Exact and approximate differential renormalization-group generators*

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In dealing with critical phenomena of complex systems that simulate realistic materials, the full structure of the renormalization group is often unnecessarily cumbersome. For approximate calculations and for systems with special properties, specialized generators are simpler to apply. We derive several such exact and approximate differential generators and solve a number of interesting practical problems to illustrate this approach: (i) We derive a new approximate differential generator based on the Wilson incomplete-integration generator. Using this generator we calculate for an n-component spin system the eigenvalues (critical-point exponents) associated with a critical point of arbitrary order $\mathcal{O}$ and "propagator exponent" $\tilde{\sigma}$ to first order in the expansion parameter $\epsilon_{\mathcal{O}}(\tilde{\sigma}) = d + \mathcal{O}(\tilde{\sigma} - d)$; this extends previous work for $\tilde{\sigma}=2$, $\mathcal{O}$ arbitrary; $\tilde{\sigma} \leq 2$, $\mathcal{O}=2$ (long-rang forces); and $\tilde{\sigma}=4$, $\mathcal{O}=2$ (the "Lifshitz point"). Our results agree with those obtainable using an approximate generator based on the Wegner-Houghton equation. The cases $\tilde{\sigma}=2L$ (L a positive integer ≥ 2) describe the onset of helical ordering for which $|\vec{k}| \sim (-p)^{\beta_k}$, where $\beta_k = 1/2(L-1) + o(\epsilon_{\mathcal{O}}^2(2L))$ and p parametrizes the hypersurface of critical points. For $p>0$, the ordered phase is uniform; for $p<0$ there is spiral order. The point $p=0$, at which such nonuniform ordering commences, we term a generalized Lifshitz point of Lifschitz character L. (ii) We consider the full Wilson and Wegner-Houghton generators in the paired spin-momenta limit and the $n \to \infty$ limit for even-order critical points. These limiting generators are identical for both full generators. This demonstrates that at least in these cases the Wilson and Wegner-Houghton generators agree exactly, without recourse to perturbation theory. These simple exact generators should provide "anchors" for calculations of exponents for higher-order critical points. (iii) We derive approximate generators which are suitable for compressible magnetic systems and more general systems with constraints for which the spin momenta are grouped in any arbitrary manner. We apply this to the case of a simple compressible magnet model and obtain the exact renormalization-group trajectories to order ϵ with $\epsilon \equiv \epsilon_2(2) = 4 - d$.

I. INTRODUCTION

The use of differential generators for the renormalization group[1,2] has several advantages over finite or recursive formulations: (i) In a recursive renormalization-group approach, the recursion relations contain the renormalization factor b explicitly. The eventually-calculated critical-point exponents are, of course, found to be independent of b; it therefore represents an unnecessary complication that is avoided with a differential generator. (ii) The differential equations obtained from a differential generator are, in general, far simpler in form than the corresponding recursion relations. This is the case because the recursion relations must exhibit all the feedback that results from the finite amount of renormalization. (iii) Differential equations are amenable to more analytic solution techniques than recursion relations. This is particularly true of the nonlinear study of renormalization-group equations.[3-6]

For many renormalization-group studies the full structure of the exact renormalization-group equations is not needed. The location and stability analysis of fixed points can be carried out to lowest order in perturbation theory with an approximate renormalization group. Studies of anisotropic systems,[7] metamagnets,[8] spin-flop and displacive transitions,[9] tricritical points,[10] critical points of arbitrary order,[11-13] coupled order parameters,[4,14] and nonlinear effects,[3-6] to name a few, can be made by means of approximations to the renormalization group. (Detailed calculations of higher-order corrections to critical-point exponents such as the calculation of[15,16] η may require the full exact equations.) In many cases, the essential physics is obtained in the lowest-order expansion. Many of the above results were obtained with the approximate recursion formula of Wilson[2,17]; they all can be discussed with an approximate differential generator. In addition, we will demonstrate (Secs. IV–VI) that exact differential generators of form much simpler than the full generator can be utilized when special or limiting features of the system Hamiltonian are incorporated into the formulation at the outset.

At present, there are two exact differential generators for the full renormalization group. The Wegner-Houghton generator represents the differential limit of the recursive formulation of Wilson.[1]

It gives the differential change of the Hamiltonian in terms of an integral over an infinitesimal shell of spin-fluctuation wave vectors (or momenta). The Wilson differential generator[1] represents an incomplete integration in which large wave vectors are more completely integrated than small wave vectors. To compare and contrast these generators, we reformulate the Wilson generator in such a way that it more closely resembles the Wegner-Houghton generator. We may then solve a large class of problems to show agreement between the two generators. For some of the problems we utilize *approximate* generators derived from the full generators and show agreement to first order in perturbation theory. For other problems we consider specific *exact* limiting forms of the two full generators and show they are identical to all orders.

Recently, we introduced[13] an approximate differential generator based on the full Wegner-Houghton generator. We wrote it in a form suitable for isotropically interacting systems; it is easily generalized to describe anisotropic systems with propagator exponent $\tilde{\sigma}$ (see discussion below),

$$\frac{\partial H}{\partial l} = dH + \frac{(\tilde{\sigma}-d)}{2}\vec{s}\cdot\vec{\nabla}H + \mathrm{tr}\ln\left(\delta_{\alpha\beta} + \frac{\partial^2 H}{\partial s^\alpha \partial s^\beta}\right), \qquad (1.1)$$

where $H(\vec{s}, l)$ is a function of an n-component spin vector $\vec{s}$ on a d-dimensional lattice, and l is the renormalization parameter. For the case of long-range forces,[18] with interaction strength $\propto 1/r^{d+\sigma}$, $\tilde{\sigma}=\sigma$ for $\sigma \leq 2$, and $\tilde{\sigma}=2$ otherwise. In the case of the Lifshitz point[19] $\tilde{\sigma}=4$. We generalize such Lifshitz systems to those at which $\tilde{\sigma}$ is an even integer, $\tilde{\sigma}=2L$; the imposition of a long-range force on such systems allows any $\tilde{\sigma} \leq 2L$.[20]

A similar approximate generator is derived in Sec. III from the Wilson incomplete integration generator. The result is

$$\frac{\partial H}{\partial l} = dH + \frac{(\tilde{\sigma}-d)}{2}\vec{s}\cdot\vec{\nabla}H + \nabla^2 H - \vec{\nabla}H\cdot\vec{\nabla}H\ . \qquad (1.2)$$

The two generators have the same linear structure, but very different nonlinear structures; for example, (1.2) lacks the propagator factors of $(1+r)^{-1}$ characteristic of (1.1).

In Sec. II, we introduce an $\epsilon_{\mathcal{O}}(\tilde{\sigma})$ expansion for critical points of order $\mathcal{O}$ ($\mathcal{O}$ phases simultaneously critical) with propagator exponent $\tilde{\sigma}$. This generalizes previous work for arbitrary order $\mathcal{O}$ with $\tilde{\sigma}=2$,[13] $\mathcal{O}=2$ with $\tilde{\sigma}\leq 2$,[18] and $\mathcal{O}=2$, $\tilde{\sigma}=4$.[19] (Cf. Fig. 1.) These calculations, which are exact to order $\epsilon_{\mathcal{O}}(\tilde{\sigma})$, agree for generators (1.1) and (1.2). We derive a generating function for the eigenvalue corrections which gives each correction as a sum of at most $[\mathcal{O}/2]$ positive terms ($n>0$). Explicit, single-term expressions are given for $n=1$ and $n=\infty$.

One finds by the method of Ref. 16 that for $\tilde{\sigma}\neq 2L$, $\eta_{\mathcal{O}}$ "sticks" to the value $\eta_{\mathcal{O}}=2L-\tilde{\sigma}$, for all $\mathcal{O}$ to order $\epsilon_{\mathcal{O}}^2(\tilde{\sigma})$, in agreement with the $\mathcal{O}=2$ result of Ref. 18.[20]

In Sec. III, we derive the approximate Wilson generator from a reformulation of the exact Wilson generator. This reformulation simplifies the renormalization-group equation when expanded around the Gaussian fixed-point solution. This facilitates the comparison with the Wegner-Houghton generator and is also the starting point of the exact calculations of Sec. IV.

In Sec. IV, we use the full generators to discuss the $n\to\infty$ limit of critical points for $\mathcal{O}$ even. We show that, in this limit, the spin fluctuations occur only in the paired momentum form: $\int s_{\vec{k}}s_{-\vec{k}} = \int s^2(x)$ [in Feynman-diagram terms, all diagrams are tree diagrams (cf. Sec. IV)]. The generators, although extremely dissimilar for finite n, become identical for $n=\infty$ and are equal to the common limit of the approximate generators. We also show that the full generators become identical [and equal to the appropriate limit of the approximate generators (cf. Sec. V)] if we make the weaker assumption that the spin momenta are paired. This limiting generator is, of course, identical to the $n=\infty$ generator for isotropically interacting systems.

