lambda_s^*\lambda_s}{\omega_s}\right|^2\right]^{1/2}. \tag{7.8}$$

After making the identification (7.7), the two solutions $\nu = 0$, $\nu \neq 0$ obtained from (7.8) for the multimode case are identical to the solutions for the single-mode case obtained from (2.14) by eliminating μ instead of ν.

C. Two-photon processes

Recent theoretical[22] and experimental[23,24] interest in double photon absorption processes suggests that it might be useful to investigate such systems for critical behavior. Since absorption of two equal-energy but oppositely propagating photons is more likely to be important for cooperative behavior (because the Doppler shift is eliminated[24]), we will investigate the following model Hamiltonian for critical behavior in the thermodynamic limit:

$$H = (a_+^\dagger a_+ + a_-^\dagger a_- + 1) + \epsilon\sum_{k=1}^{N}\tfrac{1}{2}\sigma_k^z + \frac{\lambda}{\sqrt{N}}\left(a_+^\dagger a_-^\dagger\sum_{k=1}^{N}\sigma_k^- + a_+ a_-\sum_{k=1}^{N}\sigma_k^+\right). \tag{7.9}$$

Here $a_+^\dagger$ describes photons propagating with wave vector k and $a_-^\dagger$ describes oppositely propagating photons. The photon number difference operator $\Delta = a_+^\dagger a_+ - a_-^\dagger a_-$ commutes with the Hamiltonian.

The coupled equations for the order parameters $\langle a_+ a_-\rangle$ and $\langle\sigma_k^-\rangle$ are

$$\langle a_+ a_- \rangle = -\tfrac{1}{2}\lambda \langle n_+ + n_- + 1 \rangle \frac{1}{\sqrt{N}} \sum_{k=1}^{N} \langle \sigma_k^- \rangle ,$$
$$\epsilon \langle \sigma_k^- \rangle = (\lambda/\sqrt{N}) \langle \sigma_k^z \rangle \langle a_+ a_- \rangle . \tag{7.10}$$

For fixed $\Delta = n_+ - n_-$ (Δ an integer) and at β_c, $\langle n_+ + n_- + 1 \rangle = |\Delta| + \coth\beta_c$ and $\langle \sigma_k^z \rangle = -\tanh\tfrac{1}{2}\beta_c\epsilon$. The gap equation is then

$$(\lambda^2/2\epsilon)\tanh\tfrac{1}{2}\beta_c\epsilon\,(|\Delta| + \coth\beta_c) = 1 . \tag{7.11}$$

Assuming a resonant interaction ($\epsilon = 2$), the gap equation simplifies to

$$(\tfrac{1}{2}\lambda)^2 (1 + |\Delta| \tanh\beta_c) = 1 . \tag{7.12}$$

This result clearly shows that a phase transition is possible even for very small values of the coupling constant λ, provided the photon number difference $|\Delta|$ is large enough.

A similar gap equation can be derived for the case in which two photons are absorbed from or emitted into the same mode,

$$(2\lambda^2/\epsilon)(\coth\beta_c \mp \tfrac{1}{2}) \tanh\tfrac{1}{2}\beta_c\,\epsilon = 1 . \tag{7.13}$$

The upper sign holds for the case of an even number of photons in the mode and the lower for an odd number.

VIII. CONCLUSION

The equilibrium statistical mechanical properties of the Dicke Hamiltonian have been treated from a coupled order-parameter point of view. The Dicke Hamiltonian was linearized by expanding each of the shift operators appearing in it about an undetermined c-number parameter. These parameters are chosen variationally[15] to minimize the free energy associated with the linearized Hamiltonian. The variational choice leads to an interpretation of the variational parameters as order parameters for the two coupled subsystems. In the thermodynamic limit, the free energy per particle obtained from the original Hamiltonian and its linearized version are equal.

The order parameters for the field and the atomic subsystems obey coupled nonlinear equations. These equations always support the trivial solution $\mu = 0$, $\nu_j = 0$ describing the disordered state. For large enough values of the coupling constant ($\lambda^2 > \epsilon$), a phase transition to an ordered state is possible. The gap equation (2.16) locates the critical temperature below which the onset of order is possible.

The density operator for the linearized Dicke Hamiltonian factorizes into the product of density operators, one for the field subsystem, the other for the atomic subsystem. These subsystem density operators were compared with density operators describing a statistical superposition of thermal chaos and a classical driving field. The comparison reveals that the thermodynamic state of each subsystem is a statistical superposition of thermal chaos characterized by a temperature $T = 1/K\beta$ and a classical behavior characterized by coherent state parameters α and $\tfrac{1}{2}\theta e^{-i\phi}$. For the field subsystem the order parameter μ and the coherent-state parameter α are equal. For the atomic subsystem the order parameter ν and coherent-state parameter $\tfrac{1}{2}\theta e^{-i\phi}$ are related but are not equal.

This density-operator treatment clearly reveals the mean-field nature of the equilibrium statistical mechanics of systems governed by the Dicke Hamiltonian in the thermodynamic limit. Each quantum-mechanical subsystem behaves as if it were driven classically by its counterpart. The classical driving terms are order parameters for the respective subsystems. Furthermore, the order parameters obey coupled self-consistency equations.

