Numerical results of angular cross sections of electron pair production in a point-Coulomb potential

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Numerical results of angular electron pair-production cross sections are given for the γ-ray sources ^{60}Co and ^{208}Tl. Cross sections are obtained analytically for a point-Coulomb potential, using an exact partial-wave formulation. Angular distributions are essentially unaffected in shape by the screening effects, as shown by Tseng and Pratt's calculations, so our curves can be useful to experimentalists. Moreover, they are computed at the tip region of the spectrum where screening effects are almost negligible.

I. INTRODUCTION

A little while ago Tseng and Pratt[1] reported exact screened and unscreened numerical calculations of electron-pair-production angular distributions at low energy. Comparisons between screened and unscreened results show that angular distributions are unaffected in shape by the screening; cross sections are simply renormalized by the normalizations of the positron and electron wave functions at the origin. The analyti-

TABLE I. Pair-production cross sections as computed in this work for several photon energy values.

K (MeV)	Z	Y	$\frac{d\sigma}{Z^2 dW_+}$ (μb/mc^2)
1.173 23	92	0.999	53.81
	82	0.999	51.16
	63	0.999	41.33
	47	0.999	30.15
	32	0.999	19.86
	13	0.999	10.06
	8	0.999	7.376
	3	0.999	3.505
1.332 52	92	0.999	139.2
	82	0.999	116.6
	63	0.999	77.44
	47	0.999	52.15
	32	0.999	36.07
	13	0.999	21.50
	8	0.999	15.90
	3	0.999	7.547
2.614	92	0.999	151.5
	82	0.999	126.8
	63	0.999	95.70
	47	0.999	80.70
	32	0.999	70.30
	13	0.999	46.60
	1	0.999	12.10
2.614	92	0.9	188.5
	82	0.9	166.8
	63	0.9	134.2
	47	0.9	115.8
	32	0.9	104.6
	13	0.9	96.30
	1	0.9	92.70

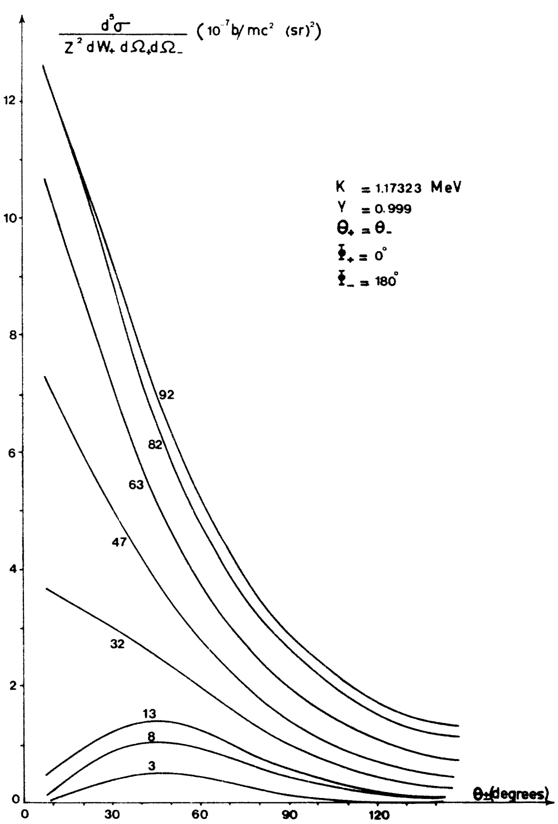

FIG. 1. Pair-production differential cross sections of a coplanar and symmetrical pair for point-Coulomb field. The numbers attached to the curves give the atomic number of the target element.

cal method we use[4,7,8] is entirely similar to that of Jaeger and Hulme,[2] and Øverbø, Mork, and Olsen,[3] and when we integrate our results over all emission angles the Øverbø, Mork, and Olsen results are recovered. The essential difference between the calculational procedures of Tseng and Pratt and ours stems from their desire to include screening effects. This necessitates numerical solution of the Dirac equation followed by numerical integration of the radial matrix elements. For a pure Coulomb field, all of this can be done analytically and only the evaluation of the final formula must be carried out numerically. It is for this rather obvious reason that their calculations require much more computer time. Since, just now, we cannot take into account analytically screening effects which may be important in some points of the spectrum, we often compute our cross sections at the tip region of the spectrum, i.e., when $Y=(W_+-1)/(K-2)\simeq 1$ where the screening effects are very small (a few percent).[1] We believe that our exact calculation is good in so far as the point-Coulomb-potential model is good.