In Sec. V, we generalize the approximate gener-

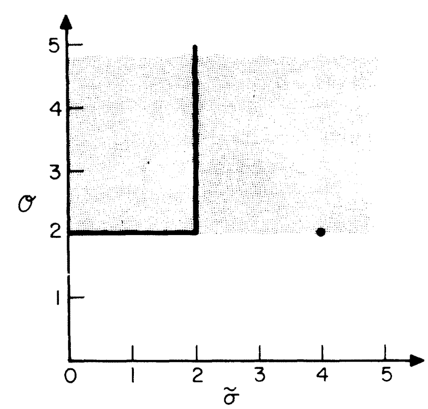

FIG. 1. Schematic plot of order $\mathcal{O}$ (defined in Sec. I) vs propagator exponent $\tilde{\sigma}$ (where the critical propagator varies with momentum as $|\vec{k}|^{\tilde{\sigma}}$). For the case of long-range forces with interaction strength $1/r^{d+\sigma}$, $\tilde{\sigma}=\sigma$ for $\sigma\leq 2$, and $\tilde{\sigma}=2$ otherwise; the case $\tilde{\sigma}=2L$ (L a positive integer ≥ 2) corresponds to a "generalized Lifshitz point." The heavy lines and solid circle correspond to previously treated special cases: (a) The vertical line indicates case $\tilde{\sigma}=2$ and $\mathcal{O}$ arbitrary (Ref. 13); (b) the horizontal line indicates the case $\mathcal{O}=2$, $\tilde{\sigma}\leq 2$ (Ref. 18); and (c) the heavy circle indicates the Lifschitz point $\tilde{\sigma}=4$ (Ref. 19). In Sec. II this previous work is extended to all meaningful values of both $\tilde{\sigma}$ and $\mathcal{O}$ (shown shaded).

ators (1.1) and (1.2) to render them applicable to systems with arbitrary groupings of spin momenta. This allows the generators to be useful, for example, in the study of compressible ferromagnets.[21,22]

In Sec. VI, we solve exactly to order ϵ_2, $\epsilon_2 \equiv 4 - d$, the nonlinear renormalization-group trajectories for a compressible ferromagnet.[21]

II. $\epsilon_{\Theta}(\tilde{\sigma})$ EXPANSION

Recently,[23] it has been shown that the full Wilson differential generator reproduces the earlier results in the $4-d$ expansion,[15] at least to lowest order. Subsequently, several authors[24-26] have shown that at least the correlation-function exponent η_2 ($\tilde{\sigma}=2$) is correctly given to second order with this generator [$\eta_2=(4-d)^2/54$ for Ising $n=1$ systems]. With the approximate generator (1.2), we will show agreement between the Wilson and other renormalization-group formulations for a wide class of systems.

In this section we will calculate the critical eigenvalues for a special class of critical points of higher order. Critical points of order Θ ($\Theta \geq 4$) were initially proposed for systems with symmetrically (or symmetrizable) competing order parameters.[27] Because of the symmetrical nature of the competing interactions, such types of critical behavior (cf. Ref. 9) are more easily achieved in the laboratory. Global features of such types of higher-order critical points have been considered in detail in Ref. 4.

Critical points of higher order can also be realized in systems representable by a single order parameter. At such a critical point of order Θ, the mean-field values of the scaling powers[28] of the Gibbs potential are given by $(2\Theta - p)/2\Theta$, with $p = 1, \ldots, 2\Theta - 2$.[12,29] Using renormalization-group recursion relations, it was suggested (and shown for $\Theta = 3, 4$) that below the borderline dimension $d_b \equiv 2\Theta/(\Theta-1)$, the scaling powers will deviate from the classical mean-field values.[12,30] The nonclassical corrections in terms of the general ϵ_Θ [$\equiv d + \Theta(2-d)$] expansion were calculated to first order[13] using Gaussian eigenoperator expansions, while the correlation function exponents η_Θ were calculated to order ϵ_Θ^2.[16]

In this work, we will calculate the generalized $\epsilon_\Theta(\tilde{\sigma})[\equiv d + \Theta(\tilde{\sigma} - d)]$ expansions for critical points of order Θ, whose Hamiltonian density in Fourier space is of the form $(\cdots + |k|^{\tilde{\sigma}} + \cdots + r)s_{\vec{k}} s_{-\vec{k}} + \cdots$; we choose our spin rescaling factor in such a way as to keep the coefficient of $|k|^{\tilde{\sigma}}$ constant. For $\tilde{\sigma} < 2$, these represent the generalizations to arbitrary order Θ of Ref. 18 for systems with long-range interactions ($\sim r^{-(d+\sigma)}$). For $\tilde{\sigma} = 2L$ (L a positive integer), these are the Θth-order critical eigenvalues for systems exhibiting a "Lifshitz point of character L."

We consider a Hamiltonian with momentum dependence in the term quadratic in the spins,

$$\mathcal{H} = \sum_{\vec{k}} \left(\sum_{j=1} A_{2j} |\vec{k}|^{2j} \right) \vec{s}_{\vec{k}} \cdot \vec{s}_{-\vec{k}} + \cdots . \tag{2.1}$$

We can obtain, in principle, many coexisting phases with different helicity vectors $\vec{k}$ (that is, each ordered phase is a "frozen" spin wave of wave vector $\vec{k}$). As in the usual Landau expansion for the order parameter, the character of the helicity at a critical point is determined by the lowest-order positive term in the wave-vector expansion. Thus, if $A_2 > 0$, we can ignore the k^4 and higher terms. The $L=2$ Lifshitz point[19] is precisely that point at which $A_2 = 0$. In general, there is a complicated competition among L possible helicity states if $A_2 = A_4 = \cdots = A_{2L-2} = 0$, $A_{2L} > 0$. By including a long-range interaction, we have additional competition between the k^σ term and the k^{2L} terms, with $\tilde{\sigma} \equiv \min(\sigma, 2L)$. For $L > 1$, a detailed renormalization-group treatment of (2.1) shows that the renormalized values of the A_{2j} ($j \leq L-1$) cannot generally be kept equal to zero.[20] This nonlinear competition occurs, however, only at order $\epsilon_\Theta^2(\tilde{\sigma})$. In the field-theoretic counterpart to this discussion, this apparent difficulty does not arise. The form of the propagator is preserved by the use of the appropriate counterterms.

We will evaluate the eigenvalues analytically to order $\epsilon_\Theta(\tilde{\sigma})$ using our approximate generators (1.1) and (1.2). The procedure follows closely the eigenoperator-expansion technique developed in Ref. 13. The renormalization-group treatment of such systems[5] is closely related to the "Gaussian" eigenfunctions of the renormalization-group equation; in the case of (1.1) and (1.2), these are the eigenfunctions when the equations are linearized around $H=0$. As noted in the context of the Wilson approximate recursion formula,[7(b)] these eigenfunctions are products of generalized Laguerre polynomials and harmonic polynomials,

$$Q_{p,m} = L_p^{m-1+n/2}\left(\frac{d-\tilde{\sigma}}{4} s^2\right) P_m(\vec{s}) , \tag{2.2}$$

with eigenvalues $\lambda_{p,m} = d + (\tilde{\sigma} - d)(p + m/2)$. Thus for $d < \tilde{\sigma}$ there are an infinite number of relevant Gaussian eigenvalues. The $(2m+n-2)(m+n-3)!\,/\,m!\,(n-2)!$ $P_m(\vec{s})$ are degenerate with respect to the linearized renormalization-group equations. These eigenfunctions are also eigenfunctions of the full Wegner-Houghton and Wilson differential generators when linearized around the Gaussian functional (cf. Sec. III).

We restrict our attention to isotropic systems ($m=0$). The renormalization-group study of an order-$\mathcal{O}$ system is simplified if the eigenvalue of the Laguerre polynomials of order $\mathcal{O}$ is small. We therefore define an expansion parameter $\epsilon_{\mathcal{O}}(\tilde{\sigma})$ by

$$\epsilon_{\mathcal{O}}(\tilde{\sigma}) = \lambda_{\mathcal{O},0} = d + \mathcal{O}(\tilde{\sigma} - d) . \tag{2.3}$$

For $\mathcal{O}=2$, this is the ϵ of Ref. 18; for $\tilde{\sigma}=2$, this is the expansion parameter for higher-order critical points discussed in Refs. 12 and 13. For this case, d_b is given by the solution of $\epsilon_{\mathcal{O}}(\tilde{\sigma})=0$, or $d_b = \tilde{\sigma}\mathcal{O}/(\mathcal{O}-1)$.