The ordered state is characterized by an enhanced condensation of the atomic subsystem into the state of maximum cooperation number r. This enhancement is due entirely to the further splitting of the atomic energy levels by the classical intensive driving field characterized by the order parameter μ' (Stark effect).

The interpretation of the variational parameters μ and ν as order parameters for the coupled subsystems suggested an alternative derivation of the coupled order-parameter equations. This involves computing the equation of motion for the order parameters $d\langle a \rangle/dt$ and $d\langle \sigma_k^- \rangle/dt$ under steady-state conditions in the thermodynamic limit. The coupled nonlinear equations obtained by this procedure appear different from the self-consistent coupled equations, although the two sets may be shown to be equivalent.

The coupled equations derived from the equation-of-motion method are more suitable for the location of the critical temperature, since they are already linearized in the order parameters at β_c. The critical temperature is easily determined from a linear eigenvalue equation.

This technique was then used to locate the critical temperatures and discuss the order-disorder behavior for three variations of the Dicke model. These models describe interaction of spin-j systems with a single field mode, interaction of two-level systems with many modes of the radiation field, and interaction of two-level systems with one or two modes of the radiation field through two-photon absorption and emission processes.

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APPENDIX: PERTURBATION EXPANSION AND SCHUR'S FORMULA

The perturbation expansion used in (2.11) involves computation of the function $f(x, y)$ defined by

$$e^{\beta(x+y)} = e^{\beta x} e^{f(x,y)} . \tag{A1}$$

The operator $e^{-\beta x} e^{(x+y)}$ obeys the first-order differential equation

$$\frac{d}{d\beta} e^{-\beta x} e^{\beta(x+y)} = e^{-\beta x} y e^{\beta(x+y)}$$

$$= e^{\mathrm{ad}(-\beta x)} y \, (e^{-\beta x} e^{\beta(x+y)}) , \tag{A2}$$

$$e^{\mathrm{ad}(x)} y \equiv e^{x} y e^{-x} = y + [x,y] + \tfrac{1}{2}[x[x,y]] .$$

Integration and iteration of (A2) lead to the perturbation expansion

$$e^{-\beta x} e^{\beta(x+y)} = I + \int_0^\beta d\beta' \, e^{-\beta' \,\mathrm{ad}\, x} \, y \left(I + \int_0^{\beta'} d\beta'' \, e^{-\beta'' \,\mathrm{ad}\, x} \, y \times (I + \cdots) \right) . \tag{A3}$$

Application of (A3) to (2.11) yields the desired expression,

$$\frac{\mathrm{Tr} e^{-\beta(\mathcal{H}_L + \mathcal{H}_Q)}}{\mathrm{Tr} e^{-\beta \mathcal{H}_L}} = 1 - \int_0^\beta d\beta' \langle e^{+\beta' \,\mathrm{ad}\, \mathcal{H}_L} \mathcal{H}_Q \rangle_L + \int_0^\beta d\beta' \int_0^{\beta'} d\beta'' \langle e^{\beta' \,\mathrm{ad}\, \mathcal{H}_L} \mathcal{H}_Q \times e^{\beta'' \,\mathrm{ad}\, \mathcal{H}_L} \mathcal{H}_Q \rangle_L + \cdots . \tag{A4}$$

The expectation value is taken with respect to the Hamiltonian $\mathcal{H}_L$.

Schur's formula[25] is essentially the first Born approximation to (A3) for the operator $e^{x+\epsilon y}$, where ϵ is assumed small.

$$\epsilon^{x+\epsilon y} = \epsilon^{x} \left(I + \int_0^1 \epsilon^{-\beta' \,\mathrm{ad}\, x} \, \epsilon y \, d\beta' \right) + O(\epsilon^2)$$

$$= e^{x} \{ I + [(e^{-\mathrm{ad}\, x} - I)/-\mathrm{ad}\, x] \, \epsilon y \} + O(\epsilon^2)$$

$$= e^{x} \exp\{ [(1 - e^{-\mathrm{ad}\, x})/\mathrm{ad}\, x] \, \epsilon y \} + O(\epsilon^2) . \tag{A5}$$

Schur's formula is particularly easy to use when y is an eigenoperator of x, $[x,y] = \alpha y$. For example, if $x = a^\dagger a + \frac{1}{2} E \sigma^z$ and $y = \lambda(a^\dagger \sigma^- + a\sigma^+)$,

$$\exp\{-\beta[a^\dagger a + \tfrac{1}{2} E \sigma^z + \lambda(a^\dagger \sigma^- + a\sigma^+)]\}$$

$$= \exp[-\beta(a^\dagger a + \tfrac{1}{2} E \sigma^z)]$$

$$\times \exp\left(\frac{1 - e^{-\beta(E-1)}}{-(E-1)} \lambda a^\dagger \sigma^- + \frac{1 - e^{\beta(E-1)}}{E-1} \lambda a \sigma^+ + O(\lambda^2) \right) . \tag{A6}$$

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