PHYSICAL REVIEW A VOLUME 13, NUMBER 5 MAY 1976

Poincare cycles and coherence of bounded thermal radiation fields

H.P. Baltes and B. Steinle
Zentrale Forschung und Entwicklung, LGZ Landis & Gyr Zug AG, CH-6301 Zug, Switzerland

M. Pabst
Institut für Kernphysik, Technische Hochschule Darmstadt, D-61 Darmstadt, W. Germany
(Received 22 September 1975)

We study the thermal radiation field in finite cavities bounded by perfectly reflecting walls. Previous analytical results on the spatially averaged temporal correlation tensors for the cube-shaped blackbody are generalized for cuboidal cavities of arbitrary edge lengths including the infinite slab resonator. Computational results for very small cubes and sufficiently long times demonstrate the almost periodic character of the coherence function. The transition from cyclic to aperiodic relaxation is shown for the visibility, the phase, and the field autocorrelation. Improved results on the spectral mode density and total radiation energy are obtained as by-products.

I. INTRODUCTION

In a previous paper[1] we introduced the spatially averaged electric, magnetic, and mixed temporal correlation tensors $\overline{\mathcal{E}}_{\mu\nu}$, $\overline{\mathfrak{B}}_{\mu\nu}$, and $\overline{\mathfrak{M}}_{\mu\nu}$ for the thermal radiation field in a finite cavity:

$$\overline{\mathcal{E}}_{\mu\nu}(t) \equiv V^{-1} \int_V d\vec{x}\, \mathrm{tr}\,[\rho E_\mu^{(-)}(\vec{x},0)\, E_\nu^{(+)}(\vec{x},t)], \qquad (1)$$

$\mu, \nu = 1,2,3$, where t denotes the time, V the volume of the cavity, ρ the canonical density operator, and $E_\mu = E_\mu^{(-)} + E_\mu^{(+)}$ the μth component of the electric field operator. The averaged tensors $\overline{\mathcal{E}}_{\mu\nu}$ and $\overline{\mathfrak{B}}_{\mu\nu}$ are related to the spectral energy density $u(\omega)\,d\omega$ of the radiation field by virtue of the quantum-optical Wiener-Khintchine theorem[2,3] leading to the relation

$$u(\omega) = \int_{-\infty}^{+\infty} dt\, e^{i\omega t}\left(\sum_{\mu=1}^{3}(\overline{\mathcal{E}}_{\mu\mu}(t) + \overline{\mathfrak{B}}_{\mu\mu}(t))\right). \qquad (2)$$

In the free-space limit $\omega V^{1/3} \to \infty$, $u(\omega) \to u_\infty(\omega)$ obeys Planck's radiation law. The spectral density $u(\omega)$ for finite blackbody cavities is relevant to the problem of radiation standards in the far-infrared and submillimeter wave regions, and the above-averaged tensors are basic to correlation experiments or Fourier-transform spectroscopy of the blackbody emissivity.[1,4-6]

We calculated the tensors for the simple case of the *cube*-shaped cavity of edge length L with perfectly reflecting walls and obtained[1]

$$\overline{\mathcal{E}}_{\mu\nu} = \delta_{\mu\nu}\Gamma = \overline{\mathfrak{B}}_{\mu\nu}, \quad \overline{\mathfrak{M}}_{\mu\nu} = 0, \qquad (3)$$

with the scalar function

$$\Gamma = \Gamma(t, L, T) = \frac{1}{3V}\sum_{k_n}\frac{\hbar c k_n e^{-ik_n ct}}{e^{\hbar c k_n/KT} - 1}, \qquad (4)$$

where T denotes the temperature, K the Boltzmann constant, $\hbar$ Planck's constant divided by 2π, and c the speed of light. The $k_n = \omega_n/c$ are the eigenvalues of the cavity resonator. We established an asymptotic expansion for $\Gamma(t)$ in powers of $1/LT$ and t/L, the first term of which corresponds to the free-space limit.[7] Unfortunately, this expansion is useful for the numerical calculation of $\Gamma(t)$ only for finite but not-too-small LT and only for not-too-long times L.

In this paper, we study the more involved case of the *cuboidal* cavity with edge lengths L_1, L_2, and L_3 as well as the almost periodic behavior of the correlation appearing in the case of *very small* cavities or *sufficiently long* times. In Sec. II, we present the correlation tensors for the cuboid in terms of exact series. We find that both the isotropy (3) and the vanishing of the surface terms of the tensor components observed for the cube[1] as well as for the free-space limit[7] do not hold for the less symmetrical cuboid. In Sec. III we present the asymptotic expansion of the tensor's trace and exploit the relation (2) for the calculation of the spectral and total radiation energies. In particular, we study the oscillatory terms of the density of states and compare it with the analogous scalar wave problem. The case of the Fabry-Perot resonator ($L_1/L_3 = L_2/L_3 \to \infty$) is discussed as well. Section IV is devoted to the numerical summation of the series of type (4) for very small cubes or very low temperatures. The corresponding Poincaré cycles are studied for the fringe visibility $|\Gamma(t)/\Gamma(0)|$, the phase angle $\Phi(t) = \arg\Gamma(t)$, and the electric field autocorrelation function

$$\langle\vec{E}(0)\,\vec{E}(t)\rangle \propto \mathrm{Re}\sum_{\mu=1}^{3}\overline{\mathcal{E}}_{\mu\mu}(t).$$