Following the methods of our bremsstrahlung calculations[5] we wish here to present some numerical results of angular distributions for several Z values of the target and for two γ-ray sources very much used by experimentalists in the laboratory: On the one hand, the ^{60}Co γ-ray, and on the other hand, the ^{208}Tl γ-ray source. In Sec. II, we give a brief survey of pair production theory and our analytical method. Numerical results and discussion are presented in Sec. III.

II. THEORY

The formalism we use for pair production[4] and bremsstrahlung calculations[5] has already been described elsewhere. Therefore, as an example, we give directly the pair-production differential cross section averaged over photon polarizations and summed over the spins of the electron and positron:

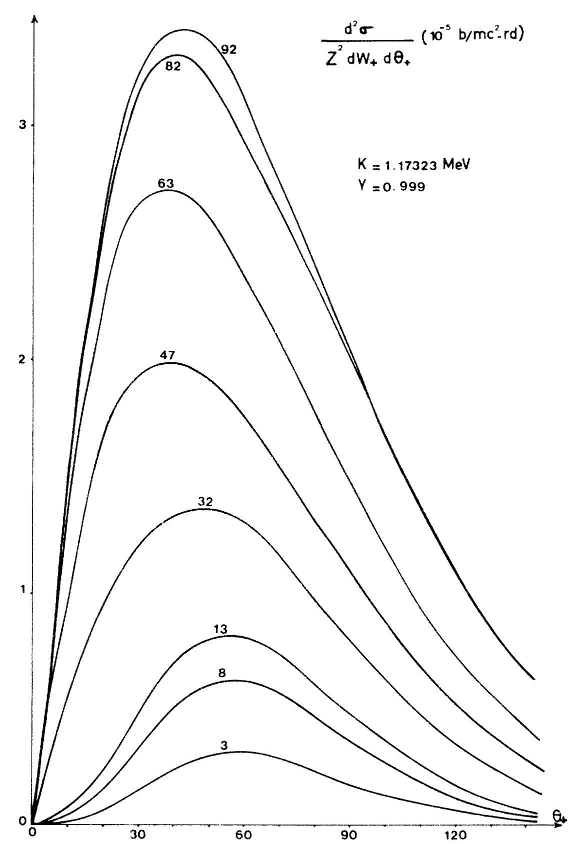

FIG. 2. Pair production angular distributions of positrons for γ-ray source $^{60}C_0$ with respect to Z values.

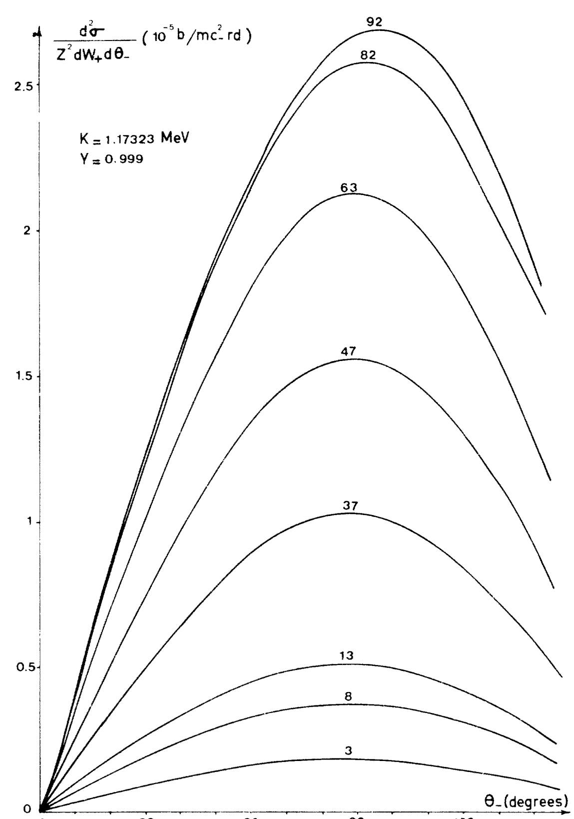

FIG. 3. Pair-production angular distributions of electrons under the same conditions as in Fig. 2.

