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Reaction rates in a relativistic plasma*

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A general theory of two-body reaction rates in a relativistic plasma is developed. The most notable results of this theory are exact expressions for total rates involving only a single integral over the cross section for the usual case of relativistic Maxwell-Boltzmann or Bose-Einstein momentum distributions. This simplification arises from the fact that in the relativistic as well as the nonrelativistic case, an effective combined-particle distribution function can be found analytically. These results are applied to calculate the rates for radiative Compton scattering, photon-photon pair creation, and pion production in proton-proton collisions.

I. INTRODUCTION

Reactions occurring in plasmas sufficiently hot for ion or electron relativistic effects to become significant, play an important role in studies of supernovas, x-ray sources, black holes, and other high-energy astrophysical phenomena.[1-3] The processes of interest include bremsstrahlung and other photon-producing reactions, lepton and meson production, and nuclear reactions. Because of the relative complexity of the equilibrium relativistic velocity distribution functions and often of the reaction cross sections themselves, most calculations to date of the rates of these reactions have utilized first-order relativistic corrections[4] or interpolations between extreme- and non-relativistic forms.[5] A general formalism for relativistic distribution-averaged reactions rates has been given by Stone and Nelson,[6] but this is difficult to utilize in practice since the results are left in the form of multidimensional integrals and the cross section is defined in a manner which is not Lorentz invariant.

In the present work, a theory of total relativistic reaction rates for two-body reactions is developed in which the originally six-dimensional integral over the phase space of the particles is reduced to a single integral over the energy-dependent total cross section for the case of relativistic Maxwell-Boltzmann distributions. Rapidly convergent sums of one-dimensional integrals occur for Bose-Einstein photon distributions. This theory is used to calculate reaction rates for radiative Compton scattering ($e\gamma \to e\gamma\gamma$), photon-photon pair creation ($\gamma\gamma \to e^+e^-$), and proton-proton pion production.

The first two rates are required for the study of supernova shock waves,[1,2] while the third is of interest in black-hole accretion studies.[3]

II. RELATIVISTIC REACTION RATE FORMALISM

We wish to find the Lorentz-invariant rate per unit volume, R_{12}, at which particles of type 1 and type 2 interact to make particles of type 3. In any given observer frame, the reactants are assumed to have a distribution of the form $n_i f_i(\vec{p}_i)$ where n_i is the total number density of particles of type i and $f_i(\vec{p}_i)\,d\vec{p}_i$ is probability that a given particle will have momentum $\vec{p}_i$. For convenience and clarity we will treat first the case where the masses of the interacting particles, m_1 and m_2, are nonzero, and later relax this restriction.

A. Particle-particle reactions

In the specific case when the type-1 particles are all at rest and the type-2 particles are all moving in a beam at relative velocity v_R, the reaction rate can be given directly in the familiar form

$$R_{12} \equiv n_1 n_2 v_R \sigma(v_R)\,, \tag{1}$$

where this relation serves to define the laboratory cross section $\sigma(v_R)$.

To find R_{12} for more general distributions, we consider the specific momentum groups, $n_1 f_1(\vec{p}_1) \times d\vec{p}_1$ and $n_2 f_2(\vec{p}_2)\,d\vec{p}_2$, in the observer frame. We transform to the rest frame of the 1 group to calculate their contribution to the reaction rate, use the R_{12} invariance to transform back to the observer frame, and then integrate over $\vec{p}_1$ and $\vec{p}_2$ to find the total reaction rate:

$$R_{12} = \frac{1}{1+\delta_{12}} \int\int \{n_1 n_2 \sigma(v_R) v_R\}_{\vec{p}_1\,\text{frame}} \times f_1(\vec{p}_1) f_2(\vec{p}_2)\,d\vec{p}_1\,d\vec{p}_2\,, \tag{2}$$

where δ_{12} is the Kronecker delta which has been introduced to compensate for the fact that interacting pairs have been counted twice when particles of type 1 and 2 are identical. To evaluate $\{n_1 n_2\}_{\vec{p}_1}$ consider the invariant scalar product of the four-vector currents, $j_1 = (n_1, n_1\vec{\beta}_1)$ and $j_2 = (n_2, n_2\vec{\beta}_2)$:

$$j_1 \cdot j_2 = n_1 n_2 (1 - \vec{\beta}_1 \cdot \vec{\beta}_2) = \{n_1 n_2\}_{\vec{p}_1\text{ or }\vec{p}_2\text{ frame}}\,. \tag{3}$$