II. CORRELATION TENSOR

A more involved version of the procedure outlined in Sec. II of Ref. 1 leads to the following series representation of the diagonal components $\mathcal{E}_{\lambda\lambda}$ and $\mathcal{B}_{\lambda\lambda}$, $\lambda=1,2,3$, of the spatially averaged electric and magnetic field temporal correlation tensors: We obtain

$$\overline{\mathcal{E}}_{\lambda\lambda}(t)=-\frac{\hbar c}{2V}\left(L_\mu\frac{\partial}{\partial L_\mu}+L_\nu\frac{\partial}{\partial L_\nu}\right)\sum_{\substack{n_\lambda=0,\\ n_\mu,n_\nu=1}}^{\infty}G(k,t) \tag{5a}$$

$$=\frac{\hbar c}{2V}\left(\sum_{n_\lambda,n_\mu,n_\nu=1}^{\infty}\frac{k^2-(\pi n_\lambda/L_\lambda)^2}{k}F(k,t)+\sum_{n_\mu,n_\nu=1}^{\infty}k_{\mu\nu}F(k_{\mu\nu},t)\right) \tag{5b}$$

and

$$\overline{\mathcal{B}}_{\lambda\lambda}(t)=-\frac{\hbar c}{2V}\left(L_\mu\frac{\partial}{\partial L_\mu}+L_\nu\frac{\partial}{\partial L_\nu}\right)\sum_{\substack{n_\lambda=1,\\ n_\mu,n_\nu=0}}^{\infty}G(k,t) \tag{6a}$$

$$=\frac{\hbar c}{2V}\left(\sum_{n_\lambda,n_\mu,n_\nu=1}^{\infty}\frac{k^2-(\pi n_\lambda/L_\lambda)^2}{k}F(k,t)+\sum_{\sigma=\mu,\nu}\sum_{n_\lambda,n_\sigma=1}^{\infty}\frac{(\pi n_\sigma/L_\sigma)^2}{k_{\lambda\sigma}}F(k_{\lambda\sigma},t)\right). \tag{6b}$$

Here we introduced $V=L_1L_2L_3$ and the wave numbers

$$k=\pi\left[\left(\frac{n_\lambda}{L_\lambda}\right)^2+\left(\frac{n_\mu}{L_\mu}\right)^2+\left(\frac{n_\sigma}{L_\sigma}\right)^2\right]^{1/2} \tag{7}$$

and

$$k_{\mu\nu}=\pi[(n_\mu/L_\mu)^2+(n_\nu/L_\nu)^2]^{1/2}. \tag{8}$$

The indices $\{\lambda,\mu,\nu\}$ are any permutation of $\{1,2,3\}$. The functions G and F are related by

$$\frac{\partial}{\partial k}G(k,t)=F(k,t)\equiv\frac{e^{-ikct}}{e^{\hbar ck/KT}-1}. \tag{9}$$

From (5a)–(6b) we learn that the isotropy of the correlation tensors as well as the identity of the corresponding electric and magnetic field tensor components valid in the free-space limit are in general lost for finite blackbodies. In the limit $t\to 0$, the representations (5a) and (6a) suggest the related anisotropy of the radiation pressure in agreement with a previous result.[8] In the case of the *cube*, $L_1=L_2=L_3=L$, however, the above six diagonal components coalesce to one scalar function,

$$\overline{\mathcal{E}}_{\lambda\lambda}(t)=\overline{\mathcal{B}}_{\lambda\lambda}(t)=\frac{\hbar c}{2V}\left(\frac{2}{3}\sum_{n_1,n_2,n_3=1}^{\infty}kF(k,t)+\sum_{n_1,n_2=1}^{\infty}\tilde{k}F(\tilde{k},t)\right), \tag{10}$$

with $\tilde{k}=(\pi/L)\,(n_1^2+n_2^2)^{1/2}$. This result is a consequence of the high symmetry.

We now have to evaluate the sums (5b) and (6b). To this end, we symmetrize the summations as follows:

$$\overline{\mathcal{E}}_{\lambda\lambda}(t)=(\hbar c/16V)$$

$$\times\left[\sum_{n_\lambda,n_\mu,n_\nu=-\infty}^{+\infty}\frac{k^2-(\pi n_\lambda/L_\lambda)^2}{k}F(k,t)\right. \tag{11a}$$

$$-\sum_{\sigma=\mu,\nu}\sum_{n_\lambda,n_\sigma=-\infty}^{+\infty}\frac{(\pi n_\sigma/L_\sigma)^2}{k_{\lambda\sigma}}F(k_{\lambda\sigma},t)+\sum_{n_\mu,n_\nu=-\infty}^{+\infty}k_{\mu\nu}F(k_{\mu\nu},t) \tag{11b}$$