$$\frac{d^5\sigma}{dW_+\,d\Omega_+\,d\Omega_-}$$

$$=\frac{r_0^2}{\alpha K^3}\sum_{\kappa_+,\bar\kappa_+,\kappa_-,\bar\kappa_-}\Big(i^{(3l_{\kappa_+}-3l_{\bar\kappa_+}+l_{\bar\kappa_-}-l_{\kappa_-})}\,e^{i(\delta'_{\kappa_+}+\delta'_{\bar\kappa_-}-\delta'_{\bar\kappa_+}-\delta'_{\kappa_-})}$$

$$\times\sum_{M,\bar M,\epsilon}[\chi^{\pm}_{\bar\kappa_-,\bar M+\epsilon/2}(\hat p_-)\,\chi_{\kappa_-,M+\epsilon/2}(\hat p_-)]\;[\chi^{+}_{\kappa_+,M-\epsilon/2}(\hat p_+)\,\chi_{\bar\kappa_+,\bar M-\epsilon/2}(\hat p_+)]\,A^{\epsilon *}_{\bar\kappa_+,\bar\kappa_-,\bar M}A^{\epsilon}_{\kappa_+,\kappa_-,M}\Big) \quad (1)$$

with

$$A^{\epsilon}_{\kappa_+,\kappa_-,M}=X^{\epsilon}\sum_L i^{L+1}\left\{\left[\left(\frac{(\kappa_- -\epsilon M)(\kappa_+ -\epsilon M)}{(2\kappa_- +1)(2\kappa_+ -1)}\right)^{1/2}\right]V(l_{\kappa_-},L,l'_{\kappa_-},M)\,R^{+}_{\kappa_-,L,\kappa_+}+\left[\left(\frac{(\kappa_- +\epsilon M)(\kappa_+ +\epsilon M)}{(2\kappa_- +1)(2\kappa_+ +1)}\right)^{1/2}\right]\right.$$

$$\left.\times V(l'_{\kappa_-},L,l_{\kappa_+},M)\,R^{-}_{\kappa_-,L,\kappa_+}\right\}\ , \quad (2)$$

$$X^{\epsilon}=\begin{cases}-\dfrac{\kappa_-}{|\kappa_-|} & \text{when } \epsilon=+1\ ,\\[2mm] \dfrac{\kappa_+}{|\kappa_+|} & \text{when } \epsilon=-1\ .\end{cases}$$

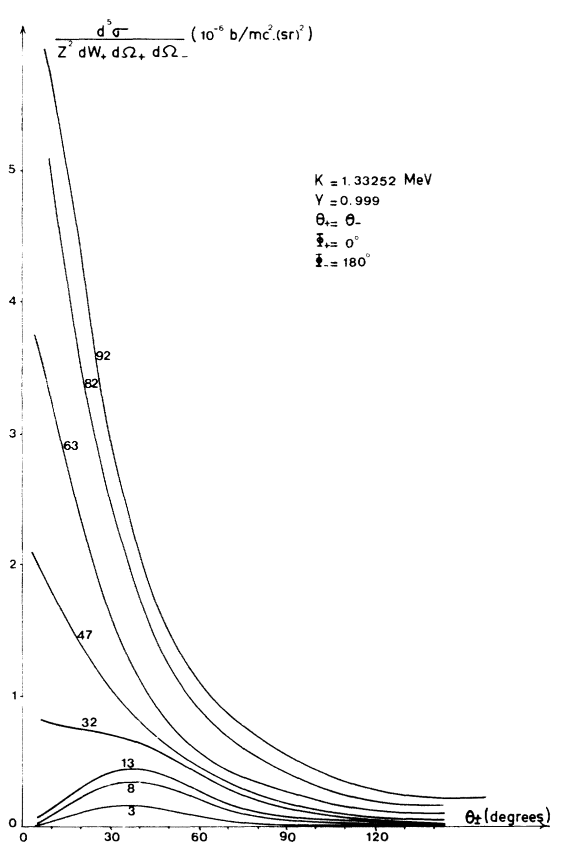

FIG. 4. The same as Fig. 1 for a different photon energy.