We first locate the fixed-point Hamiltonian H^*. We write

$$H^* = a\epsilon_{\mathcal{O}}(\tilde{\sigma})Q_{\mathcal{O}} + \epsilon_{\mathcal{O}}^2(\tilde{\sigma})H^{(2)} + \cdots , \tag{2.4}$$

and substitute this into the fixed-point equation $\partial H^*/\partial l = 0$. To the order required we may represent both generators (1.1) and (1.2) in the form

$$\frac{\partial H}{\partial l} = \mathfrak{L}H + \mathfrak{D}_{(m)}(H,H) , \quad m=1,2 \tag{2.5}$$

where $\mathfrak{L}$ is the linear differential operator common to both generators and $\mathfrak{D}_{(1)}$ and $\mathfrak{D}_{(2)}$ are the quadratic parts of generators (1.1) and (1.2), respectively. Upon inserting (2.4) into (2.5) we find

$$0 = aQ_{\mathcal{O}} + a^2\mathfrak{D}_{(m)}(Q_{\mathcal{O}}, Q_{\mathcal{O}}) + \mathfrak{L}H^{(2)} + \cdots . \tag{2.6}$$

To determine a, we choose $H^{(2)}$ orthogonal to $Q_{\mathcal{O}}$ and take the inner product of (2.6) with $Q_{\mathcal{O}}$,

$$0 = \langle Q_{\mathcal{O}} | Q_{\mathcal{O}} \rangle + a \langle \mathfrak{D}_{(m)}(Q_{\mathcal{O}}, Q_{\mathcal{O}}) | Q_{\mathcal{O}} \rangle . \tag{2.7a}$$

It is convenient to define $[i,j;k]$ by

$$[i,j;k]_{(m)} = \langle \mathfrak{D}_{(m)}(Q_i, Q_j) | Q_k \rangle / \langle Q_k | Q_k \rangle . \tag{2.7b}$$

Then we may write (2.7a) as

$$1 = -a[\mathcal{O},\mathcal{O};\mathcal{O}]_{(m)} . \tag{2.7c}$$

We now determine the eigenvalues of the new eigenfunctions when the generators are linearized around H^*. Since the fixed-point Hamiltonian H^* differs from the Gaussian fixed point $H=0$ only slightly, we expect the eigenfunctions and eigenvalues to be changed only by order $\epsilon_{\mathcal{O}}(\tilde{\sigma})$ amounts. We set

$$\overline{Q}_p \equiv Q_p + \epsilon_{\mathcal{O}}(\tilde{\sigma})q_p \tag{2.8a}$$

and

$$\overline{\lambda}_p \equiv \lambda_p + \epsilon_{\mathcal{O}}(\tilde{\sigma})\delta\lambda_p . \tag{2.8b}$$

Inserting $H = H^* + \overline{Q}_p$ into (2.4) and linearizing, we find that

$$\delta\lambda_p Q_p = \mathfrak{L}q_p + 2a\mathfrak{D}_{(m)}(Q_{\mathcal{O}}, Q_p) + \cdots . \tag{2.9}$$

Choosing q_p orthogonal to Q_p and taking the inner product of (2.9) with Q_p gives

$$\delta\lambda_p = 2a[\mathcal{O},p;p]_{(m)} . \tag{2.10a}$$

Eliminating the quantity a, we have

$$\overline{\lambda}_p = d + p(\tilde{\sigma} - d) - 2\epsilon_{\mathcal{O}}(\tilde{\sigma})[\mathcal{O},p;p]_{(m)}/[\mathcal{O},\mathcal{O};\mathcal{O}]_{(m)} . \tag{2.10b}$$

Although the expressions which define $[\mathcal{O}, l; l]$ differ for generators (1.1) and (1.2), the ratio which appears in (2.10b) is the same. We write

$$\frac{[\mathcal{O},p;p]_{(m)}}{[\mathcal{O},\mathcal{O};\mathcal{O}]_{(m)}} = \frac{\langle \mathcal{O},p;p \rangle}{\langle \mathcal{O},\mathcal{O};\mathcal{O} \rangle} , \tag{2.11a}$$

where

$$\langle \mathcal{O},p;p \rangle = \sum_{j=0}^{[\mathcal{O}/2]} \binom{p}{j} \binom{p+n/2-1}{j} \binom{2p-2j}{\mathcal{O}-2j} . \tag{2.11b}$$

For $n=1$ and $n=\infty$, (2.11b) can be simplified to give for (2.10b)

$$\overline{\lambda}_p = d + p(\tilde{\sigma} - d) - 2\epsilon_{\mathcal{O}}(\tilde{\sigma}) \binom{2p}{\mathcal{O}} \Big/ \binom{2\mathcal{O}}{\mathcal{O}} \tag{2.12a}$$

for $n=1$, and

$$\overline{\lambda}_p = d + p(\tilde{\sigma} - d) - 2\epsilon_{\mathcal{O}}(\tilde{\sigma}) \binom{p}{[(\mathcal{O}+1)/2]} \Big/ \binom{\mathcal{O}}{[(\mathcal{O}+1)/2]} \tag{2.12b}$$

for $n=\infty$.

Unsymmetric Landau expansions without rotational symmetry for Ising systems have been utilized to describe higher-order critical behavior in fluid mixtures.[31,32] The nonclassical corrections for $a_p = \overline{\lambda}_p/d$ are given by (2.12a) with $p = \frac{1}{2}, 1, \frac{3}{2}, \ldots, 2\mathcal{O}-1$.

To derive (2.11), we first postulate that $[\mathcal{O},p;p]_{(1)} = 4(\mathcal{O}-1)[\mathcal{O},p;p]_{(2)}$. In terms of Laguerre-polynomial integrals, this reads

$$\int_0^\infty e^{-x}x^\alpha f[(\alpha+1)f'g' + x(f'g')' + 2x^2f''g'']\,dx = (\mathcal{O}-1)\int_0^\infty e^{-x}x^{\alpha+1}ff'g'\,dx , \tag{2.13a}$$

where we have set $\alpha = n/2 - 1$, $f = L_p^\alpha(x)$, and $g = L_{\mathcal{O}}^\alpha(x)$. Using that fact that $g' = L_{\mathcal{O}-1}^{\alpha+1}$ eliminates the explicit appearance of $\mathcal{O}$. Integrating by parts twice, we have

$$\int e^{-x}x^{\alpha+1}g'[f(f - f' - xf''' - (\alpha+2-x)f'') + f'(f'(\alpha+2-x) - xf'')] = 0 . \tag{2.13b}$$

The bracketed quantity vanishes for all $f = L_p^\alpha$.

Since the two generators agree, we will use the Wilson-based generator (1.2) to calculate the eigenvalue corrections. Consider the integral

$$I(\Theta,p)\equiv\int_0^\infty e^{-x}x^{\alpha+1}L_\Theta^\alpha(x)[L_p^\alpha(x)]^2\,dx\,, \qquad (2.14a)$$

We form a generating function for $I(\Theta,p)$ by defining $G_p(t)=\sum(-t)^\Theta I(\Theta,p)$. Using a generating function for Laguerre polynomials, we can write

$$G_p(t)=\int_0^\infty e^{-x}x^\alpha L_p^\alpha(\lambda x)L_p^\alpha(\lambda x)\,dx\,, \qquad (2.14b)$$

where $\lambda=1+t$. Expressing $L_p^\alpha(\lambda x)$ in terms of a sum of $L_m^\alpha(x)$, we perform the integrals to obtain

$$G_p(t)=\sum_{m=0}^{p}\binom{p}{m}\binom{p+\alpha}{m}(1+t)^{2p-2m}t^{2m}\,. \qquad (2.14c)$$

Equation (2.11b) immediately follows.

Equations (2.11) and (2.12) give the lowest-order corrections to most critical-point exponents. However, the shift of the critical-point exponent η_Θ (equivalently, the shift in the propagator exponent) cannot be calculated from the approximate generators. In Ref. 16, η_Θ is calculated by a field-theoretic technique for critical points of order Θ and $\tilde\sigma=2$. The result is

$$\eta_\Theta=4f[\epsilon_\Theta^2(2)]\Big/\binom{2\Theta}{\Theta}^3\,, \qquad (2.15a)$$

$$f=\frac{\prod_{j=1}^{\Theta-1}(n+2j)}{(2\Theta-1)!!}\left(\frac{\langle\Theta,\Theta;\Theta\rangle_{n=1}}{\langle\Theta,\Theta;\Theta\rangle_n}\right)^2. \qquad (2.15b)$$

Thus, $f=1$ for $n=1$. For large n, we can extract the leading dependence of η_Θ from (2.12b) and write

$$\eta_\Theta=4\epsilon_\Theta^2(2)\Big/n\binom{\Theta}{\Theta/2}^2+O\left(\frac{1}{n^2}\right), \qquad (2.16a)$$

for Θ even, and

$$\eta_\Theta=4\epsilon_\Theta^2(2)\Big/\binom{\Theta}{(\Theta-1)/2}^3(\Theta+1)^3+O\left(\frac{1}{n}\right),$$

for Θ odd. By an extension of the method used in Ref. 16, one can show that for $\tilde\sigma\neq 2L$ there is no shift in the value of η_Θ at least to order $\epsilon_\Theta(\tilde\sigma)^2$, as found in Ref. 18 for the special case $\Theta=2$.[20]

III. DERIVATION OF THE APPROXIMATE DIFFERENTIAL GENERATOR (1.2) FROM THE FULL WILSON EQUATIONS

To derive (1.2) from the Wilson partial-integration generator, we put the full Wilson equation into a form more closely resembling the full Wegner-Houghton equation. This reformulation is more convenient for perturbation expansions from the Gaussian fixed-point solution as illustrated below. We begin with the full Wilson equation for Ising-like systems ($n=1$),

$$\frac{\partial\mathcal{H}}{\partial l}=d\mathcal{H}+\int_k s_k\left(\frac{\tilde\sigma-\delta\eta-d}{2}+\beta(k^2)-k\frac{\partial}{\partial k}\right)\frac{\delta\mathcal{H}}{\delta s_k}+\int\left(\frac{\tilde\sigma-\delta\eta}{2}+\beta(k^2)\right)\left(\frac{\delta^2\mathcal{H}}{\delta s_k\delta s_{-k}}-\frac{\delta\mathcal{H}}{\delta s_k}\frac{\delta\mathcal{H}}{\delta s_{-k}}\right), \qquad (3.1)$$

where $\tilde\sigma$ is the propagator exponent. Here $\delta\eta$ is the shift in the propagator exponent; e.g., in the case of non-Lifshitz systems with long-range-force exponent σ, $\tilde\sigma=\sigma$ and $\delta\eta=\eta-2+\sigma$. For compactness, we write k instead of $\vec k$ whenever it is not confusing. This expression differs from that given in Ref. 1(b), since in (3.1) the operator $k\partial/\partial k$ does not act on the s_k or momentum conserving δ functions in the expansion of $\mathcal{H}$, while in Ref. 1(b) $k\partial/\partial k$ acts only on the s_k. The difference is simply an integration by parts. The function β is an arbitrary (increasing) function of k^2 (commonly, $\beta=k^2$).