$$\left.-\sum_{\sigma=\mu,\nu}\sum_{n_\sigma=-\infty}^{+\infty}\left|\frac{\pi n_\sigma}{L_\sigma}\right|F\left(\left|\frac{\pi n_\sigma}{L_\sigma}\right|,t\right)\right] \tag{11c}$$

$$+KT/12V, \tag{11d}$$

with k and $k_{\mu\nu}$ as given by (7) and (8). $\overline{\mathcal{B}}_{\lambda\lambda}$ looks like $\overline{\mathcal{E}}_{\lambda\lambda}$ with the exception that the two-dimensional sums (11b) have the opposite signs. From the dimensionality of the summations we easily infer that (11a) is related to the cavity volume, whereas (11b) corresponds to the faces and (11c) to the edges of the cuboid. The edge-length-independent term (11d) is supposed to be related to the corners of the cuboid. For instance, the last term in (11b) stems from the faces that are orthogonal to the λ direction. In (11c), only the edges orthogonal to the λ direction play a role. Obviously, the surface terms of

$$\Gamma(t)=\frac{1}{3}\sum_{\lambda=1}^{3}\overline{\mathcal{E}}_{\lambda\lambda}(t)=\frac{1}{3}\sum_{\lambda=1}^{3}\overline{\mathfrak{B}}_{\lambda\lambda}(t) \tag{12}$$

vanish.

III. ASYMPTOTIC EXPANSION FOR THE TRACE AND SPECTRAL DENSITY

The sums occurring in(11a)–(11c) are evaluated by means of the Poisson summation technique. The detailed results for the single tensor components are very complicated and are therefore compiled in the Appendix. In this section, we restrict our considerations to the trace (12) related to the spectral energy density by the relation (2). Introducing the reduced reciprocal temperature $\alpha\equiv\hbar c/KT$ and the reduced time $\tau\equiv ct/\alpha=KTt/\hbar$, the Poisson summation for the trace (12) leads to

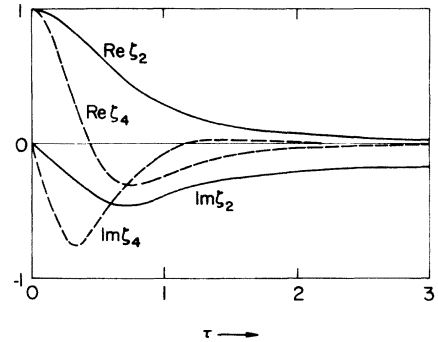

FIG. 1. Real and imaginary parts of the normalized generalized Riemann ζ functions $\zeta_4(\tau)=90\pi^{-4}\zeta(4,1+i\tau)$ (dashed lines) and $\zeta_2(\tau)=6\pi^{-2}\zeta(2,1+i\tau)$ (solid lines). They occur in the temporal coherence functions (13) and (15). For $\tau\gtrsim 2$, these functions are approximately described by $-15\pi^{-4}(3\tau^{-3}-2i\tau^{-3})$ and $3\pi^{-2}(\tau^{-2}-2i\tau^{-1})$.

$$\begin{aligned}\Gamma(t,\alpha)=&\frac{\hbar c}{\pi^2\alpha^4}\zeta(4,1+i\tau)-\frac{\hbar c}{12\pi\alpha^2}\frac{L_1+L_2+L_3}{V}\zeta(2,1+i\tau)+\frac{KT}{12V}\\&-\frac{i\hbar c}{12\pi^2\alpha^3}\sum_{\nu_1,\nu_2,\nu_3=-\infty}^{+\infty}{}' \mu^{-1}\left[\zeta\left(3,1+i\left(\tau-\frac{2\mu}{\alpha}\right)\right)-\zeta\left(3,1+i\left(\tau+\frac{2\mu}{\alpha}\right)\right)\right]\\&-\frac{\hbar c}{24\pi\alpha^2V}\sum_{\lambda=1}^{3}\left\{L_\lambda\sum_{m=-\infty}^{+\infty}{}'\left[\zeta\left(2,1+i\left(\tau-\frac{2mL_\lambda}{\alpha}\right)\right)+\zeta\left(2,1+i\left(\tau+\frac{2mL_\lambda}{\alpha}\right)\right)\right]\right\},\end{aligned} \tag{13}$$

with

$$\mu\equiv[(\nu_1L_1)^2+(\nu_2L_2)^2+(\nu_3L_3)^2]^{1/2}, \tag{14}$$

and where $\zeta(s,z)$ denotes the generalized Riemann ζ function. The prime indicates that the terms with $\mu=0$ and $m=0$, respectively, are omitted in the summation. The result (13) is valid for any finite α and any τ.