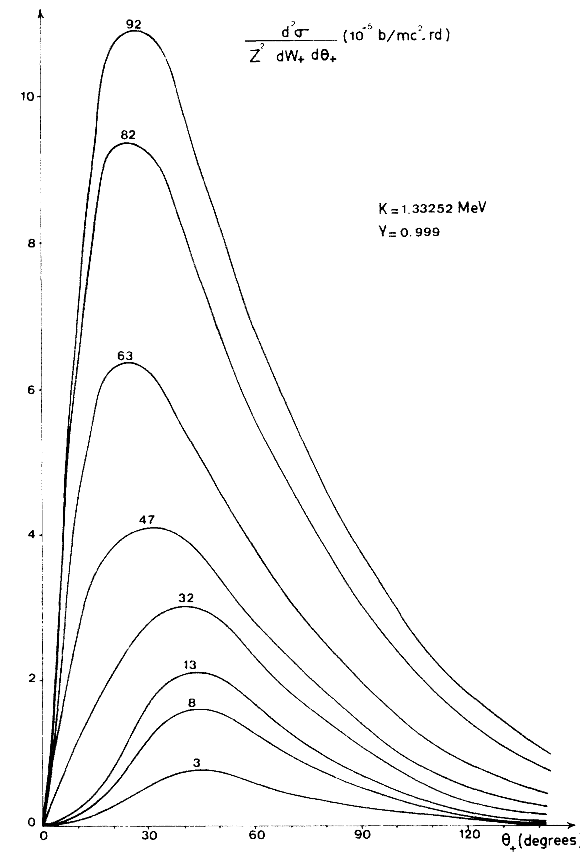

FIG. 5. Pair-production angular distributions of positrons.

The constant V is a combination of Clebsch-Gordan coefficients,

$$V(l_{\kappa_-}, L, l'_{\kappa_+}, M) = (2L+1)\left(\frac{2l_{\kappa_-}+1}{2l_{\kappa_+}+1}\right)^{1/2} C(l_{\kappa_-}, L, l_{-\kappa_+}; 0, 0, 0)\, C(l_{\kappa_-}, L, l_{-\kappa_+}; M, M, 0). \tag{3}$$

The R function is given by

$$R^{\pm}_{\kappa_-, L, \kappa_+} = d_{\kappa_+, \kappa_-} \begin{Bmatrix} [(W_+ + 1)(W_- + 1)]^{1/2} \\ -[(W_+ - 1)(W_- - 1)]^{1/2} \end{Bmatrix} \sum_{n=0}^{L} \frac{(L+n)!}{n!(L-n)!}\left(\frac{1}{2k}\right)^n \frac{\Gamma(a)}{(k+p_+ + p_-)^a} S_n^{\pm}, \tag{4}$$

where $S_n^{\pm}$ is a combination of generalized Appell hypergeometric functions[12] $F_2(a; b_1, b_2; c_1, c_2; x_1, x_2)$:

$$S_n^{\pm} = \mathrm{Im}\{\exp[-\tfrac{1}{2} i\pi(\gamma_+ + \gamma_- - L - 1)][K_+K_- F_2(a; b_+, b_-; c_+, c_-; x_+, x_-) \pm K_+K_-^* F_2(a; b_+, b_- - 1; c_+, c_-; x_+, x_-)$$

$$\mp K_+^*K_-^* F_2(a; b_+ - 1, b_-; c_+, c_-; x_+, x_-) - K_+^*K_-^* F_2(a; b_+ - 1, b_- - 1; c_+, c_-; x_+, x_-)]\}.$$

Using the definition of $\chi_{\kappa,\mu}(\hat{p})$ we can write

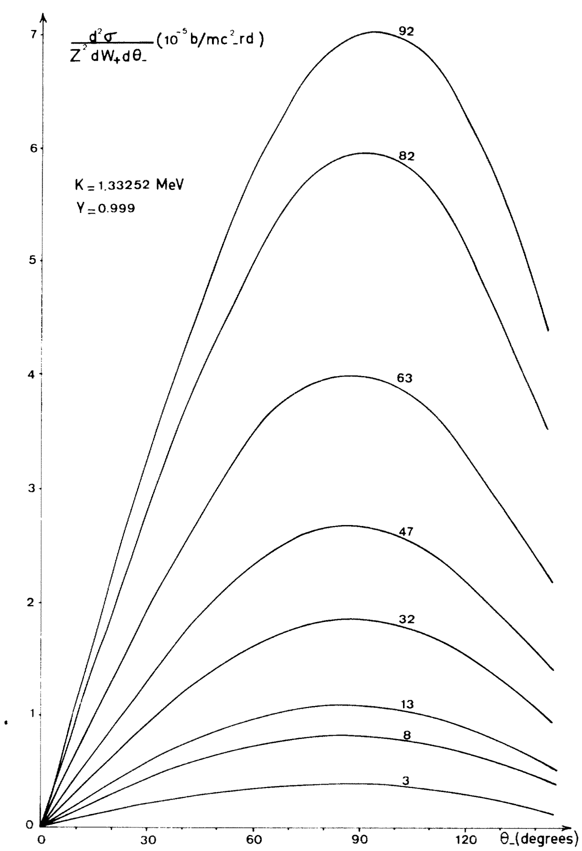

FIG. 6. Pair-production angular distributions of electrons.

