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# Landau theory of a moderately dense Boltzmann gas

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The moderately dense quantum Boltzmann gas at equilibrium is shown to be equivalent to a gas of "dressed particles," or "quasiparticles," like those of Landau's theory of a Fermi liquid. The Landau quasiparticle interaction energy $F(p)$ is related to the second virial coefficient by means of an inverse Laplace transform. Expressions are obtained for $F(p)$, and consequently the "renormalized" single-quasiparticle energy $\epsilon_p$, in terms of both scattering phase shifts and the Møller operator. The theory is compared with the Hartree approximation, the results of Allen, Nicolis *et al.*, the work of Baerwinkel and Grossmann, and with expressions from Fermi-fluid theory.

## I. INTRODUCTION

A successful approach to the study of an interacting many-body system is the Landau theory of a Fermi liquid.[1] The theory has primarily been applied to degenerate Fermi systems. This article discusses the application of Landau theory to a moderately dense Boltzmann gas.

In Landau's theory, a fluid of $N$ particles with kinetic energy $p^2/2m$ and two-body interaction $V(r)$ is viewed as a system of interacting *quasiparticles* with a momentum-dependent single-quasiparticle energy $\epsilon_p$ and a momentum-dependent energy of interaction $F_{\vec{p}_1\vec{p}_2}$. The effects of the position-dependent potential $V$ are absorbed into the functions $\epsilon_p$ and $F_{\vec{p}_1\vec{p}_2}$.

The applications of Landau's theory have been directed primarily to the study of liquid helium-3 at temperatures between 2 and 50 mK. Although it was originally thought that Landau's quasiparticle picture is valid only in the extreme low-temperature limit of a Fermi fluid,[2] in fact, a quasiparticle picture may be constructed at arbitrarily high temperatures.[3] In general, $\epsilon_p$ and $F_{\vec{p}_1\vec{p}_2}$ are temperature dependent, thus reflecting the many-body nature of the transformation of particles to quasiparticles.

We expect that the Landau quasiparticle energy, whose temperature-dependent form is understood for degenerate Fermi fluids, can be analytically continued above the Fermi degeneracy temperature $T_F$ and into the region $T \gg T_F$ where the effects of statistics are negligible, that is, into the region of an interacting Boltzmann gas. Investigations along these lines have been made by Baerwinkel and Grossmann[4-6]; however, their formalism is valid only to first order in the interaction potential $V$ or in the scattering amplitude $t$. For most realistic intermolecular potentials a power series in $V$ is inappropriate because of hard-core effects, and an ordering in powers of $t$ is ambiguous. Furthermore, Baerwinkel and Grossmann find it necessary to make *ad hoc* corrections to Landau's equation for the pressure tensor,[6] whereas, as shall be shown, Landau's pressure-tensor equation at equilibrium is correct as it stands, provided the quasiparticle picture is properly formulated.

A theory of the moderately dense Boltzmann gas has also been developed[7-9] in terms of dressed or "physical" particles. An expression for the low-momentum limit of the dressed-particle energy has been derived classically by Allen and Nicolis[7] and verified for the leading quantum corrections by Colinet.[8] Moreover, Clavin and Wallenborn[9] have demonstrated the equivalence of the "physical particle" formalism to Landau's theory. As is shown in Sec. IV, the low-momentum limit of the "quasiparticle" energy derived in this paper agrees with that of Refs. 7–9 to within a constant.

In addition, certain previous articles concerning low-temperature Fermi fluids[10,11] contain expressions for the Landau interaction energy in terms of scattering phase shifts which, as is shown in Sec. V, are applicable to the moderately dense Boltzmann gas. The present paper therefore unifies several previous treatments.

The object of this paper is to derive a Landau-like theory of a moderately dense Boltzmann gas for arbitrarily strong, isotropic, short-range potentials, with the exception that the potential is not allowed to support bound states. The density regime is such that the contribution of the second virial coefficient to the equation of state is important, but contributions of third and higher virial coefficients are negligible. It is shown, in fact, that the Landau interaction energy $F_{\vec{p}\vec{p}'}$ is most closely associated with the second virial coefficient $B$. Reasonably, at such low densities $F_{\vec{p}\vec{p}'}$ reduces to a function $F(\frac{1}{2}|\vec{p}-\vec{p}'|)$ of the *relative* momentum of a pair of quasiparticles. $F$ is also temperature independent and so describes purely dynamical aspects of the interacting gas.