For any time $t<2L_{\min}/c$, with $L_{\min}=\min(L_1,L_2,L_3)$, the following asymptotic expansion in the limit $\alpha/L_{\min}\to 0$ can be established:

$$\begin{aligned}\Gamma(t)\sim&\frac{\hbar c}{\pi^2\alpha^4}\zeta(4,1+i\tau)-\frac{\hbar c}{12\pi\alpha^2}\frac{L_1+L_2+L_3}{V}\zeta(2,1+i\tau)+\frac{KT}{12V}\\&-\frac{\hbar c}{16\pi\alpha^4}\sum_{n=0}^{\infty}\tau^{2n}2^{-2n}(2n+1)q_n+\frac{i\hbar c}{8\pi\alpha^4}\sum_{n=0}^{\infty}\tau^{2n+1}\sum_{m=n}^{\infty}(-1)^{m-n}2^{-2m}B_{2(m-n)}\binom{2m+1}{2n+1}q_m,\end{aligned} \tag{15}$$

with the Bernoulli numbers $B_{2(m-n)}$ and with

$$q_m\equiv\frac{\alpha^3}{3V}\sum_{\lambda=1}^{3}\left(\frac{\alpha}{L_\lambda}\right)^{2m+1}\zeta(2m+2)-\frac{m+1}{6\pi}\alpha^{2m+4}\sum_{\nu_1,\nu_2,\nu_3=-\infty}^{+\infty}{}'\mu^{-2m-4}, \tag{16}$$

where $\zeta(s)$ denotes the ordinary Riemann ζ function and where μ is given by (14).

For not-too-large $\alpha/L_{\min}$ and for not-too-long times t, $\Gamma(t)$ is sufficiently well described by the two leading terms of (13) or (15). The corresponding generalized Riemann ζ functions are displayed in Fig. 1. Apparently the decay of the correlation is already slowed down by the first correction term proportional to $\zeta(2,1+i\tau)$. We mention that this term becomes minimal in the case of the cube-shaped cavity showing the same volume.

Applying (2) to the exact result (13), we calculate the spectral energy density. Omitting the Bose-Einstein factor $\hbar ck(e^{\hbar ck/KT}-1)$ we obtain the spectral mode density

$$D(k)=\frac{Vk^2}{\pi^2}\sum_{\nu_1,\nu_2,\nu_3=-\infty}^{+\infty}\frac{\sin(2\mu k)}{2\mu k}-\frac{1}{2\pi}\sum_{\lambda=1}^{3}\left(L_\lambda\sum_{m=-\infty}^{+\infty}\cos 2mL_\lambda k\right)+\tfrac{1}{2}\delta(k)\,, \tag{17}$$

with μ defined by (14). The corresponding total radiation energy U reads

$$\frac{U(T)}{\hbar c}=\frac{\pi^2}{15}\frac{L_1L_2L_3}{\alpha^4}-\frac{\pi}{12}\frac{L_1+L_2+L_3}{\alpha^2}+\frac{1}{2\alpha}-a$$
$$+\frac{\pi}{2}\frac{L_1L_2L_3}{\alpha^3}\sum_{\nu_1,\nu_2,\nu_3=-\infty}^{+\infty}{}'\mu^{-1}\left\{\left(\frac{2\pi\mu}{\alpha}\right)^{-3}-\cosh\left(\frac{2\pi\mu}{\alpha}\right)\left[\sinh\left(\frac{2\pi\mu}{\alpha}\right)\right]^{-3}\right\}$$
$$+\frac{4\pi}{\alpha}\sum_{\lambda=1}^{3}\sum_{m=-\infty}^{+\infty}{}'|m|^{-1}\left[\sinh\left(\frac{2\pi|m|L_\lambda}{\alpha}\right)\right]^{-2}, \tag{18}$$

where

$$a=\frac{\pi}{16}\left[\frac{1}{3}\left(\frac{1}{L_1}+\frac{1}{L_2}+\frac{1}{L_3}\right)-\frac{L_1L_2L_3}{\pi^2}\sum_{\nu_1,\nu_2,\nu_3=-\infty}^{+\infty}{}'\mu^{-4}\right]. \tag{19}$$

For $L_1=L_2=L_3=L$, (17) and (18) reproduce the known results for the cube-shaped cavity.[1,6,9] We observe that LU scales with $KTL/\hbar c$ in the case of the cube,[6] whereas no analogous scaling law in terms of some length $L_{\rm eff}$ characterizing the cuboid can be established for the cuboid. Equation (18), as well as the analogous result for the cube,[1,6,9] represent the total energy in terms of an *asymptotic expansion* around $T\to\infty$ ("high-temperature expansion"). By definition,[10] the difference between the exact value of the total energy and a finite number of leading terms of (18) is thus of order T^{-M}, with some $M>0$, where M increases with the number of terms taken into account. A finite number of leading terms is an excellent approximation of the total energy for finite but not-too-small T. In the case of the cube the four leading terms represent the total energy to within 1% error for $LT\gtrsim 0.3$ cm K.[11,12] We cannot, however, expect that these four or any finite number of leading terms of (18) will describe the limit $T\to 0$ correctly. In particular, Eq. (18) does not imply $U\to$ const or $dU/dT\to$ const as $T\to 0$.[4,11] For extremely small values, say $LT\lesssim 1$ cm K, it is more feasible to use Eq. (10) for the numerical calculation of the total energy.[11,13] The formula (10) is likewise an asymptotic expansion around $T=0$ ("low-temperature expansion"). This low-temperature approach is considered further in Sec. IV. The range of validity of both the high- and low-temperature expansions of the total energy of the cube-shaped blackbody was studied numerically in Ref. 11.