The presence of β in the first integral in (3.1) as well as in the second integral is an inconvenience for some calculations. If we make the change of variable

$$s_k\to s_k\left(\frac{(\tilde\sigma-\delta\eta)/2+\beta(k^2)}{C(k^2)}\right)^{1/2}, \qquad (3.2)$$

Eq. (3.1) can be rewritten as

$$\frac{\partial\mathcal{H}}{\partial l}=d\mathcal{H}+\int_k s_k\left(\frac{\tilde\sigma-d-\delta\eta}{2}+q(k^2)-k\frac{\partial}{\partial k}\right)\frac{\delta\mathcal{H}}{\delta s_k}+\int C(k^2)\left(\frac{\delta^2\mathcal{H}}{\delta s_k\delta s_{-k}}-\frac{\delta\mathcal{H}}{\delta s_k}\frac{\delta\mathcal{H}}{\delta s_{-k}}\right), \qquad (3.3a)$$

where q is related to β and C by

$$q(k^2)=\beta(k^2)+\frac{k}{2}\frac{\partial}{\partial k}\ln\left(\frac{(\tilde\sigma-\delta\eta)/2+\beta(k^2)}{C(k^2)}\right). \qquad (3.3b)$$

The function $C(k^2)$ is a cutoff function in the usual sense; for example, $C=\Theta(1-k^2)$, a Brillouin-zone cutoff, or $C=e^{-k^2}$, a smooth cutoff. In passing from (3.1) to (3.3) we have increased the number of arbitrary functions from one (β) to two (q and C). We reduce the number to one again by examining the Gaussian fixed-point solution.

The Gaussian solution is defined as the fixed-point solution of the form

$$\mathcal{H}_G=\frac{1}{2}\int w(|k|)s_ks_{-k} \qquad (3.4a)$$

for $\delta\eta=0$. The function w satisfies

$$\left(\tilde\sigma+2q-k\frac{\partial}{\partial k}\right)w=2C(k^2)w^2\,. \qquad (3.4b)$$

If we expand around the Gaussian solution, $\mathcal{H}=\mathcal{H}_G+\mathcal{H}'$, we have

$$\delta\eta\mathcal{H}_G+\frac{\partial\mathcal{H}'}{\partial l}=d\mathcal{H}'+\int_k s_k\left(\frac{\tilde{\sigma}-d-\delta\eta}{2}+q-2wC(k^2)-k\frac{\partial}{\partial k}\right)\frac{\delta\mathcal{H}'}{\delta s_k}+\int_k C(k^2)\left(\frac{\delta^2\mathcal{H}'}{\delta s_k\delta s_{-k}}-\frac{\delta\mathcal{H}'}{\delta s_k}\frac{\delta\mathcal{H}'}{\delta s_{-k}}\right). \quad (3.5)$$

Equation (3.5) is simplified if q and C are related by $q=2wC$. Combining this with (3.4b) we have

$$w=\frac{|k|^{\tilde{\sigma}}}{\int_{k^2}^{\infty}C(x)x^{(\tilde{\sigma}-2)/2}\,dx}. \quad (3.6)$$

In (3.6) the upper limit of the integral has been chosen so that q and β are increasing functions of $|k|$ for large $|k|$.

In this formulation, the generator (3.5) when linearized around the Gaussian fixed point has the same form as the similarly linearized Wegner-Houghton generator

$$\frac{\partial\mathcal{H}}{\partial l}=d\mathcal{H}+\int_k s_k\left(\frac{\tilde{\sigma}-d}{2}-k\frac{\partial}{\partial k}\right)\frac{\delta\mathcal{H}}{\delta s_k}+\int_k C(k^2)\frac{\delta^2\mathcal{H}}{\delta s_k\delta s_{-k}}. \quad (3.7)$$

For the Wegner-Houghton generator, the cutoff function $C=\delta(|k|-1)$. Equation (3.7) admits solutions with momentum-independent expansion coefficients, in contrast with the usual formulations of the Wilson generator.[2,23-26]

If we now set $\delta\eta=0$ in (3.5) and neglect momentum dependence by considering the limit of all k's $\to 0$, we have the following equation for $H(s,l)$:

$$\frac{\partial H}{\partial l}=dH+\left(\frac{\tilde{\sigma}-d}{2}\right)s\frac{\partial H}{\partial s}+\frac{\partial^2 H}{\partial s^2}-\left(\frac{\partial H}{\partial s}\right)^2, \quad (3.8)$$

where we have normalized C by $\int d^dk\,C(k^2)=1$. The linearized eigenfunctions of (3.8) are eigenfunctions of (3.7) if we identify s^m with $\int_{k_1}\cdots\int_{k_m}\delta(\vec{k}_1+\cdots+\vec{k}_m)s_{k_1}\cdots s_{k_m}$.

The above discussion may be repeated for general n, leading to the approximate generator (1.2).

The reformulated equation (3.5), just as the original generator (3.1), may be used for momentum-dependent calculations. As discussed in Refs. 23–26, at least the leading contribution to η_2 (for $\sigma=2,\tilde{\sigma}=2$) is independent of the function β in (3.1). Thus it is to be expected that η_2 is independent of the choice of C in (3.5).

This is somewhat easier to demonstrate than the independence of β because of the simplification of the equation given by the constraint $q=2wC$ used in (3.5). It is straightforward to express η_2 in terms of the function C. The method is similar to that of Ref. 23 and will not be detailed here. The result is

$$\eta_2=-\frac{\epsilon_2^2}{9}\left(\int_0^{\infty}C(x)\,dx\right)^{-3}I, \quad (3.9a)$$

where

$$I=\int_0^1 dt\int_0^{\infty}p^2\,dp\int_0^{\infty}q^2\,dq\int_0^{\pi}\frac{2}{\pi}\sin^2\theta\,d\theta\;C(p^2)C(q^2)C'(p^2+tq^2+2pqt^{1/2}\cos\theta), \quad (3.9b)$$

with $C'(x)=dC/dx$.

To compute I, we rewrite the multiple integrals of (3.9b) in terms of the Fourier transform of C, $\hat{C}(z)=\int_0^{\infty}C(x)e^{-izx}dx$. Performing the angular integration gives a Bessel function and (3.9b) becomes

$$I=\frac{i}{(2\pi)^3}\int_{-\infty}^{\infty}dz_1\,dz_2\,dz_3\int_0^1 dt\int_0^{\infty}dx\,dy\,\hat{C}(z_1)\hat{C}(z_2)\hat{C}(z_3)\left(\frac{xy}{t}\right)^{1/2}J_1(2z_3(txy)^{1/2})\exp\{i[xz_1+yz_2+(x+ty)z_3]\}, \quad (3.10)$$

where $x\equiv p^2$, $y\equiv q^2$. We now assume (as in Ref. 26) that the z integrals can be deformed off the real axis so that each z has a small positive imaginary part. With the aid of this convergence factor, the integral over x gives

$$I=\frac{-i}{(2\pi)^3}\int_{-\infty}^{\infty}dz_1\,dz_2\,dz_3\int_0^1 dt\int_0^{\infty}dy\,\hat{C}(z_1)\hat{C}(z_2)\hat{C}(z_3)\,y\frac{z_3}{(z_1+z_2)^2}\exp\left(iyz_2+\frac{ityz_1z_3}{(z_1+z_3)}\right). \quad (3.11)$$

The y and t integrations are now elementary and we obtain (after symmetrizing in z_1, z_2, and z_3)

$$I=\frac{i}{6(2\pi)^3}\left(\int_{-\infty}^{\infty}\frac{dz}{z}\hat{C}(z)\right)^3.$$

Because $C(x)\equiv 0$ for $x<0$, $\hat{C}(z)$ is analytic in the lower half plane. We may, therefore, close the contour down and write $I=-\frac{1}{6}\hat{C}^3(0)$. However, $\hat{C}(0)=\int_0^{\infty}C(x)\,dx$. Hence, from (3.9), we find $\eta_2=\epsilon_2^2/54$, independent of the cutoff function C.

IV. EXACTNESS OF THE APPROXIMATE GENERATORS FOR PAIRED SPIN MOMENTA AND FOR EVEN-ORDER CRITICAL POINTS IN THE $n\to\infty$ LIMITS

In this section we will show that in some circumstances (in particular, the even-order critical points) the $n=\infty$ limit of the full Wilson and Wegner-Houghton differential generators can be written in simpler form. This exact limiting form is the same for both generators, demonstrating their equivalence to all orders of perturbation theory. This limiting generator is also identical to that obtained from either of the approximate generators, showing that the approximate generators become exact in the $n=\infty$ limit, Θ even. A similar exact limit is obtained when all the spins are paired, $\mathcal{H}=\mathcal{H}(\int s_p^\alpha s_{-p}^\beta)$.