Explicit forms for $F(p)$ are deduced. The derived forms for $\epsilon_p$ and $F(p)$ are shown to agree with both those of Grossmann[4] and the well-known Hartree approximation[12] in the weak-potential limit.

## II. STATISTICAL THERMODYNAMICS

The Landau quasiparticle picture was originally conceived within the framework of time-dependent perturbation theory. In the infinite past the fluid is assumed to be a gas of free particles in a volume $\mathcal{V}$, so that the particles occupy discrete states with momentum index $\vec{p}$ and energy $p^2/2m$. The interparticle potential $V(r)$ is turned on adiabatically, i.e., so slowly that no particles change states. What results is a fluid of "quasiparticles," or "dressed" particles, which occupy states with the original momentum labels $\vec{p}$ but with a "renormalized" energy $\epsilon_p \neq p^2/2m$. The total system energy $E$ is then a functional of the quasiparticle state occupation numbers $n_p$.

A difficulty encountered with this picture for Fermi fluids at low but nonzero temperatures is that the quasiparticle lifetimes are no longer infinite. The quasiparticle states decay in a time shorter than that needed to turn on the interaction adiabatically. Hence, the entire picture breaks down.[2] However, Balian and De Dominicis[3] have devised an alternative Landau-like formalism which is valid at an arbitrary temperature. The "statistical" quasiparticle energies of Balian and De Dominicis are always real, in contrast to the earlier "dynamical" quasiparticle energies which have a positive imaginary part. The two forms of quasiparticle energies coincide only at zero temperature and on the Fermi surface.[11]

It is shown in Sec. V that the Landau picture of adiabatically shifted energy levels may be justified explicitly for a moderately dense Boltzmann gas. However, for purposes of initial discussion the precise transformation from particles to quasiparticles is left unspecified except for certain general requirements: (1) the single-quasiparticle states must obey the same boundary conditions as the free-particle states and must obey the same statistics (Boltzmann, Fermi, or Bose) as the particles; (2) the number of quasiparticles must equal the number of particles; and (3) the total system energy must be a functional of only the quasiparticle occupation numbers $n_p$.

With these assumptions, the arguments of Landau[1] suffice to specify the statistical thermodynamics of the system. The entropy is found by the same combinatorial considerations as that of an ideal gas, namely

$$S = -k_B \sum_p Y(n_p), \tag{1}$$

where

$$\begin{aligned} Y(n_p) &= n_p \ln n_p - n_p \quad \text{(Boltzmann)} \\ &= n_p \ln n_p + (1-n_p)\ln(1-n_p) \quad \text{(Fermi)} \\ &= n_p \ln n_p - (1+n_p)\ln(1+n_p) \quad \text{(Bose)}. \end{aligned} \tag{2}$$

The sum is over quasiparticle states and includes a sum over spin where appropriate. The Boltzmann entropy has been defined so that it is additive. According to Landau, the single-quasiparticle energy $\epsilon_p$ is defined to be the partial derivative of the energy with respect to the distribution function,

$$\epsilon_p \equiv \partial E/\partial n_p . \tag{3}$$

In general, $\epsilon_p$ depends on density and temperature, as well as the interaction potential.

In the infinite-volume limit, the states are no longer discrete, and as usual, the sum over $p$ is replaced by an integral,

$$\sum \to \mathcal{V} h^{-3}\overline{\omega} \int d\vec{p}, \tag{4}$$

where $\overline{\omega}$ is a spin degeneracy factor. In this limit, the partial derivative of Eq. (3) becomes a functional derivative; i.e., $\epsilon_p \equiv \delta E/\delta n_p$. Note that Eq. (3) does *not* imply $E = \sum_p \epsilon_p n_p$ [see Eq. (15)]. Although the spin degeneracy factor $\overline{\omega}$ may easily be incorporated into the formalism, it adds nothing conceptually, but complicates various equations with an extra factor. For this reason, $\overline{\omega}$ is here replaced by 1 whenever dealing with Boltzmann gases.