We mention that the electromagnetic mode density (17) differs from the mode densities derived for the lattice-point problem[14] and the scalar wave Dirichlet and Neumann problems.[15] The scalar mode density shows terms proportional to $L_\lambda L_\mu$, including oscillatory terms in the form of two-dimensional sums over the zero-order Bessel function. Similar surface terms are obtained in the electromagnetic case only if one calculates the partial mode densities corresponding to the modes contributing to some specific field component E_λ. This can be seen from Fourier transforming the terms (A2) and (A3) in the Appendix.

In the limit of flat cavities with $L_2=L_3=L\to\infty$, but finite $L_1=l$, we find the tensor components

$$\overline{\mathcal{E}}_{11}(\tau)=\frac{\hbar c}{\pi^2\alpha^4}\zeta(4,1+i\tau)-\frac{\hbar c}{16\pi^2}\frac{1}{\alpha^2l^2}\sum_{\nu=-\infty}^{+\infty}{}'\left\{\frac{1}{\nu^3}\left[\zeta\left(2,1+i\left(\tau+\frac{2\nu l}{\alpha}\right)\right)+\zeta\left(2,1+i\left(\tau-\frac{2\nu l}{\alpha}\right)\right)\right]\right.$$
$$\left.+\frac{i\alpha}{2l\nu^3}\left[\psi\left(1+i\left(\tau+\frac{2\nu l}{\alpha}\right)\right)-\psi\left(1+i\left(\tau-\frac{2\nu l}{\alpha}\right)\right)\right]\right\}+\frac{\hbar c}{4\pi\alpha^3l}\zeta(3,1+i\tau)\,, \tag{20}$$

$$\overline{\mathfrak{B}}_{11}(\tau)=\overline{\mathcal{E}}_{11}(\tau)-(\hbar c/2\pi\alpha^3l)\zeta(3,1+i\tau)\,, \tag{21}$$

and

$$\overline{\mathcal{E}}_{22}(\tau)=\overline{\mathcal{E}}_{33}(\tau)=\frac{\hbar c}{\pi^2\alpha^4}\zeta(4,1+i\tau)+\frac{\hbar c}{8\pi^2\alpha^2 l^2}\sum_{\nu=-\infty}^{+\infty}{}' \left\{\frac{il}{\alpha\nu}\left[\zeta\left(3,1+i\left(\tau+\frac{2\nu l}{\alpha}\right)\right)-\zeta\left(3,1+i\left(\tau-\frac{2\nu l}{\alpha}\right)\right)\right]\right.$$

$$+\frac{1}{4\nu^2}\left[\zeta\left(2,1+i\left(\tau+\frac{2\nu l}{\alpha}\right)\right)+\zeta\left(2,1+i\left(\tau-\frac{2\nu l}{\alpha}\right)\right)\right]$$

$$\left.+\frac{i\alpha}{8l\nu^3}\left[\psi\left(1+i\left(\tau+\frac{2\nu l}{\alpha}\right)\right)-\psi\left(1+i\left(\tau-\frac{2\nu l}{\alpha}\right)\right)\right]\right\}$$

$$-\frac{\hbar c}{8\pi\alpha^3 l}\zeta(3,1+i\tau)\,, \tag{22}$$

$$\overline{\mathcal{B}}_{22}(\tau)=\overline{\mathcal{B}}_{33}(\tau)=\overline{\mathcal{E}}_{22}(\tau)+(\hbar c/4\pi\alpha^3 l)\,\zeta(3,1+i\tau)\,, \tag{23}$$

where ψ denotes the ψ function

$$\psi(1+z)=\sum_{n=1}^{\infty}\frac{z}{n(n+z)}-C\,,$$

with C denoting Euler's constant. For the trace as defined by (12) we obtain

$$\Gamma(\tau)=\frac{\hbar c}{\pi^2\alpha^4}\zeta(4,1+i\tau)+i\frac{\hbar c}{12\pi^2\alpha^3 l}\sum_{\nu=-\infty}^{+\infty}{}'\frac{1}{\nu}\left[\zeta\left(3,1+i\left(\tau+\frac{2\nu l}{\alpha}\right)\right)-\zeta\left(3,1+i\left(\tau-\frac{2\nu l}{\alpha}\right)\right)\right], \tag{24}$$

leading to

$$D(k)=\frac{Vk^2}{\pi^2}\sum_{m=-\infty}^{+\infty}\frac{\sin 2mlk}{2mlk} \tag{25}$$

and

$$\frac{U(\tau)}{\hbar c}=\frac{\pi^2}{15}\frac{V}{\alpha^4}+\frac{\pi V}{2\alpha^3 l}\sum_{m=-\infty}^{+\infty}{}'\frac{1}{|m|}\left\{\left(\frac{2\pi|m|l}{\alpha}\right)^{-3}-\cosh\left(\frac{2\pi|m|l}{\alpha}\right)\left[\sinh\left(\frac{2\pi|m|l}{\alpha}\right)\right]^{-3}\right\}, \tag{26}$$

thus confirming previous results for the mode density[16,17] and the total radiation energy[18] of the flat parallelepiped with perfectly conducting walls.