If we assume spin isotropy, we can write the full Wilson generator as

$$\delta\eta\mathcal{H}_G+\frac{\partial\mathcal{H}}{\partial l}=d\mathcal{H}+\int_p\int_q x_{p,q}\left(\tilde\sigma-d-\delta\eta-p\frac{\partial}{\partial p}-q\frac{\partial}{\partial q}\right)\frac{\delta\mathcal{H}}{\delta x_{p,q}}$$
$$+2n\int_p C(p^2)\frac{\delta\mathcal{H}}{\delta x_{p,-p}}+4\int_p\int_k\int_{k'}C(p^2)x_{k,k'}\frac{\delta^2\mathcal{H}}{\delta x_{p,k}\delta x_{-p,k'}}-4\int_p\int_k\int_{k'}C(p^2)x_{k,k'}\frac{\delta\mathcal{H}}{\delta x_{p,k}}\frac{\delta\mathcal{H}}{\delta x_{-p,k'}},\qquad(4.1)$$

where $x_{p,q}=\vec{s}_p\cdot\vec{s}_q$ and $\mathcal{H}_G$ is the Gaussian functional defined in (3.4)–(3.6). Consider the transformation

$$z_{p,q}=\left(x_{p,q}-\frac{2nC(p^2)\delta(\vec p+\vec q)}{d-\tilde\sigma-\delta\eta}\right)\left(\frac{d-\tilde\sigma-\delta\eta}{8n}\right)^{1/2}.\qquad(4.2\text{a})$$

This transformation is suggested by an examination of the approximate generators; in the large-n limit, it proves convenient to use the Laguerre polynomial expansion,

$$L_p^{n/2-1}(x)=\left(\frac{n^{p/2}}{2^p p!}\right)\left[H_p\left(\frac{(x-n/2)}{n^{1/2}}\right)+O(n^{-1/2})\right],\qquad(4.2\text{b})$$

where H_p is the pth Hermite polynomial. In terms of the variables $z_{p,q}$, the generator (4.1) becomes

$$\delta\eta\mathcal{H}_G+\frac{\partial\mathcal{H}}{\partial l}=d\mathcal{H}+\int_p\int_q z_{p,q}[\tilde\sigma-d-\delta\eta]\frac{\delta\mathcal{H}}{\delta z_{p,q}}+\int_p\int_q C(p^2)C(q^2)\left(\frac{\delta^2\mathcal{H}}{\delta z_{p,q}\delta z_{-p,-q}}-\frac{\delta\mathcal{H}}{\delta z_{p,q}}\frac{\delta\mathcal{H}}{\delta z_{-p,-q}}\right)$$
$$-\int\int[z_{p,q}+C(p^2)\delta(\vec p+\vec q)(2n)^{1/2}]\left(p\frac{\partial}{\partial p}+q\frac{\partial}{\partial q}\right)\frac{\delta\mathcal{H}}{\delta z_{p,q}}+O(n^{-1/2}).\qquad(4.3)$$

The Gaussian functional term is

$$\delta\eta\mathcal{H}_G=(2n)^{1/2}\delta\eta\int_k w(|k|)z_{k,-k}+\frac{\delta\eta n\Omega}{d-\tilde\sigma-\delta\eta}\int_k w(|k|)C(k^2),\qquad(4.4)$$

where Ω is the volume of the system. The second term in (4.4) is just a constant which we may ignore. The coefficients of the momentum derivatives and the remaining $\delta\eta$ term in $\mathcal{H}_G$ are apparently $O(n^{1/2})$. We cannot directly take the limit $n\to\infty$ unless $\delta\eta$ and the momentum derivatives are $O(n^{-1/2})$ or smaller. From (2.16) we expect that this is the case for *even*-order critical points (we at least know that it cannot be true for *odd*-order points). As an ansatz, we assume that the troublesome terms are $O(n^{-1})$. With this assumption we obtain from (4.3)

$$\frac{\partial\mathcal{H}}{\partial l}=d\mathcal{H}+(\tilde\sigma-d)\int_p\int_q z_{p,q}\frac{\partial\mathcal{H}}{\partial z_{p,q}}+\int_p\int_q C(p^2)C(q^2)\frac{\delta^2\mathcal{H}}{\delta z_{p,q}\delta z_{-p,-q}}-\int_p\int_q C(p^2)C(q^2)\frac{\delta\mathcal{H}}{\delta z_{p,q}}\frac{\delta\mathcal{H}}{\delta z_{-p,-q}}+O(n^{-1/2}).\qquad(4.5)$$

All reference to n is absorbed in the transformation (4.2). Hence, there can be no momentum dependence in the expansion coefficients of $\mathcal{H}(z_{p,q})$. Equation (4.5) is inconsistent with this momentum independence unless $\mathcal{H}$ is a functional of only the paired momentum spins, $z_{p,-p}$. We therefore define

$$z=\frac{\int_p z_{p,-p}}{[\Omega\int_k C^2(k^2)]^{1/2}}.\qquad(4.6)$$

In terms of this variable, the generator (4.5) becomes

$$\frac{\partial H}{\partial l}=dH+(\tilde\sigma-d)z\frac{\partial H}{\partial z}+\frac{\partial^2 H}{\partial z^2}-\frac{\partial H}{\partial z}\frac{\partial H}{\partial z}.\qquad(4.7)$$

This is precisely the limiting form of the approximate generator (1.2) under the transformation $z=[s^2-2n/(d-\tilde\sigma)][(d-\tilde\sigma)/8n]^{1/2}$.

A similar analysis of the full Wegner-Houghton equation again leads to the pairing of spin momenta and (4.7). It can be obtained from the corresponding approximate generator (1.1) by the matrix transformation $z^{\alpha\beta}=[s^\alpha s^\beta - 2\delta_{\alpha\beta}/(d-\tilde{\sigma})][(d-\tilde{\sigma})/8n]^{1/2}$ ($z\equiv\sum_\alpha z^{\alpha\alpha}$) and a change of scale of H, $H\to 4H/(d-\tilde{\sigma})$.

The limiting generator (4.7) is obviously not exact for odd-order critical points since for $\tilde{\sigma}=2L$, η has a finite limit as $n\to\infty$.[20] It is *also* not the correct *approximate* generator in this limit for such fixed points. The nonlinear term in (4.7) has the wrong parity for odd-order points to produce the $O(\epsilon_\Theta)$ corrections. [In this respect, it resembles the "odd-dominated" Hamiltonians treating in Ref. 13. It is possible to obtain the correct first-order corrections by retaining the $O(n^{-1/2})$ corrections in the approximate generators.] This failure is also related to the fact that for even-order critical points, the fixed-point Hamiltonians (2.3) have a finite limit as $n\to\infty$ (in terms of the variable z), while for odd-order points, the fixed point grows like $n^{1/2}$. In the derivation above, no n-dependent scaling of the Hamiltonian was utilized.

Equation (4.7) differs greatly from the expression derived in Ref. 1 and obtained by simply setting $n=\infty$ in Eq. (1) of Ref. 13:

$$\frac{\partial\overline{H}}{\partial l}=d\overline{H}+(\tilde{\sigma}-d)x\frac{\partial\overline{H}}{\partial x}+\ln\left(1+\frac{\partial\overline{H}}{\partial x}\right), \tag{4.8}$$

where $x=s^2/n$ and $\overline{H}=2H/n$. This form is inadequate for general even-order fixed points. First, it presumes that the Hamiltonian H is $O(n)$, rather than $O(1)$ as shown above. Second, the use of s^2/n as a variable, while it properly treats the gross behavior of s^2, loses the $O(n^{1/2})$ character of the fluctuations. This corresponds to taking the $n\to\infty$ limit of Laguerre polynomials with *fixed* argument,

$$\lim_{n\to\infty}L_p^{n/2-1}(x)=\frac{(1-x)^p}{p!}, \tag{4.9}$$

rather than the Hermite-polynomial limit (4.2b). These functions are the eigenfunctions of (4.8) when linearized around $H=0$. Although complete, these functions do not have any convenient orthogonality relations. This results in the loss of the ϵ_Θ expansion for all $\Theta\neq 2$. In Ref. 1(a), it was in fact shown that (4.9) has only one fixed-point solution, corresponding to $\Theta=2$, the spherical model.[33] The change of variables used in this work preserves the second-order character of the linearized equations in the $n\to\infty$ limit. In this manner, the eigenfunctions of the limiting generator are still orthogonal polynomials.