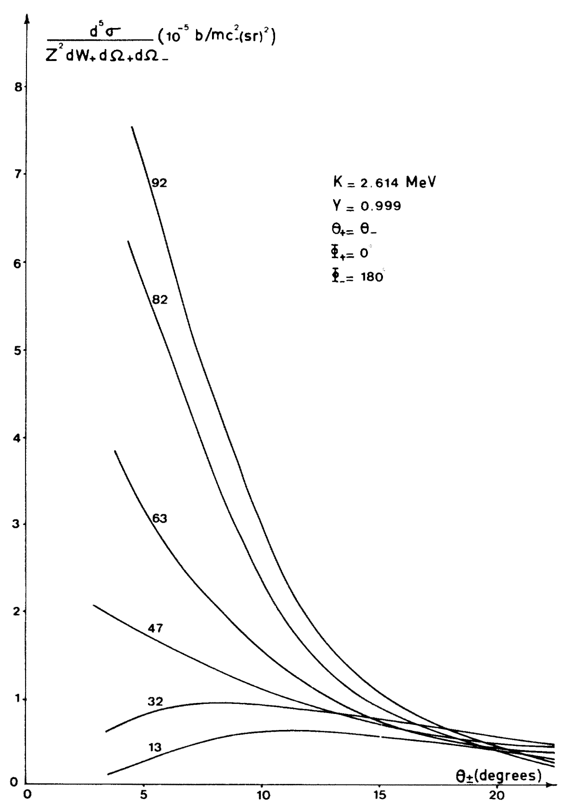

FIG. 7. Pair-production differential cross sections of a coplanar and symmetrical pair for γ-ray source ^{208}Tl with respect to Z values for $Y = 0.999$.

$$\chi^{\pm}_{\bar{K},\bar{\mu}}(\hat{p})\,\chi_{K,\mu}(\hat{p})=\sum_{m} C(\bar{l},\tfrac{1}{2},\bar{j};\,\bar{\mu}-m,m)\,C(l,\tfrac{1}{2},j;\,\mu-m,m)\,Y_{\bar{l}}^{\bar{\mu}-m\,*}(\hat{p})\,Y_{l}^{\mu-m}(\hat{p})\,. \tag{5}$$

According to the preceding relations, it is easy to find the angular distribution of one particle using the condition of the orthonormality for the states $\chi_{K,\mu}$:

$$\int \chi^{\pm}_{\bar{K},\bar{\mu}}(\hat{p})\,\chi_{K,\mu}(\hat{p})\,d\Omega=\delta_{\bar{K},K}\,\delta_{\bar{\mu}-\mu,0}\,. \tag{6}$$

For example, the angular distribution of the positron will be

$$\frac{d^3\sigma}{dW_+d\Omega_+}=\frac{r_0^2}{\alpha K^3}\sum_{\bar{K}_+,K_+,K_-}\left(i^{3l_{K_+}-3l_{\bar{K}_+}}e^{i(\delta_{K_+}-\delta_{\bar{K}_+})}\sum_{M,\epsilon}\left[\chi^{+}_{\bar{K}_+,M-\epsilon/2}(\hat{p}_+)\,\chi_{K_+,M-\epsilon/2}(\hat{p}_+)\right]A^{\epsilon\,*}_{\bar{K}_+,K_-,M}\,A^{\epsilon}_{K_+,K_-,M}\right)\,. \tag{7}$$

The other cross sections would be determined in the same way.

III. NUMERICAL RESULTS AND DISCUSSION

We present here some cases (Table I), specified by K, Z, $Y=(W_+-1)/(K-2)$, which we have computed for the γ-ray source ^{60}Co ($K=1.173\,23 \pm 0.000\,03$ MeV and $K=1.332\,52\pm 0.000\,03$ MeV) with absolute photon intensity per 100 decays of ^{60}Co being 99.88 and 100, respectively,[9] and for the γ-ray source ^{208}Tl ($K=2.614$ MeV).