IV. QUASIPERIODIC CORRELATION FOR SMALL CUBES

We now go back to the special case $L_1=L_2=L_3=L$ of the cube-shaped cavity. In our previous paper[1] we emphasized the asymptotic expansions for the averaged complex correlation tensors $\overline{\mathcal{E}}_{\mu\nu}=\delta_{\mu\nu}\Gamma=\overline{\mathcal{B}}_{\mu\nu}$, useful for finite but not-too-small TL, $TL\gtrsim\hbar c/K$, and finite but not-too-long times $t\lesssim L/c$. Now we demonstrate the quasiperidoic behavior of the correlation occuring for sufficiently small values of TL and large values of t. To this end, we display curves for the three quantities of physical interest, namely, the degree of first-order coherence $|\gamma(t)|=|\Gamma(t)/\Gamma(0)|$, the phase angle $\Phi(t)=\arg\Gamma(t)$, and the electric field autocorrelation function $\langle\vec{E}(0)\vec{E}(t)\rangle$ proportional to $\mathrm{Re}\Gamma(t)$.

The result (13) and the analogous formula (19) in Ref. 1 are valid for any cavity dimension and temperature as well as for any time t. For very small cavities or very low temperatures as well as for very long times, however, the computational evaluation of these formulas becomes more tedious than that of the sum representation (10). We thus compute the above-mentioned quantities for a variety of values of TL below 1 Kcm as functions of the reduced time in the range $0\le\tau\lesssim 5$. We com-

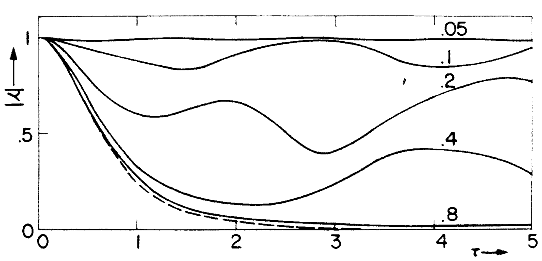

FIG. 2. Temporal coherence $|\gamma|=|\Gamma(\tau)/\Gamma(0)|$ for the cube-shaped blackbody plotted as a function of $\tau=KTt/\hbar$ for $TL=0.05$, 0.1, 0.2, 0.4, and 0.8 Kcm (solid lines). The thermodynamic limit $TL\to\infty$ is shown for comparison (dashed line).

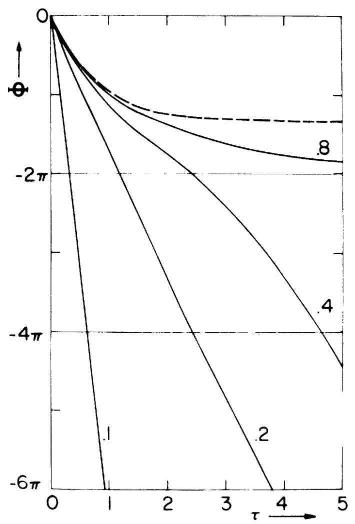

FIG. 3. Phase angle $\Phi = \arg[\Gamma(\tau)/\Gamma(0)]$ for the temporal coherence function belonging to the cube-shaped blackbody plotted for $TL = 0.1$, 0.2, 0.4, and 0.8 Kcm (solid lines) as a function of $\tau = KTt/\hbar$. The thermodynamic limit $TL \to \infty$ is shown for comparison (dashed line).

pare the results with the aperiodic free-space limit $LT \to \infty$.[19]

The results for the *degree of first-order temporal coherence* are shown in Fig. 2. Almost complete first-order coherence, $|\gamma| \approx 1$, is found in the limit of very small LT, where the lowest mode with the wave number $k = 2^{1/2}\pi/L$ is dominant. However, $|\gamma|$ relaxes rapidly to very small values for $TL \gtrsim 0.8$ Kcm. We point out that the results for $TL = 0.8$ Kcm and $TL \to \infty$ are similar only for $\tau \lesssim 1$. Substantial relative deviations occur, however, for larger values of τ, where the free-space limit is described by

$$|\gamma_\infty| \sim 30\pi^{-4}\tau^{-3}. \tag{27}$$

The time evaluation of the *phase angle* is displayed in Fig. 3. A free rotation $\Phi \propto \tau$ is found for very small TL, where the fundamental mode is dominant. This periodic motion is more and more slowed down with increasing TL, and the monotonic relaxation towards the stationary value $-\frac{3}{2}\pi$ finally appears in the limit $TL \to \infty$. For sufficiently large τ, this free-space limit is described by

$$\Phi_\infty \sim -\tfrac{3}{2}\pi + \tfrac{3}{2}\tau^{-1}. \tag{28}$$

Finally, we consider the *electric field autocorrelation function*

$$\mathrm{Re}\,\gamma = \langle \vec{E}(0)\vec{E}(\tau)\rangle / \langle \vec{E}^2(0)\rangle$$

$$= \sum_{\lambda=1}^{3} \overline{\mathcal{E}}_{\lambda\lambda}(\tau)\left(\sum_{\lambda=1}^{3}\overline{\mathcal{E}}_{\lambda\lambda}(0)\right)^{-1}. \tag{29}$$