Here, $\vec{\beta}_i = \vec{v}_i/c$, where as usual $\vec{v}_i$ is the velocity of particle i and c is the speed of light. In a similar fashion the scalar product of the four-vector velocities, $(\gamma_i, \gamma_i\vec{\beta}_i)$, gives the relation

$$\gamma_1\gamma_2(1-\vec{\beta}_1\cdot\vec{\beta}_2) = \gamma_R \text{ (an invariant)}, \tag{4}$$

where $\gamma_i = (1-\beta_i^2)^{-1/2}$ and γ_R and β_R are now specifically understood to be measured in the rest frame of the type-1 particle group.

The expression for R_{12} thus becomes

$$R_{12} = \frac{(n_1n_2)_{\text{obs}}c}{1+\delta_{12}}\int\int f_1(\vec{p}_1)f_2(\vec{p}_2)\frac{\gamma_R\beta_R}{\gamma_1\gamma_2}\sigma(\beta_R)\,d\vec{p}_1\,d\vec{p}_2\,, \tag{5}$$

where the subscript "obs" denotes measurement in the observer frame.

If we now restrict our attention to distributions that are isotropic in the observer frame and consider any specified $\vec{p}_1$ to be fixed alone the z axis, (5) reduces to

$$R_{12} = \frac{c(n_1n_2)_{\text{obs}}}{2(1+\delta_{12})} \times\int_0^\infty\int_0^\infty\int_{-1}^1 f_1(p_1)f_2(p_2)\frac{\gamma_R\beta_R}{\gamma_1\gamma_2}\sigma(\beta_R)du\,dp_1\,dp_2\,, \tag{6}$$

where $u=\cos\theta = (\vec{\beta}_1\cdot\vec{\beta}_2)/\beta_1\beta_2$ and $f_i(p_i) = 4\pi p_i^2 f_i(\vec{p}_i)$ so that $\int_0^\infty f_i(p_i)\,dp_i = 1$.

Utilizing relation (4) to eliminate u in favor of $p_R = \gamma_R\beta_R m_2 c$, we obtain R_{12} in the convenient form

$$R_{12} = \frac{(n_1n_2)_{\text{obs}}}{1+\delta_{12}}\int_0^\infty \beta_R c\,\sigma(\beta_R)F(p_R)\,dp_R\,, \tag{7}$$

where

$$F(p_R) = \frac{\gamma_R\beta_R}{2m_2c} \times\int_0^\infty \frac{f_1(p_1)}{\beta_1\gamma_1^2}\left[\int_{\gamma_1\gamma_R|\beta_1-\beta_R|m_2c}^{\gamma_1\gamma_R(\beta_1+\beta_R)m_2c}\frac{f_2(p_2)}{\beta_2\gamma_2^2}\,dp_2\right]dp_1\,. \tag{8}$$

To further simplify the expression for $F(p_R)$, it is necessary to know the forms of $f_2(p_2)$ and/or $f_1(p_1)$. We first treat the case where $f_1(p_1)$ is arbitrary, while $f_2(p_2)$ is a relativistic Maxwell-Boltzmann distribution (RMB) given by

$$f_2(p_2) = \frac{p_2^2\exp[-(p_2^2c^2+m_2^2c^4)^{1/2}/kT_2]}{m_2^2ckT_2K_2(m_2c^2/kT_2)} \tag{9}$$

where T_2 is the temperature of the type-2 particles, K_2 is the second-order modified Bessel function of the second kind, and k is Boltzmann's constant. We then find

$$F(p_R)\big|_{\text{Arb. }f_1,\text{RMB }f_2} = \frac{\gamma_R\beta_R}{2m_2cK_2(m_2c^2/kT_2)} \times\int_0^\infty\frac{f_1(p_1)}{\beta_1\gamma_1^2}\left[\exp\left(-[\gamma_1^2\gamma_R^2(\beta_1-\beta_R)^2+1]^{1/2}\frac{m_2c^2}{kT_2}\right)-\exp\left(-[\gamma_1^2\gamma_R^2(\beta_1+\beta_R)^2+1)]^{1/2}\frac{m_2c^2}{kT_2}\right)\right]dp_1\,. \tag{10}$$