The errors of calculation are estimated in the

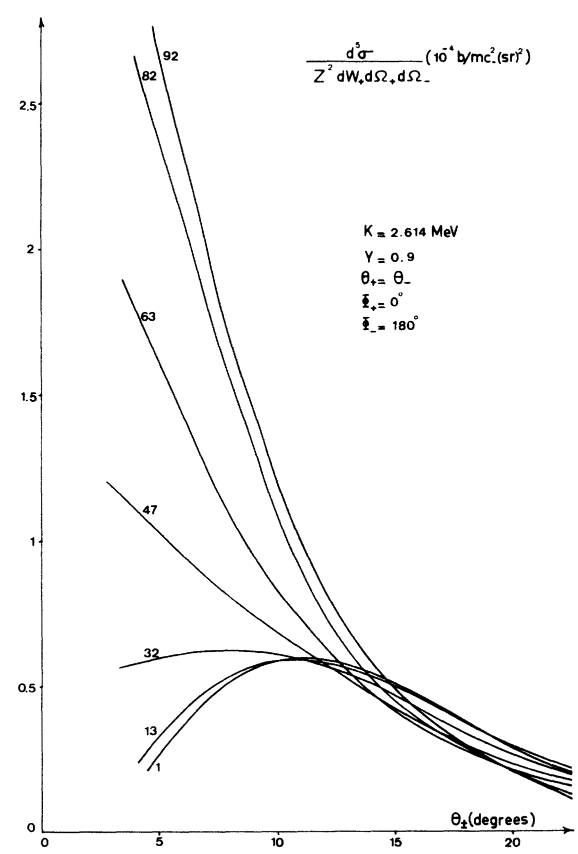

FIG. 8. The same as Fig. 7 for $Y=0.9$.

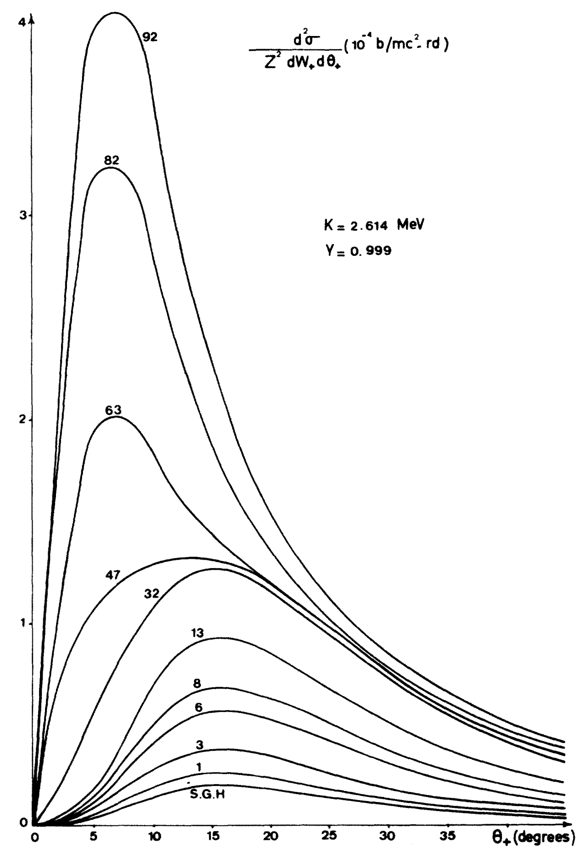

FIG. 9. Angular distributions of positron for ^{208}Tl and $Y=0.999$ compared with the Born approximation results (S.G.H.).

worst cases to 1% for $d^2\sigma/(Z^2)dW_+d\theta_\pm$ and $d^5\sigma/(Z^2)dW_+d\Omega_+d\Omega_-$, and about to 0.5% for $d\sigma/(Z^2)dW_+$. For the γ-ray source ^{60}Co we have computed cross sections near the tip of the pair production spectrum, i.e., at $Y=0.999$, so screening effects are quite negligible, as shown by Tseng and Pratt[1] (see, for example, their results for $K=2.60\ mc^2$ with $Z=79$, and $Y=0.95$; screening effects are around 3/1000, i.e. smaller than errors introduced by their numerical calculation). We do know that at the tip region of the spectrum, the Bethe and Heitler calculation predicts zero for the cross sections since that theory does not correctly take into account the Coulomb effect of the nucleus, hence we cannot compare our results with that theory. We can only note the large influence of the targets on the exact cross-section values. (See for example Figs. 1–6.)