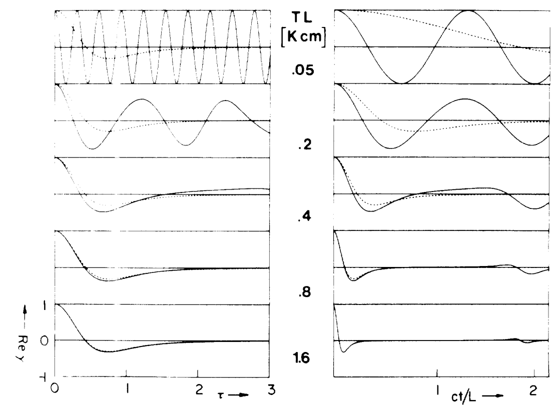

FIG. 4. Field autocorrelation function Reγ for the cube-shaped blackbody plotted as a function of $\tau = KTt/\hbar$ (left-hand side) and ct/L (right-hand side), respectively, for $TL = 0.05$, 0.2, 0.4, 0.8, and 1.8 Kcm (solid lines). Each of the ten plots is meant for the range $-1 \le \mathrm{Re}\gamma \le 1$, and in each plot the thermodynamic limit $TL \to \infty$ is displayed for comparison (dashed lines).

This quantity is of physical interest because of the analogy with autocorrelation functions occuring in time-dependent statistical mechanics, e.g., Brownian motion in a finite box.[20] We have plotted the field autocorrelation in Fig. 4 for various values of TL as a function of the "thermodynamic" reduced time $\tau = KTt/\hbar$ and the "traveling" reduced time ct/L. Apparently, τ is the natural variable for the thermodynamic limit, where γ scales with tT, whereas γ is a function of both tT and t/L in the case of finite TL. The corresponding plot of γ as a function of ct/L reveals the Poincaré cycles: The first "return" of the autocorrelation Reγ occurs at about the time $t = 2L/c$, i.e., twice the time the radiation needs for traveling from one face of the cube to the opposite one. On the other hand, our results demonstrate that the computer simulation of correlation functions carried through for a finite subsystem in a small box yields reliable information on the corresponding thermodynamic limit for times small compared to the signal traveling time.

V. CONCLUSION

In this paper we have obtained the complete analytical expressions for the spatially averaged temporal correlation tensors, the spectral mode density, and the total radiation energy of the electromagnetic field in a cuboidal cavity with perfectly reflecting walls. These results modify the Jeans number and the Planck and Stefan-Boltzmann radiation laws valid in the thermodynamic limit as well as the corresponding blackbody coherence properties.[3,19] We have discussed the time correlation for the cube numerically, demonstrating the transition from the quasiperiodic behavior in the dominant single-mode case to the aperiodic behavior in the thermodynamic limit. Thus all of the nonlocal properties of the cuboidal blackbody with lossless walls have been calculated.

We should, however, like to point out that the position dependence of the temporal correlation as well as of the total and spectral energy densities is also of great physical interest. The position-dependent temporal correlation tensors have recently been calculated in the case of the half-space bounded by a perfect conductor.[21] The starting point for the analogous calculation in the case of the cube is given by the equations (8)–(11) in Ref. 1.

APPENDIX

We apply the Poisson summation technique to the various sums appearing in the terms (11a)–(11c) of the tensor components $\mathcal{E}_{\lambda\lambda}(t)$ and $\overline{\mathfrak{B}}_{\lambda\lambda}(t)$, respectively, and obtain the following results:

The volume term (11a) yields

$$\frac{\hbar c}{\pi^2\alpha^4}\zeta(4,1+i\tau)+\frac{\hbar c}{8\pi^2\alpha}\sum_{\nu_1,\nu_2,\nu_3=-\infty}^{+\infty}{}' \left\{\frac{i[\mu^2-(\nu_\lambda L_\lambda)^2]}{\alpha^2\mu^3}\left[\zeta\left(3,1+i\left(\tau+\frac{2\mu}{\alpha}\right)\right)-\zeta\left(3,1+i\left(\tau-\frac{2\mu}{\alpha}\right)\right)\right]\right.$$
$$-\frac{3(\nu_\lambda L_\lambda)^2-\mu^2}{4\alpha\mu^4}\left[\zeta\left(2,1+i\left(\tau+\frac{2\mu}{\alpha}\right)\right)+\zeta\left(2,1+i\left(\tau-\frac{2\mu}{\alpha}\right)\right)\right]$$
$$\left.-i\frac{3(\nu_\lambda L_\lambda)^2-\mu^2}{8\mu^5}\left[\psi\left(1+i\left(\tau+\frac{2\mu}{\alpha}\right)\right)-\psi\left(1+i\left(\tau-\frac{2\mu}{\alpha}\right)\right)\right]\right\}, \tag{A1}$$

where $\alpha=\hbar c/KT$, $\tau=ct/\alpha$, and $\mu=[(\nu_1L_1)^2+(\nu_2L_2)^2+(\nu_3L_3)^2]^{1/2}$, and where ψ denotes the ψ function [see (20)–(23)].