In the case where $f_1(p_1)$ is also an RMB, Eq. (10) becomes

$$F(p_R)\big|_{\text{RMB }f_1,\text{RMB }f_2} = \frac{\gamma_R\beta_R\phi_1}{2m_2cK_2(\phi_1)K_2(\phi_2)}\int_0^\infty\beta_1\Big[\exp\left(-[\gamma_1^2\gamma_R^2(\beta_1-\beta_R)^2+1]^{1/2}\phi_2-\gamma_1\phi_1\right) - \exp\left(-[\gamma_1^2\gamma_R^2(\beta_1+\beta_R)^2+1]^{1/2}\phi_2-\gamma_1\phi_1\right)\Big]d(\gamma_1\beta_1)\,, \tag{11a}$$

where $\phi_i = m_ic^2/kT_i$. To evaluate the integral I in this expression, we make the hyperbolic substitutions:

$$\gamma_1=\cosh\theta_1,\quad \gamma_R=\cosh\theta_R,\quad \beta_1=\tanh\theta_1,\quad \beta_R=\tanh\theta_R,$$

yielding

$$I=\int_0^\infty\{\exp[-\phi_2\cosh(\theta_1-\theta_R)-\phi_1\cosh\theta_1]-\exp[-\phi_2\cosh(\theta_1+\theta_R)-\phi_1\cosh\theta_1]\}\sinh\theta_1\,d\theta_1\,.$$

Now,

$$\phi_2\cosh(\theta_1\pm\theta_R)+\phi_1\cosh\theta_1 = z\cosh(\theta_1\pm\psi)$$

where

$$z=(\phi_1^2+2\phi_1\phi_2\gamma_R+\phi_2^2)^{1/2}\,,$$

$$\psi=\sinh^{-1}(\phi_2\sinh\theta_R/z)\,.$$

Using this identity, I can be written in the form

$$I=\int_{-\psi}^\infty e^{-z\cosh\omega}\sinh(\omega+\psi)d\omega - \int_\psi^\infty e^{-z\cosh\omega}\sinh(\omega-\psi)d\omega\,,$$

where we have made the substitution $\omega = \theta_1 - \psi$ in the first integral and $\omega = \theta_1 + \psi$ in the second. Expanding the $\sinh(\omega \pm \psi)$ expressions, and using the symmetry of the resulting terms, the integral reduces to

$$I = 2\sinh\psi \int_0^\infty e^{-z\cosh\omega}\cosh\omega\, d\omega = 2\sinh\psi K_1(z),$$

where the integral definition[7] of $K_1(z)$, the first-order modified Bessel function of the second kind, has been used.

We thus find

$$F(p_R)\big|_{\mathrm{RMB}\, f_1,\, \mathrm{RMB}\, f_2} = \frac{\gamma_R^2\beta_R^2\phi_1\phi_2 K_1(z)}{m_2 c K_2(\phi_1)K_2(\phi_2) z} \tag{11b}$$

and

$$R_{12}\big|_{\mathrm{RMB}\, f_1,\, \mathrm{RMB}\, f_2} = \frac{(n_1 n_2)_{\mathrm{obs}}}{1+\delta_{12}} \frac{\phi_1\phi_2 c}{K_2(\phi_1)K_2(\phi_2)} \times \int_0^\infty \frac{\sigma(\chi)\chi^3 K_1(z)\, d\chi}{z(1+\chi^2)^{1/2}}, \tag{12a}$$

where $\chi \equiv \gamma_R\beta_R$.