As for the γ-ray source ^{208}Tl, we give some curves at $Y=0.999$, avoiding in that way screening effects, but we give, too, some curves at $Y=0.9$ because when the photon energy is about 5 mc^2 in the tip region of the spectrum screening effects become small, surely less than 1%, as shown by Tseng and Pratt's results.[1,13] For $Y=0.9$ we can compare our results with those obtained by the Sauter-Gluckstern-Hull formula[10]; that calculation is valid in the Born approximation (i.e., when $\alpha Z/\beta_\pm \ll 1$), without screening and with negligible nuclear recoil (i.e., for small- or large-angle pair production). We do see that for high-Z values, we cannot be confident in the Born approximation results, neither for the emission angles of the particles, nor for the values of the cross sections (Figs. 11, 12).

Finally we give angular distributions of the electron and positron at $K=2.614$ MeV with $Z=10$ (Ne) for several values of Y (Figs. 7–10). We believe these curves are not too much perturbed by screening effects (less than 1%) because Z is small and the photon energy is far enough from the threshold.[13] These curves are interesting because they display very well the shifting of the maximal angle value for different Y values (Figs. 13, 14).

The big difficulty in checking theoretical calculations for pair production near threshold is the lack of experimental measurements,[6] unlike for the bremsstrahlung effect where at small energies we

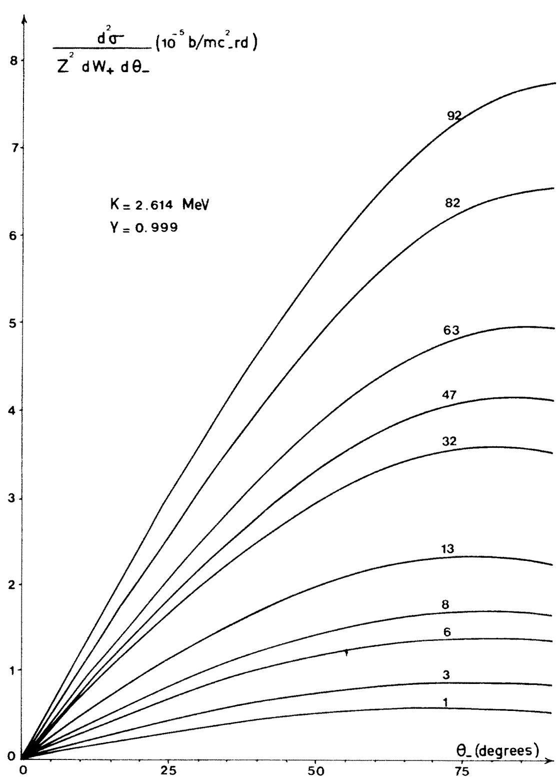

FIG. 10. Angular distributions of electrons for ^{208}Tl and $Y=0.999$.

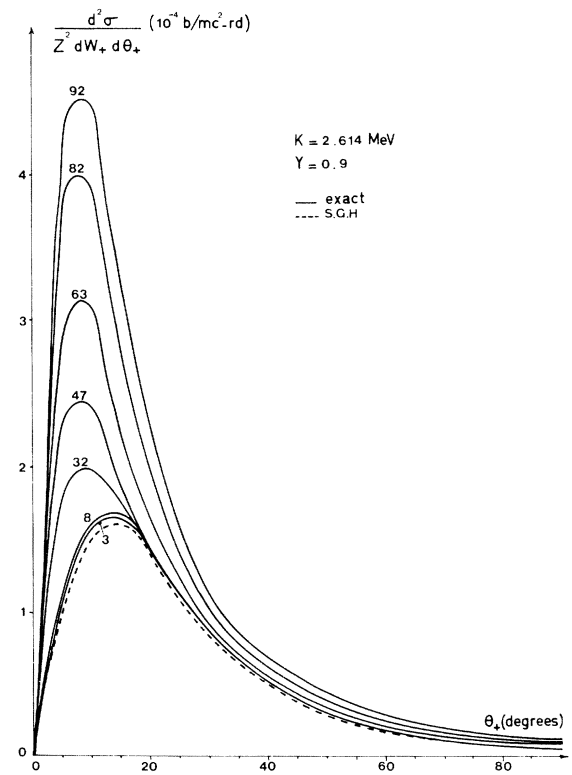

FIG. 11. Angular distributions of positrons for ^{208}Tl and $Y=0.9$ compared with the Born approximation results (broken line).