The first two terms of (11b) are equivalent to

$$\frac{\hbar c L_\lambda L_\sigma}{8\pi\alpha^3 V}\left[\zeta(3,1+i\tau)-\frac{\alpha^2}{8\pi}\sum_{\rho_\lambda,\rho_\sigma=-\infty}^{+\infty}{}' \frac{\partial^2}{\partial(\rho_\sigma L_\sigma)^2}\int_0^\infty \frac{e^{-ir\tau}}{e^r-1}J_0\left(\frac{2[(\rho_\lambda L_\lambda)^2+(\rho_\sigma L_\sigma)^2]^{1/2}r}{\alpha}\right)dr\right], \tag{A2}$$

and the last term of (11b) leads to

$$\frac{\hbar c}{4\pi\alpha^3 L_\lambda}\left[\zeta(3,1+i\tau)+\frac{1}{2}\sum_{\rho_\mu,\rho_\nu=-\infty}^{+\infty}{}' \int_0^\infty \frac{r^2}{e^r-1}e^{-i\tau r}J_0\left(\frac{2[(\rho_\mu L_\mu)^2+(\rho_\nu L_\nu)^2]^{1/2}r}{\alpha}\right)dr\right]. \tag{A3}$$

The series of (11c) finally yields ($\sigma\in\{1,2,3\}$, $\lambda\neq\sigma$)

$$\frac{\hbar c L_\sigma}{8\pi\alpha^2 V}\left\{\zeta(2,1+i\tau)+\frac{1}{2\pi}\sum_{\nu=-\infty}^{+\infty}{}' \left[\zeta\left(2,1+i\left(\tau-\frac{2\nu L_\sigma}{\alpha}\right)\right)+\zeta\left(2,1+i\left(\tau+\frac{2\nu L_\sigma}{\alpha}\right)\right)\right]\right\}. \tag{A4}$$

Using the asymptotic expressions for $\zeta(z,s)$ given, e.g., in Ref. 10, one can easily evaluate the asymptotic behavior of (A1) and (A4) for $ct<2L_{\min}$ in the limit $\alpha/L_{\min}\to 0$, where $L_{\min}=\min(L_1,L_2,L_3)$, and obtain expressions analogous to those occurring in (15).

[1]B. Steinle, H. P. Baltes, and M. Pabst, Phys. Rev. A 12, 1519 (1975).

[2]R. Glauber, Phys. Rev. 131, 2766 (1963).

[3]C. L. Mehta and E. Wolf, Phys. Rev. 157, 1183 (1967).

[4]H. P. Baltes and F. K. Kneubühl, Helv. Phys. Acta 45, 481 (1972).

[5]H. P. Baltes, Am. J. Phys. 42, 505 (1974).

[6]H. P. Baltes, International Conference on Infrared Physics, Zürich, August 11–15, 1975 (to be published).

[7]C. L. Mehta and E. Wolf, Phys. Rev. 134, A1143 (1964); 134, A1149 (1964).

[8]W. Lukosz, Physica (Utr.) 56, 109 (1971).

[9]K. M. Case and S. C. Chiu, Phys. Rev. A 1, 1170 (1970).

[10]M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions* (Dover, New York, 1965).

[11]H. P. Baltes, Appl. Phys. 1, 39 (1973).

[12]H. P. Baltes and E. R. Hilf, *Spectra of Finite Systems* (Bibliograph. Inst., Zürich, 1976), p. 87.

[13]H. P. Baltes and E. R. Hilf, Solid State Commun. 12, 369 (1973).

[14]A. Oppenheim, Proc. Lond. Math. Soc. 26, 295 (1926).

[15]R. Balian and C. Bloch, Ann. Phys. (N.Y.) 69, 76 (1972).
[16]S. R. Barone, Microwave Res. Inst. Symp. Ser. 20, 649 (1970); and Air Force Cambridge Res. Lab. Report No. AFCRL-65-228 (1965) (unpublished).
[17]G. S. Agarwal, Phys. Rev. A 11, 253 (1975).
[18]M. Fierz, Helv. Phys. Acta 33, 855 (1960).
[19]Y. Kano and E. Wolf, Proc. R. Soc. 80, 1273 (1962).
[20]H. P. Baltes, E. R. Hilf, and M. Pabst, Appl. Phys. 3, 21 (1974).
[21]G. S. Agarwal, Phys. Rev. A 11, 230 (1975).