In the case when $T_1 = T_2 = T$, we note kTz is just the center of mass energy of the two interacting particles. In the limit when all energies and temperatures of interest are non-relativistic, (12a) reduces to the familiar form (cf. Clayton[8])

$$R_{12}\big|_{\mathrm{NRMB}\, f_1,\, \mathrm{NRMB}\, f_2} = \frac{n_1 n_2}{1+\delta_{12}} 4\pi \left[\frac{m_1 m_2}{2\pi k(m_1 T_2 + m_2 T_1)}\right]^{3/2} \int_0^\infty \exp\left[-\frac{m_1 m_2 v_R^2}{2k(m_1 T_2 + m_2 T_1)}\right]\sigma(v_R) v_R^3\, dv_R. \tag{12b}$$

B. Particle-photon interactions

The assumption of massive reactants made in the previous section can easily be relaxed in the case when $m_2 = 0$. [For notational convenience in considering this case, we shall discuss photons (i.e., $2 \to \gamma$) though no loss of generality is implied.] The analogous expression to (4) is then

$$p_\gamma \gamma_1 (1 - \beta_1 \cos\theta) = p_\gamma', \text{ an invariant}, \tag{13}$$

where p_γ' is the momentum of the photon in the rest frame of particle 1, and $\cos\theta = (\vec\beta_1 \cdot \vec\beta_\gamma)/\beta_1\beta_\gamma = u$ where $\vec\beta_1$ and $\vec\beta_\gamma$ are measured in the observer frame. Expressions (1), (2), and (3) are changed only in that $2 \to \gamma$, $v_R \to c$, $\sigma(v_R) \to \sigma(p_\gamma')$, and $\delta_{12} = 0$, and so for general distributions, we find

$$R_{1\gamma} = (n_1 n_\gamma)_{\mathrm{obs}}\, c \int\int f_1(\vec{p}_1) f_\gamma(\vec{p}_\gamma) \frac{p_\gamma'}{\gamma_1 p_\gamma} \times \sigma(p_\gamma')\, d\vec{p}_1\, d\vec{p}_\gamma, \tag{14}$$

and for isotropic distributions we find

$$R_{1\gamma} = \frac{(n_1 n_\gamma)_{\mathrm{obs}}\, c}{2} \int_0^\infty \int_0^\infty \int_{-1}^1 f_1(p_1) f_\gamma(p_\gamma) \frac{p_\gamma'}{\gamma_1 p_\gamma} \times \sigma(p_\gamma')\, du\, dp_1\, dp_\gamma \tag{15}$$

or

$$R_{1\gamma} = (n_1 n_\gamma)_{\mathrm{obs}} \int_0^\infty c\sigma(p_\gamma') F_\gamma(p_\gamma')\, dp_\gamma', \tag{16}$$

where

$$F_\gamma(p_\gamma') = \frac{p_\gamma'}{2} \int_0^\infty \frac{f_1(p_1)}{\beta_1 \gamma_1^2} \left(\int_{\gamma_1 p_\gamma'(1-\beta_1)}^{\gamma_1 p_\gamma'(1+\beta_1)} \frac{f_\gamma(p_\gamma)}{p_\gamma^2}\, dp_\gamma \right) dp_1 = \frac{p_\gamma'}{2} \int_0^\infty \frac{f_\gamma(p_\gamma)}{p_\gamma^2} \times \left(\int_{|p_\gamma'/p_\gamma - p_\gamma/p_\gamma'| m_1 c/2}^\infty \frac{f_1(p_1)}{\beta_1\gamma_1^2}\, dp_1 \right) dp_\gamma. \tag{17}$$

We now take $f_\gamma(p_\gamma)$ to be a relativistic Bose-Einstein distribution (RBE) given by

$$f_\gamma(p_\gamma) = \frac{8\pi}{h^3 n_\gamma} \frac{p_\gamma^2}{C e^{p_\gamma c/kT_\gamma} - 1}, \tag{18}$$

where h is Planck's constant, T_γ is the photon temperature, and $C \equiv \exp(-\mu_\gamma/kT)$, where μ_γ is the photon chemical potential, is a dimensionless degeneracy parameter which is equal to 1 for a blackbody distribution and goes to infinity in the nondegenerate limit. n_γ and C are related by

$$n_\gamma = \frac{16\pi(kT_\gamma)^3}{c^3 h^3} \sum_{n=1}^\infty \frac{1}{n^3 C^n} = \begin{cases} \zeta(3) n_\gamma^0, & C = 1 \\ n_\gamma^0/C, & C \to \infty \end{cases} \tag{19}$$

where

$$\zeta(3) = \sum_{n=1}^\infty \frac{1}{n^3} = 1.2021\cdots \text{ and } n_\gamma^0 = \frac{16\pi(kT_\gamma)^3}{c^3 h^3}.$$