Since Eq. (4.7) is exact, it provides an "anchor" for calculations of exponents for higher-order critical points, much as the spherical model has been used to test the accuracy of high-temperature series and $4-d$ expansions for ordinary critical points. Perturbational or exact solutions of (4.7) will be vastly simpler to achieve than the corresponding solutions of the full renormalization-group equations for finite n and can therefore be carried to higher order and used to check finite-n results. In the special case of the spherical model, $\Theta=2$, (4.7) can easily be solved exactly. The exact fixed point for an isotropic system is

$$H^*=\frac{2\tilde{\sigma}-d}{4}\left(z^2-\frac{2}{d}\right). \tag{4.10}$$

Linearizing around this fixed point, we again obtain Hermite polynomials, but with eigenvalues

$$\lambda_p=d-\tilde{\sigma}p\,. \tag{4.11}$$

For $\tilde{\sigma}=2$, we recover the eigenspectrum of the spherical model.[33]

This exact solution suggests that for $d\leq\tilde{\sigma}$, the spherical model has no phase transition at finite temperature (e.g., $1/\delta\leq 0$) as is known for $\tilde{\sigma}=2$. This corresponds to the onset of infrared divergences in the direct Feynman-diagram expansion. The possible existence of a phase transition at finite n is, however, not entirely excluded. Perturbation expansions for the usual critical theory ($\Theta=2,\tilde{\sigma}=2$) fail at $d=2$ for all n. However, the $d=2$ Ising model ($n=1$) has a "normal" transition at its critical point, while the $d=2$ XY model ($n=2$) has a Stanley-Kaplan transition.[34] For each Θ and $\tilde{\sigma}$, a similar situation may prevail at this "infrared boundary dimension" $d_{\rm ir}\equiv\tilde{\sigma}$. For the generalized Lifshitz points ($\tilde{\sigma}=2L$) the physically most interesting case of three dimensions is always below $d_{\rm ir}=2L$ ($L\geq 2$), and it would seem that the theory must be modified. As mentioned above, this corresponds to the occurrence of infinitely many relevant Gaussian eigenvalues. If we admit anisotropy in the $\vec{k}$ dependence, both the dimension from which we perturb and the infrared boundary dimension are lowered. For example, if only one component of the wave vector contributed to the k^{2L} term while the other components retain k^2 dependence, then $d_{\rm ir}=3-1/L$. Discussions of perturbation expansions for such systems are given elsewhere.[20]

In taking the $n\to\infty$ limit, we found that it was necessary to pair the spin momenta. We may reverse the order of this procedure, and first pair the spin momenta. For *isotropic* systems, this limit is the same as the $n\to\infty$ limit, but for systems with only spin-reversal symmetry. On the other hand, paired Hamiltonians of *anisotropic* spins are also of interest; they represent classical analogs of BCS pairing in superconductivity and in anisotropic superfluidity, where complex tensor order parameters are formed through momentum pairing. If the initial Hamiltonian is a functional of paired-momenta spins only, it remains so for all l. This corresponds to the fact that in a diagrammatic representation of such a theory, only tree diagrams occur. The coincidence with the $n\to\infty$ limit arises because each loop of the trees carries a combinatorial factor of n. Thus, the tree diagrams dominate all other diagrams as n grows large.

For this case, we will give the derivation from the exact Wegner-Houghton generator; the Wilson case is entirely similar. If we assume the Hamiltonian is of the form $\mathcal{H}=\mathcal{H}(s_p^\alpha s_{-p}^\beta)$, the exact Wegner-Houghton generator can be written as

$$\frac{\partial\mathcal{H}}{\partial l}=d\mathcal{H}+(\bar{\sigma}-d)\int_p(s_p^\alpha s_{-p}^\beta)\frac{\delta\mathcal{H}}{\delta(s_p^\alpha s_{-p}^\beta)}+\Omega\int_p C(p^2)\,\mathrm{tr}\ln\left(\delta_\beta^\alpha+2\frac{\delta^2\mathcal{H}}{\delta(s_p^\alpha s_{-p}^\beta)}+\frac{4s_p^\gamma s_{-p}^\nu}{\Omega}\frac{\delta^2\mathcal{H}}{\delta(s_p^\alpha s_{-p}^\beta)\delta(s_p^\gamma s_{-p}^\nu)}\right), \tag{4.12}$$

where $C(p^2)=\delta(|p|-1)$ in the original formulation of Ref. 1; it is convenient for our purposes to let it be more general. In the paired limit, we can drop the momentum derivatives and $\delta\eta$.

We make the transformations

$$z_p^{\alpha\beta}\equiv\left(s_p^\alpha s_{-p}^\beta+2\frac{\delta_\beta^\alpha C(p^2)\Omega}{d-\bar{\sigma}}\right)\left(\frac{d-\bar{\sigma}}{8\Omega}\right)^{1/2}, \tag{4.13a}$$

$$\mathcal{H}\to\frac{4}{d-\bar{\sigma}}\mathcal{H}. \tag{4.13b}$$

Equation (4.12) becomes

$$\frac{\partial\mathcal{H}}{\partial l}=d\mathcal{H}+(\bar{\sigma}-d)\int_p z_p^{\alpha\beta}\frac{\delta\mathcal{H}}{\delta z_p^{\alpha\beta}}+\int_p C^2(p^2)\frac{\delta^2\mathcal{H}}{\delta z_p^{\alpha\beta}\delta z_{-p}^{\alpha\beta}}-\int_p C(p^2)\frac{\delta\mathcal{H}}{\delta z_p^{\alpha\beta}}\frac{\delta\mathcal{H}}{\delta z_{-p}^{\alpha\beta}}+O(\Omega^{-1/2}). \tag{4.14}$$

Now we make the further transformations

$$z^{\alpha\beta}\equiv\frac{\int_p z_p^{\alpha\beta}}{[\int_k C^2(k^2)]^{1/2}}, \tag{4.15a}$$

$$\mathcal{H}\to\frac{\int_k C^2(k^2)}{\int_p C(p^2)}H. \tag{4.15b}$$

Inserting these into (4.14) we obtain

$$\frac{\partial H}{\partial l}=dH+(\bar{\sigma}-d)z^{\alpha\beta}\frac{\partial H}{\partial z^{\alpha\beta}}+\frac{\partial^2 H}{\partial z^{\alpha\beta}\partial z^{\alpha\beta}}-\frac{\partial H}{\partial z^{\alpha\beta}}\frac{\partial H}{\partial z^{\alpha\beta}}. \tag{4.16}$$

Under similar transformations, (4.16) is the limiting expression of the Wilson generator. The form of the approximate generators stated in (1.1) and (1.2) made it difficult to see that (4.16) is also their common limit since the approximate generators do not contain factors of the volume Ω. This defect is remedied in Sec. V. From the expressions for the approximate generators given there [cf. (5.5)], it is easy to derive that (4.16) is the appropriate limit of the approximate generators as well. If we assume isotropy and define $z=z^{\alpha\alpha}/n^{1/2}$ we return to (4.7).

V. GENERALIZATION OF THE APPROXIMATE GENERATORS TO INCLUDE ALL POSSIBLE SPIN-MOMENTA GROUPINGS

The approximate generators given in (1.1) and (1.2) are appropriate for Hamiltonians of the form

$$H=\int d^dx\,H(\vec{s}(\vec{x})). \tag{5.1}$$

The momentum-dependent term in $s_k s_{-k}$ is assumed but not written out explicitly. The momentum-space representation of this Hamiltonian contains one momentum-conserving δ function for each term. However, for other problems a more general form of momentum dependence is required and yet the full equations need not be used. For example, in the large-n and paired-momenta limit discussed in Sec. IV, we considered $H=H(\int\vec{s}(x)^2d^dx)$. In this case, all the spins occur in momentum pairs. A still more complicated example is that of Hamiltonians suitable for the study of compressible magnets,[21] in which some spin momenta are paired and some are not. The following form is used[21]:

$$H=\frac{r}{2}\int\vec{s}^{\,2}(\vec{x})\,d^dx+\frac{u}{8}\int[\vec{s}^{\,2}(\vec{x})]^2d^dx+\frac{v}{8\Omega}\left[\int\vec{s}^{\,2}(\vec{x})\,d^dx\right]^2, \tag{5.2}$$

where, as in Sec. IV, Ω is the volume of the system.

The most general form which can be subsumed in the approximate generators is

$$H=H(\{\langle s^{\alpha_1}\cdots s^{\alpha_m}\rangle\}), \tag{5.3a}$$

where

$$\langle s^{\alpha_1}\cdots s^{\alpha_m}\rangle\equiv\int d^dx\,s^{\alpha_1}(\vec{x})\cdots s^{\alpha_m}(\vec{x}). \tag{5.3b}$$

Here $\alpha_1\cdots\alpha_m$ runs over all possible sets of coordinate indices ($\alpha_i\le n$). The usual case (5.1) corresponds to H linear in each $\langle s^{\alpha_1}\cdots s^{\alpha_m}\rangle$; that is, $H = \langle H(\vec{s})\rangle$. The paired-momenta and large-n limit corresponds to $H = H(\langle s^\alpha s^\beta\rangle)$. In this notation, the magnetoelastic problem has the Hamiltonian

$$H = \frac{r}{2}\langle \vec{s}^{\,2}\rangle + \frac{u}{8}\langle \vec{s}^{\,2}\vec{s}^{\,2}\rangle + \frac{v}{8\Omega}\langle \vec{s}^{\,2}\rangle^2 . \tag{5.4}$$

More general spin groupings in Hamiltonians can occur when the systems are subjected to arbitrary global constraints. By examining the exact generators, we find that the appropriate approximate generators are

$$\frac{\partial H}{\partial l} = dH + \tfrac{1}{2}(\tilde{\sigma} - d)\langle \vec{s}\cdot\vec{\nabla}H\rangle + \left\langle \operatorname{tr}\ln\left(\delta_{\alpha\beta} + \frac{\partial^2 H}{\partial s^\alpha \partial s^\beta}\right)\right\rangle , \tag{5.5a}$$

$$\frac{\partial H}{\partial l} = dH + \tfrac{1}{2}(\tilde{\sigma} - d)\langle \vec{s}\cdot\vec{\nabla}H\rangle + \langle \nabla^2 H\rangle - \langle \vec{\nabla}H\cdot\vec{\nabla}H\rangle . \tag{5.5b}$$