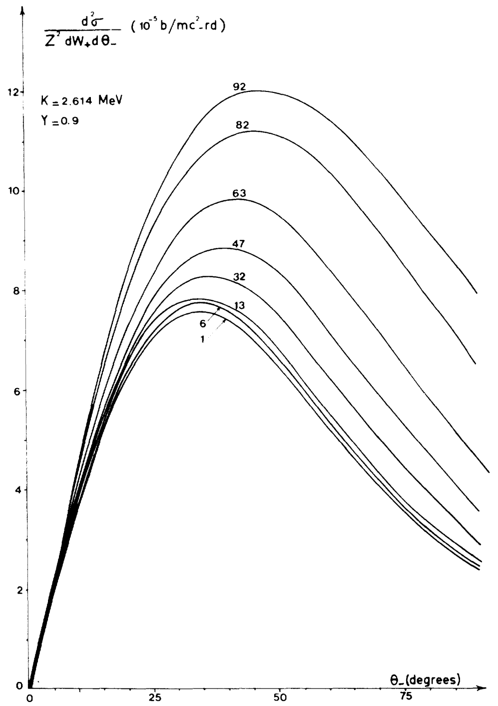

FIG. 12. Angular distributions of electrons for ^{208}Tl and $Y=0.9$.

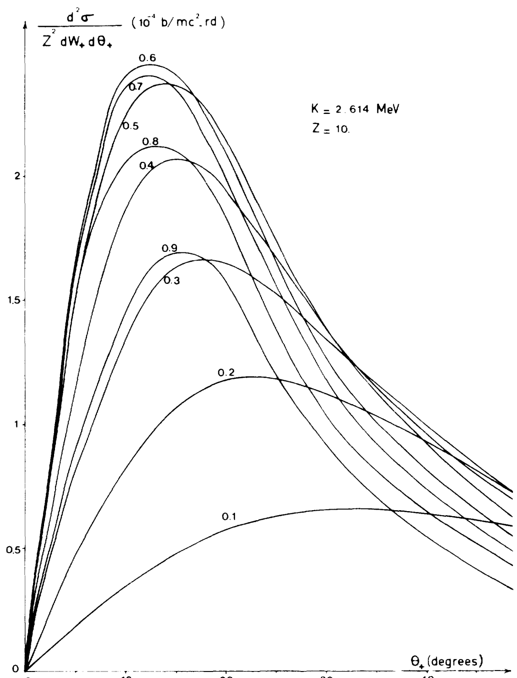

FIG. 13. Angular distributions of positrons for ^{208}Tl and $Z=10$. The numbers attached to the curves give the values of Y.

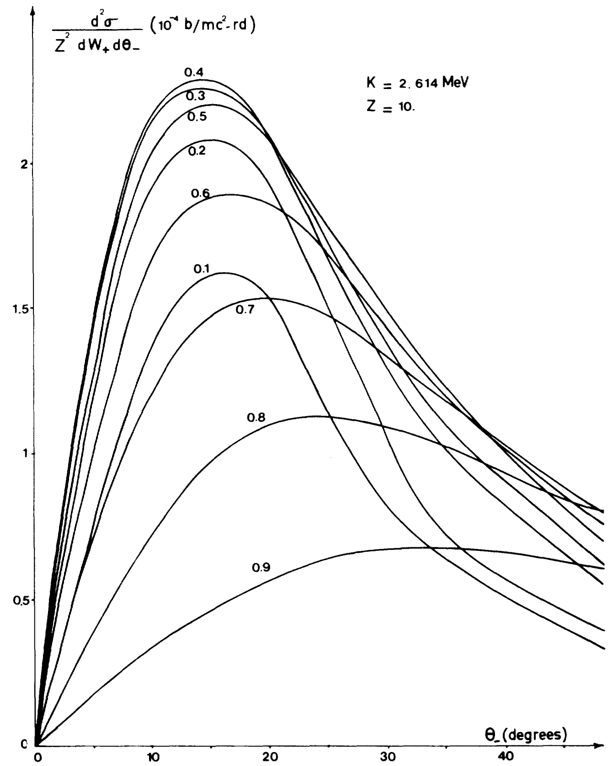

FIG. 14. Angular distributions of electrons in the same conditions as Fig. 13.

have some energy distributions. We recognize the difficulty in measuring the angular position of a particle in space without making a large experimental error when cross-section values are small; however, that is the only way to determine what the correct potential model is to use in calculations, i.e., to discover what is really happening to the photon near the nucleus. In pair production some energy distributions have been obtained, for example, by Rao *et al.*[11] but they were unable to note significant deviations between their experimental data and the point-Coulomb results from the γ-ray source ^{60}Co and the screened results of Tseng and Pratt, because of their large experimental error (5–10% or more).

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