For this case, we find

$$F_\gamma(p_\gamma')\big|_{\mathrm{Arb}\, f_1,\, \mathrm{RBE}\, f_\gamma} = \frac{n_\gamma^0}{2n_\gamma} \frac{p_\gamma' c^2}{(kT_\gamma)^2} \sum_{n=1}^\infty \frac{1}{nC^n} \int_0^\infty \frac{f_1(p_1)}{\beta_1\gamma_1^2} e^{-\gamma_1 p_\gamma' nc/kT_\gamma} \sinh\left(\frac{\gamma_1 p_\gamma' \beta_1 nc}{kT_\gamma}\right) dp_1. \tag{20}$$

For the case when f_1 is an RMB, while f_γ is an arbitrary isotropic distribution, we find

$$F_\gamma(p'_\gamma)\,|_{\mathrm{RMB}f_1,\,\mathrm{Arb}f\gamma} = \frac{p'_\gamma}{2K_2(\phi_1)} \int_0^\infty \frac{f(p_\gamma)}{p_\gamma^2} \exp\left[-\frac{1}{2}\left(\frac{p_\gamma}{p'_\gamma}+\frac{p'_\gamma}{p_\gamma}\right)\phi_1\right] dp_\gamma . \tag{21}$$

Finally, when f_1 is an RMB and f_γ is an RBE, we find $R_{1\gamma}$ to be

$$R_{1\gamma}\,|_{\mathrm{RMB}f_1,\,\mathrm{RBE}f\gamma} = \frac{(n_1 n_\gamma^0)_{\mathrm{obs}}}{2} \frac{c\phi_1}{K_2(\phi_1)} \sum_{n=1}^\infty \frac{1}{C^n} \int_0^\infty \frac{\zeta^2\sigma(\zeta)K_1(z'_n)}{z'_n}\, d\zeta , \tag{22}$$

where

$$\zeta = p'_\gamma c/kT_\gamma \text{ and } z'_n = (\phi_1^2 + 2n\phi_1\zeta)^{1/2},$$

and we have used integral 3.324 # 1 of Gradshteyn and Ryzhik.[9]

The sum involved here is rapidly convergent, and vanishes in the nondegenerate limit.

In the limit when the energies and temperatures of interest are nonrelativistic (except in the case of the inherently relativistic photon where we require $p'_\gamma \ll m_1 c$), (22) reduces as required to

$$R_{1\gamma}|_{\mathrm{NRMB}f_1,\,\mathrm{RBE}\,f\gamma} = n_1 c\,\frac{8\pi}{h^3} \int_0^\infty \frac{p'^2_\gamma\,\sigma(p'_\gamma)\,dp'_\gamma}{Ce^{p'_\gamma c/kT_\gamma} - 1} . \tag{23}$$

In the nondegenerate case, expressions (13)–(23) can also be obtained as the limits of the equivalent massive-particle forms when $\gamma_R \to p'_\gamma c/m_2 c^2$ and $m_2 \to 0$.

C. Photon-photon reactions

The above formalism is still not in a suitable form to treat reactions between massless particles, since it is impossible to transform to a massless particle's rest frame. It is therefore necessary to generalize the definition of the cross section. This could be done by directly assuming that the definition in (1) holds in a general reference frame[6], but this results in a generalized cross section that is not Lorentz invariant. It is more conventional[10] to define the cross section by the invariant relation

$$R_{12} \equiv \frac{n_1 n_2 (1-\vec{\beta}_1\cdot\vec{\beta}_2)[(\mathcal{P}_{1\mu}\mathcal{P}_2^\mu)^2 - m_1^2 m_2^2 c^4]^{1/2}}{\mathcal{P}_{1\mu}\mathcal{P}_2^\mu}\, c\sigma \tag{24a}$$

$$= n_1 n_2 \left((1-\vec{\beta}_1\cdot\vec{\beta}_2)^2 - \frac{m_1^2 m_2^2 c^8}{E_1^2 E_2^2}\right)^{1/2} c\sigma \tag{24b}$$