In (5.5) we have defined

$$\frac{\partial}{\partial s^\beta}\langle s^{\alpha_1}\cdots s^{\alpha_m}\rangle = \frac{\partial}{\partial s^\beta}(s^{\alpha_1}\cdots s^{\alpha_m}) , \tag{5.6a}$$

$$\begin{aligned}\frac{\partial}{\partial s^\beta}&\langle s^{\alpha_1}\cdots s^{\alpha_m}\rangle\langle s^{\gamma_1}\cdots s^{\gamma_j}\rangle \\ &= \langle s^{\alpha_1}\cdots s^{\alpha_m}\rangle\frac{\partial}{\partial s^\beta}\langle s^{\gamma_1}\cdots s^{\gamma_j}\rangle \\ &\quad + \langle s^{\gamma_1}\cdots s^{\gamma_j}\rangle\frac{\partial}{\partial s^\beta}\langle s^{\alpha_1}\cdots s^{\alpha_m}\rangle , \text{etc.,}\end{aligned} \tag{5.6b}$$

$$\langle 1\rangle = \Omega . \tag{5.6c}$$

Of course, for any particular problem, the final result should be independent of Ω in the limit $\Omega\to\infty$. With the generators (5.5), it is easy to verify the paired-momenta limit of (4.16) is the limit of the approximate generators as well.

As an example of the use of (5.5), we give the nonlinear differential equations for the Hamiltonian (5.4) correct to $O(\epsilon)$, $\epsilon \equiv 4-d$, using the approximate Wegner-Houghton generator,

$$\frac{\partial r}{\partial l} = 2r + (n+2)\frac{u}{1+r} + n\frac{v}{1+r} , \tag{5.7a}$$

$$\frac{\partial u}{\partial l} = \epsilon u - (n+8)\frac{u^2}{(1+r)^2} , \tag{5.7b}$$

$$\frac{\partial v}{\partial l} = \epsilon v - 2(n+2)\frac{uv}{(1+r)^2} - n\frac{v^2}{(1+r)^2} . \tag{5.7c}$$

These represent the differential limit of the recursion equations for the coupled order parameters s^2 and $\langle s^2\rangle$ studied in Ref. 21. It is possible to give a nonlinear solution of (5.7) similar to our other nonlinear problems involving coupled order parameters[4]; this solution is given in Sec. VI.

We also point out that the approximate generators (5.5) can be written as a functional differential equation in a single function $\vec{s}(\vec{x})$, rather than as an ordinary differential equation in an infinite number of variables $\langle s^{\alpha_1}\cdots s^{\alpha_m}\rangle$. However, we must respect the underlying lattice structure of the system by noting that

$$\delta s^\beta(\vec{x})/\delta s^\alpha(\vec{x}) = \delta_{\alpha\beta} , \tag{5.8a}$$

instead of the usual formula

$$\delta s^\beta(\vec{x})/\delta s^\alpha(\vec{x}) = \delta_{\alpha\beta}\delta^d(0) . \tag{5.8b}$$

The resolution of this seeming disparity is that $\delta^d(0)$ is essentially $(1/a)^d$, where a is the smallest length scale in the system. For true continuum systems $a=0$, but for lattice systems in which natural units for length are used (i.e., the lattice spacing) $a=1$. With this *caveat*, the approximate generators can be written as

$$\frac{\partial H}{\partial l} = dH + \tfrac{1}{2}(\tilde{\sigma} - d)\int_x d^d x\, s^\alpha(\vec{x})\frac{\delta H}{\delta s^\alpha(\vec{x})} + \int_x d^d x\,\operatorname{tr}\ln\left(\delta_{\alpha\beta} + \frac{\delta^2 H}{\delta s^\alpha(\vec{x})\delta s^\beta(\vec{x})}\right) , \tag{5.9a}$$

$$\frac{\partial H}{\partial l} = dH + \tfrac{1}{2}(\tilde{\sigma} - d)\int_x d^d x\, s^\alpha(\vec{x})\frac{\delta H}{\delta s^\alpha(\vec{x})} + \int_x d^d x\frac{\delta^2 H}{\delta s^\alpha(\vec{x})\delta s^\beta(\vec{x})} - \int_x d^d x\frac{\delta H}{\delta s^\alpha(\vec{x})}\frac{\delta H}{\delta s^\beta(\vec{x})} . \tag{5.9b}$$

VI. NONLINEAR SOLUTION TO COMPRESSIBLE FERROMAGNET RENORMALIZATION-GROUP EQUATIONS (5.7) TO FIRST ORDER IN ϵ

To solve (5.7) we make the change of variables $y_n \equiv (n+8)u/\epsilon[(1+r)^2]$, $y_e \equiv nv/\epsilon[(1+r)^2]$, $\bar{r}\equiv r/(1+r)$, and $x \equiv \bar{r} + \epsilon y_n\Delta_n/2 + \epsilon y_e/2$, where $\Delta_n \equiv (n+2)/(n+8)$. The subscripts n and e refer to the normal n-component and elastic portions of the Hamiltonian (5.2). We then have the equations, to $O(\epsilon)$,

$$\frac{\partial x}{\partial l} = 2x\left(1 - x - \frac{\epsilon\Delta_n y_n}{2} - \frac{\epsilon y_e}{2}\right) , \tag{6.1a}$$

$$\frac{\partial y_n}{\partial l} = y_n[\epsilon(1-y_n) - 4x] , \tag{6.1b}$$

$$\frac{\partial y_e}{\partial l} = y_e[\epsilon(1-y_e) - 2\epsilon\Delta_n y_n - 4x] . \tag{6.1c}$$

In terms of these variables the four critical $(x=0)$ fixed points and the corresponding temperature-like eigenvalues are (cf. Fig. 2) as follows: (i) the Gaussian fixed point, $y_n = y_e = 0$, $\lambda_T^{\mathrm{i}} = 2$; (ii) the Wilson-Fisher n-spin point,[35] $y_e = 0$, $y_n = 1$,

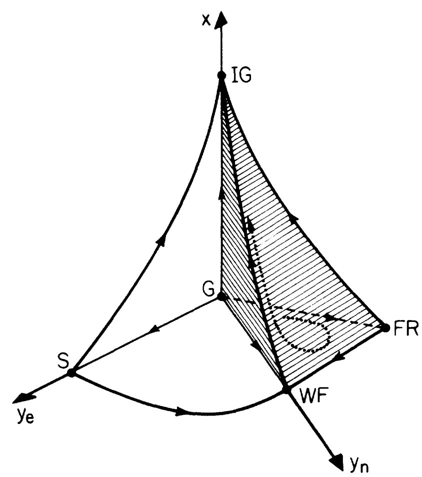

FIG. 2. Solution region for isotropic compressible ferromagnets for $\alpha_{WF}<0$ (shown shaded). Here G, WF, S, FR, and IG denote the (i) Gaussian, (ii) Wilson-Fisher, (iii) spherical, (iv) Fisher-renormalized, and (v) infinite-Gaussian fixed points. The variables y_e, y_n, and x are defined at the outset of Sec. VI.

$\lambda_T^{ii}=2-\epsilon\Delta_n$; (iii) a "pure elastic" point, $y_n=0$, $y_e=1$, $\lambda_T^{iii}=2-\epsilon$ (identical to the spherical model[33]); (iv) the Fisher-renormalized n-spin point[36] (z point), $y_n=1$, $y_e=1-2\Delta_n$, $\lambda_T^{iv}=2-\epsilon\Delta_z$, where $\Delta_z=6/(n+8)$. In addition, there is the "infinite-Gaussian" fixed point at $x=1$, $y_n=y_e=0$.

The existence of the z point corresponds to the possibility of defining a linear combination of y_n and y_e, $z\equiv y_e-(1-2\Delta_n)y_n$, such that $z=0$ is the trajectory connecting the Gaussian fixed point to the z point. The equation for z is

$$\frac{\partial z}{\partial l}=z[\epsilon(1-y_n-y_e)-4x]. \qquad (6.2)$$

Fixed points (i) and (iii) and fixed points (ii) and (iv) are Fisher-renormalization[36] pairs. That is, $\lambda_T^{i}+\lambda_T^{iii}=d$, and $\lambda_T^{ii}+\lambda_T^{iv}=d$. As expected from the arguments of Ref. 36, the relative stability of two fixed points which form a Fisher-renormalization pair is determined by the sign of the specific-heat exponent ($\alpha\equiv 2-d/\lambda_T$); the stabler fixed point is that which has a negative α, that is, $\lambda_T<d/2$.

We define the auxiliary functions R, X, Y_n, and Y_e in terms of which we can give the solution,[4]

$$\frac{\partial R}{\partial l}=2(1-\bar{r})R, \qquad (6.3a)$$

$$\frac{\partial X}{\partial l}=-2xX, \qquad (6.3b)$$

$$\frac{\partial Y_n}{\partial l}=-\epsilon y_n Y_n, \qquad (6.3c)$$

$$\frac{\partial Y_e}{\partial l}=-\epsilon y_e Y_e. \qquad (6.3d)$$

We find $X=(1-\bar{r})$, $R=xY_n^{-2\Delta_n}Y_e^{-2}$, and the following relationship between Y_n and Y_e:

$$y_e-(1-2\Delta_n)y_n=y_eY_n^{1-2\Delta_n}-(1-2\Delta_n)y_nY_e. \qquad (6.4)$$

Various renormalization ("scaling") invariants can be formed from these functions as in Ref. 4. We can advance beyond this stage here for the compressible magnet, since (6.1) can be solved exactly in the $x=0$ plane.