$$= \begin{cases} n_1 n_2 (1-\vec{\beta}_1\cdot\vec{\beta}_2)\beta_R c\sigma & \text{for } m_1, m_2 \neq 0 \quad (24c) \\ n_1 n_2 (1-\vec{\beta}_1\cdot\vec{\beta}_2) c\sigma & \text{for } m_1 \text{ or } m_2 = 0 \quad (24d) \end{cases}$$

where $\mathcal{P}_{1\mu}\mathcal{P}_2^\mu$ is the scalar product of the four-momenta of particles of type 1 and 2 (assumed to be in beams) and E_i is the *total* energy of particle i. Equation (24c) can be interpreted as defining the cross section as the ratio of the reaction rate to the proper particle flux, and reduces to (1) in the rest frame of one of the particles; while (24d) reduces to the natural result, $R = n_1 n_2 2c\sigma$, in the center-of-momentum frame for two massless particles.

The case of reactions between distributions of massless particles can now be treated in a fashion analogous to the massive-particle case, except that a transformation to the center-of-momentum frame is used. The invariants analogous to (3) and (4) are then

$$n_{\gamma 1} n_{\gamma 2}(1-\cos\theta) = 2n_{\gamma 1}^* n_{\gamma 2}^* \tag{25}$$

and

$$p_{\gamma 1} p_{\gamma 2}(1-\cos\theta) = 2p_{\gamma 1}^{*2} = 2p_{\gamma 2}^{*2} \equiv 2p_\gamma^{*2} \tag{26}$$

where $p_{\gamma 1}$ and $p_{\gamma 2}$ are the magnitudes of the momenta of two specific photon groups and θ is the angle between them, while $n_{\gamma i}$ is the photon number density of group i. The superscript asterisk denotes measurement in the center-of-momentum frame. Here again the trivial but convenient specialization to the photon case has been made.

The reaction rate for general distributions can then be written:

$$R_{\gamma\gamma} = (n_\gamma^2)_{\mathrm{obs}}\, c \int\!\!\int f(\vec{p}_{\gamma 1}) f(\vec{p}_{\gamma 2}) \frac{p_\gamma^{*2}}{p_{\gamma 1} p_{\gamma 2}}\, \sigma(p_\gamma^*)\, d\vec{p}_{\gamma 1}\, d\vec{p}_{\gamma 2} , \tag{27}$$

and for isotropic distributions, we find

$$R_{\gamma\gamma} = (n_\gamma^2)_{\mathrm{obs}}\, c \int_0^\infty \sigma(p_\gamma^*) F_\gamma^*(p_\gamma^*)\, dp_\gamma^* , \tag{28}$$

where

$$F_\gamma^*(p_\gamma^*) = 2p_\gamma^{*3} \int_0^\infty \frac{f(p_{\gamma 1})}{p_{\gamma 1}^2} \left(\int_{p_\gamma^{*2}/p_{\gamma 1}}^\infty \frac{f(p_{\gamma 2})}{p_{\gamma 2}^2}\, dp_{\gamma 2} \right) dp_{\gamma 1} . \tag{29}$$

Specializing to a relativistic Bose-Einstein photon distribution, we find

$$R_{\gamma\gamma}|_{\mathrm{RBE}} = (n_\gamma^0)^2 c \sum_{n=1}^{\infty} \sum_{l=1}^{\infty} \frac{1}{\sqrt{nl}\, C^{n+l}} \times \int_0^{\infty} \sigma(\xi)\xi^4 K_1(2\sqrt{nl}\,\xi)\, d\xi \quad (30)$$

where $\xi = p_\gamma^* c/kT_\gamma$ and, as before, the sums are generally rapidly convergent.

In this case also, the nondegenerate form of (30) can be obtained as the limit of (12a) if a transform from lab to center-of-momentum coordinates is made and the invariance of the cross section is used.

D. Numerical considerations

In applying the above theory to numerically calculating specific reaction rates, it is convenient to calculate the quantity

$$\langle\sigma v\rangle \equiv (1+\delta_{12}) R_{12}/n_1 n_2 \quad (31)$$

Power-series fits to the required K Bessel functions are given by Abramowitz and Stegun.[11] The integrands in the rate integrals are often quite sharply peaked, and the use of an integration method where the increment size is adaptively determined[12] is helpful.

III. APPLICATION TO REACTION RATES OF ASTROPHYSICAL INTEREST

A. Radiative Compton scattering

The differential cross section for radiative Compton scattering was calculated by Mandl and Skyrme,[13] and has been numerically integrated by Ram and Wang[14] to give the total cross section for the emission of a photon with an energy $h\nu$ greater than 5 keV.

For the case of an RBE photon distribution and an RMB electron distribution at the same temperature T, the total photon emission rate ($h\nu > 5$ keV) can readily be found from Eq. (22), and the corresponding $\langle\sigma v\rangle$ is plotted in Fig. 1.

It is evident that the process only becomes important for $kT \gtrsim 0.1 m_e c^2$ where the photons carry enough momentum to significantly accelerate an electron in a Compton collision. In the regime $0.1 m_e c^2 \lesssim kT \lesssim 10 m_e c^2$, however, the $\langle\sigma v\rangle$ for radiative Compton scattering becomes comparable to that for bremsstrahlung. The small degeneracy effects apparent in Fig. 1 are primarily due to the augmentation of the low-energy tail of the blackbody distribution ($C = 1$) relative to the nondegenerate distribution ($C \to \infty$) and the peak in the total cross section near $m_e c^2$.

The role of this reaction in accelerating radiative equilibrium in supernova shock waves is treated in Refs. 1 and 2.

B. Photon-photon pair creation

Jauch and Rohrlich[10] give the relativistically correct γ-γ pair-production cross section in the form

$$\sigma = \pi r_0^2 \Phi^2 [(2 + 2\Phi^2 - \Phi^4)\cosh^{-1}(1/\Phi) - (1+\Phi^2)(1-\Phi^2)^{1/2}], \quad \Phi > 1 \quad (32)$$

where $\Phi = m_e c^2/\epsilon_\gamma$, ϵ_γ is the center-of-momentum energy of one of the photons, r_0 is the classical electron radius, and σ is meant in the Lorentz-invariant sense of Eq. (24). The density normal-

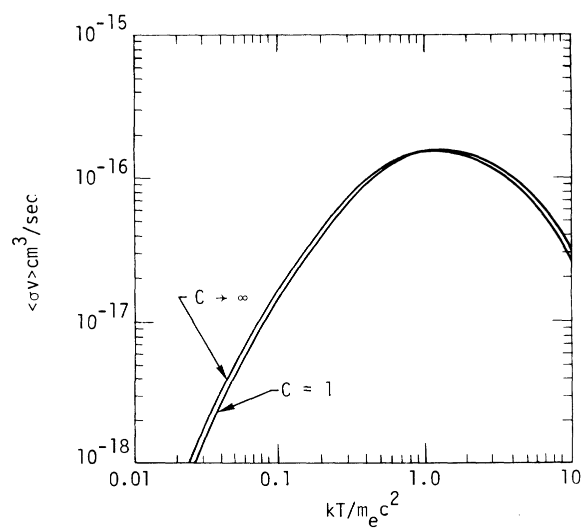

FIG. 1. Radiative Compton scattering emission rate ($h\nu > 5$ keV).

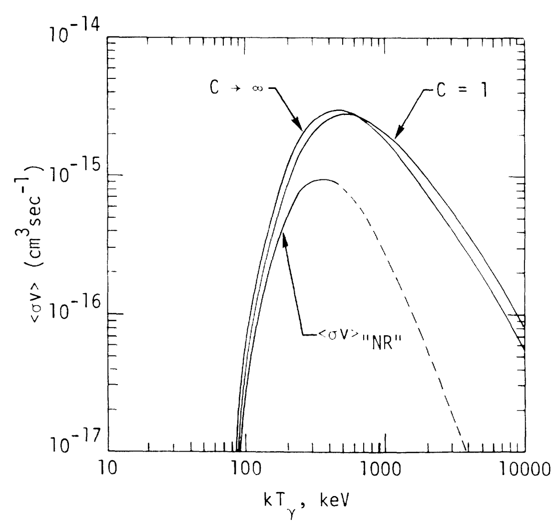

FIG. 2. Rate of the $\gamma\gamma \to e^+e^-$ reaction.