We have immediately that $Y_n=1-y_n$ and from (6.4)

$$Y_e=\frac{y_e[(1-y_n)^{1-2\Delta_n}-1]+(1-2\Delta_n)y_n}{(1-2\Delta_n)y_n}. \qquad (6.5)$$

The separatrix joining the n-spin point to the z point is the line $y_n=1$. The separatrix connecting the pure elastic to the stablest fixed point (the z point for $n<4$ and the n-spin point otherwise) is given by $Y_e=0$ (cf. Ref. 4), or

$$y_e=\frac{(1-2\Delta_n)y_n}{1-(1-y_n)^{1-2\Delta_n}}. \qquad (6.6)$$

Thus, the two separatrices join smoothly at the stablest point, but the boundary as a whole is not analytic at the stablest point. This is to be expected from the general theory of nonlinear differential equations; each region bounded by separatrices must be handled separately.

We will consider the region bounded in the $x=0$ plane by the lines $z=0$, $y_e=0$, $y_n=1$ (cf. Fig. 2); this region always includes the stablest fixed point.[37] The two-dimensional separ-surfaces that bound the region for $x=0$ are $z=0$, $y_e=0$ and that given by $y_n=\varphi_n(x,y_e/y_n)$, where φ_n is found to $O(\epsilon)$ by the methods of Ref. 4 to be

$$\varphi_n=(1-x)^{d/2}\exp\left\{\epsilon\frac{x}{2}\left[1-2\left(\Delta_n+\frac{y_e}{y_n}\right)\right]\right\}. \qquad (6.7)$$

The auxiliary function Y_n is similarly determined to be [again to $O(\epsilon)$]

$$Y_n=\left(1-\frac{y_n}{\varphi_n}\right)\exp\left[\epsilon x\frac{y_n}{\varphi_n}\left(\Delta_n+\frac{y_e}{y_n}\right)\right]. \qquad (6.8)$$

Y_e can now be determined by use of (6.4) and the solution is complete.

We now define the nonlinear scaling fields

$$S_{(i)}\equiv\frac{x}{XY_n^{\Delta_n}Y_e}, \qquad (6.9a)$$

$$S_{(ii)}\equiv\frac{x}{X^{1-2\Delta_n}y_n^{\Delta_n}Y_e}, \qquad (6.9b)$$

$$S_{(\text{iii})} \equiv \frac{x\, Y_n^{\Delta_n}\, X}{y_e}, \tag{6.9c}$$

$$S_{(\text{iv})} \equiv \frac{x}{X^{1-2\Delta_g}\, y_n^{\Delta_g}} \frac{y_n}{y_e}. \tag{6.9d}$$

These scaling fields satisfy

$$\frac{\partial S_{(j)}}{\partial l} = \lambda_T^j\, S_{(j)}. \tag{6.9e}$$

The free energy is a generalized homogeneous function[28] of the ordering field h and the $S_{(j)}$:

$$G(h, \{S_{(j)}\}) = \mu^{-d} G(\mu^{\lambda_h} h, \{\mu^{\lambda_T^j} S_{(j)}\}), \tag{6.10}$$

where

$$j = \text{i}, \text{ii}, \text{iv} \quad \text{and} \quad \lambda_h = d + \tfrac{1}{2}(2-d).$$

A general solution of this form will have singularities on the bounding surfaces, in particular, on the separatrix leaving the stablest fixed point.[4] If these singularities are to be eliminated, we must choose the form of the function in (6.10) carefully. This is equivalent to defining new scaling fields which avoid the singularities. From a *phenomenological* point of view, such a "super-scaling-field" is not unique. It need only satisfy certain limiting properties. For example, when $n < 4$ and the Fisher-renormalized fixed point is stablest, we must have[4] the limiting critical behavior of that point everywhere on the interior of the critical surface. Thus, we wish to define S_T such that

$$S_T \to S_{(\text{iv})}, \tag{6.11a}$$

for $x \to 0$, $y_e \neq 0$. On the other hand, if $y_e \to 0$ with y_n fixed, we require that

$$S_T \to S_{(\text{ii})}^{\lambda_T^{\text{iv}}/\lambda_T^{\text{ii}}}. \tag{6.11b}$$

Finally, as $y_n \to 0$ ($y_e \to 0$ also, to stay in the solution region) we require

$$S_T \to S_{(\text{i})}^{\lambda_T^{\text{iv}}/\lambda_T^{\text{i}}}. \tag{6.11c}$$

It is always possible to define such a function. With S_T satisfying (6.11) $G(h, S_T)$ has no singularities,

$$G(h, S_T) = \mu^{-d} G(\mu^{\lambda_h} h, \mu^{\lambda_T^{\text{iv}}} S_T). \tag{6.12}$$

The proper scaling field can be found by calculating the equation of state using the nonlinear scaling fields. The solution in the remaining region (bounded by $z = 0$, $y_n = 0$, and the separ-surface) and the equation of state is treated elsewhere.[38]

VII. EPILOGUE

In this work we have shown that simplifications can be achieved in renormalization-group calculations by altering the structure of the renormalization-group equations prior to detailed calculation. In each case treated here, we reduced the full renormalization-group equations to a nonlinear differential equation. While such equations are not always easy to handle, they are far more tractable and familiar than nonlinear functional integro-differential equations.

The most general generator derived, (5.9), is, of course, approximate. However, many renormalizations-group problems are studied in lowest order of perturbation theory; the use of the full renormalization equations is clearly superfluous. Moreover, in special circumstances [such as the paired-spin-momenta and large-n (even-$\mathcal{O}$) limits], these simpler generators become exact.

On a more philosophical level, it seems likely that approximate generators accurate to higher order in perturbation theory can also be constructed. In the Callan-Symanzik approach to critical phenomena, first-order linear partial differential equations for the spatially uniform limit of the thermodynamic functions (i.e., constant magnetic field and magnetization) are constructed. The coefficients in these equations are nonlinear functions of the renormalized Hamiltonian parameters which must be calculated by field-theoretic Feynman-diagram expansions. In a separate work, we show that renormalization-group generators such as the Wegner-Houghton generator can be reformulated as generators for the free energy and equation of state. The approximate generator (5.5a) can in the same way be converted into an approximate generator for the thermodynamic functions.[38] The equations are highly nonlinear, but, in contrast to the Callan-Symanzik method, they are self-contained equations. The nonlinear structure automatically incorporates the results of the diagram expansion. Since we are usually interested in the spatially uniform limit, we might expect to be able to construct renormalization equations involving only spatially uniform quantities. The approximate generators given in this work include these effects to lowest order; the higher-order effects of the spin fluctuations might be incorporated by increasing the degree of nonlinearity of the generators. This would correspond to being able to give operator definitions for the coefficients of the Callan-Symanzik equations. The generators studied here represent the first stage in the development of a hierarchy of generators, each incorporating a higher order of spin fluctuations and, hence, a more precise description of thermodynamic functions.

Note added in proof. After the submission of this manuscript, it came to our attention that F. J. Wegner [Phys. Lett. 54A, 1 (1975)] has independently indicated the equivalence of the Wegner-Houghton and Wilson generators to first order and has recalculated the values of η given in Ref. 16.

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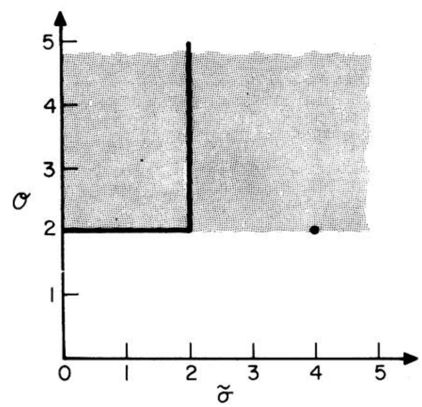

FIG. 1. Schematic plot of order $\mathcal{O}$ (defined in Sec. I) vs propagator exponent $\tilde{\sigma}$ (where the critical propagator varies with momentum as $|\vec{k}|^{\tilde{\sigma}}$). For the case of long-range forces with interaction strength $1/r^{d+\sigma}$, $\tilde{\sigma}=\sigma$ for $\sigma \leq 2$, and $\tilde{\sigma}=2$ otherwise; the case $\tilde{\sigma}=2L$ (L a positive integer ≥ 2) corresponds to a "generalized Lifshitz point." The heavy lines and solid circle correspond to previously treated special cases: (a) The vertical line indicates case $\tilde{\sigma}=2$ and $\mathcal{O}$ arbitrary (Ref. 13); (b) the horizontal line indicates the case $\mathcal{O}=2$, $\tilde{\sigma} \leq 2$ (Ref. 18); and (c) the heavy circle indicates the Lifschitz point $\tilde{\sigma}=4$ (Ref. 19). In Sec. II this previous work is extended to all meaningful values of both $\tilde{\sigma}$ and $\mathcal{O}$ (shown shaded).