ized emission rates for a nondegenerate RBE and blackbody photon distribution have been calculated from Eq. (30) and are plotted in Fig. 2, along with a "nonrelativistic" expression for this rate, obtained by assuming $kT_\gamma \ll m_e c^2$ in (30) and $1-\Phi \ll 1$ in (32) to find

$$\langle \sigma v \rangle_{\text{"NR"}} = \frac{\pi^2 r_0^2}{4} \left(\frac{m_e c^2}{kT_\gamma} \right) e^{-2m_e c^2/kT_\gamma} . \tag{33}$$

It is apparent that higher-order relativistic effects are important even at low temperatures, owing to the requirement that $\epsilon_\gamma > m_e c^2$ for a pair to be formed. The degeneracy effects are principally due to the relative depopulation of high-energy photons in the blackbody distribution by a factor $\simeq \zeta(3)$ with respect to a nondegenerate distribution, and the peak in the cross section near 1 MeV photon energies. For $10 < kT < 100$ keV, the nondegenerate reaction rate $R^{\pm}_{\gamma\gamma}$ can be fit to within 2% by the expression

$$R^{\pm}_{\gamma\gamma} \simeq 1.042 \left[1 + 0.728 \left(\frac{kT_\gamma}{m_e c^2} \right) \right]^{7/2} \frac{n_\gamma^2 c}{2} \langle \sigma v \rangle_{\text{"NR"}} \tag{34}$$

while the blackbody rate is smaller by a factor $\simeq [\zeta(3)]^2$.

References 1 and 2 discuss an astrophysical application of this reaction.

C. Proton-proton pion production

The experimentally measured cross sections for pion production in proton-proton collisions are given by Lock and Measday,[15] along with Mandelstam theory[16] fits to cross sections near threshold for the case of single π^0 or π^+ production.

The rates for reactions giving rise to one or more pions ("total"), single π^+ production ($pp \to pn\pi^+$), and single π^0 production ($pp \to pp\pi^0$) have been calculated using Eq. (12a) and are given in Fig. 3. The total rate is only accurate for $kT > 50$ MeV because of the lack of a detailed specification of the threshold dependence of the total pion cross section.

The importance of π^0 mesons in producing γ-ray emission from matter accreting onto a black hole is discussed in Ref. 3. In addition, these rates are of interest in treating pion production resulting from shock waves in nuclear matter[17,18] which may occur in relativistic heavy-ion collisions.

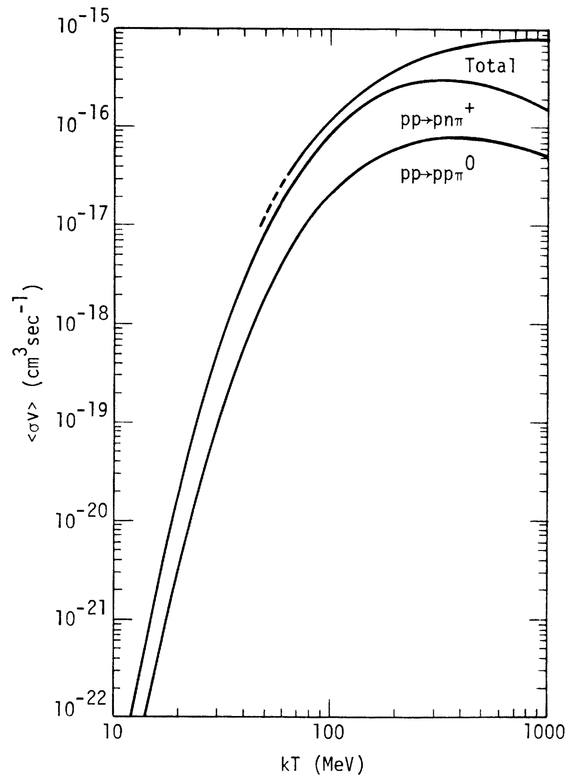

FIG. 3. Rates for pion-producing reactions in proton-proton collisions. The curve labeled "Total" is the sum of the rates for all pion-producing reactions in such collisions.

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