The entropy is maximized subject to the constraints of constant total energy and particle number,

$$k_B^{-1}\delta S - \alpha\,\delta N - \beta \delta E = -\mathcal{V} h^{-3}\overline{\omega} \int d\vec{p}\left(\frac{\partial Y}{\partial n_p} + \alpha + \beta\epsilon_p\right)\delta n_p = 0 . \tag{5}$$

The equilibrium quasiparticle distribution function is thus found to be

$$\begin{aligned} n_p &= \exp[-\beta(\epsilon_p - g)] \quad \text{(Boltzmann)} \\ &= \{\exp[\beta(\epsilon_p - g)] + 1\}^{-1} \quad \text{(Fermi)} \\ &= \{\exp[\beta(\epsilon_p - g)] - 1\}^{-1} \quad \text{(Bose)}, \end{aligned} \tag{6}$$

where the Lagrange multipliers have been identified in terms of the chemical potential $g$ and temperature $T$ according to $\alpha = -\beta g$ and $\beta = (k_B T)^{-1}$. Although these expressions are formally identical to their ideal-gas counterparts, interactions are included implicitly by means of the quasiparticle energy $\epsilon_p$. Equation (1) for the Fermi case becomes a useful starting point for deriving the thermodynamic properties of a low-temperature Fermi fluid.[13]

Landau has also developed a hydrodynamic theory in quasiparticle form.[1] Here the quasiparticles are assumed to obey a Boltzmann equation in which the quasiparticle energy plays the role of a single-particle Hamiltonian. Equations of change for momentum and energy are then derived, and the pressure (momentum-flux) tensor is found to have the form

$$\Pi_{ik}=h^{-3}\varpi\int d\vec{p}\,p_i n_p\frac{\partial\epsilon_p}{\partial p_k}+\delta_{ik}\left(h^{-3}\varpi\int d\vec{p}\,n_p\epsilon_p-\frac{E}{\mathcal{V}}\right). \tag{7}$$

It turns out that, at equilibrium, $\Pi_{ik}=P\delta_{ik}$ where $P$ is the pressure. This result is actually somewhat surprising, since in the derivation of Eq. (7), Landau assumed that the collision integral conserves momentum, whereas the generalized collision integral for a moderately dense Boltzmann gas is nonlocal and so does not conserve momentum.[14] The non-momentum-conserving terms must then be included in a consistent derivation of the pressure tensor.[14] The proof that Eq. (7) gives the equilibrium pressure has been presented previously for a Fermi fluid,[13] but is repeated here in order to demonstrate its validity for all statistics.

At equilibrium, $\Pi_{ik}=\overline{P}\delta_{ik}$ by isotropy, where

$$\begin{aligned}\overline{P}&\equiv h^{-3}\varpi\int_0^\infty 4\pi p^2\,dp\,n_p\left(\frac{1}{3}p\frac{\partial\epsilon_p}{\partial p}+\epsilon_p\right)-\frac{E}{\mathcal{V}}\\&=-h^{-3}\varpi\int_0^\infty\tfrac{4}{3}\pi p^3\,dp\,\epsilon_p\frac{\partial n_p}{\partial p}-\frac{E}{\mathcal{V}}.\end{aligned} \tag{8}$$

The last line follows from an integration by parts.

The condition of maximum entropy, Eq. (5), implies that

$$\frac{\partial Y}{\partial p}=-\beta(\epsilon_p-g)\frac{\partial n_p}{\partial p}. \tag{9}$$

As a consequence, the entropy, Eq. (1), may be rewritten as

$$\begin{aligned}S&=-k_B\mathcal{V}\varpi h^{-3}\int_0^\infty 4\pi p^2 Y\,dp\\&=-\mathcal{V}\varpi T^{-1}h^{-3}\int_0^\infty\tfrac{4}{3}\pi p^3\,dp\,(\epsilon_p-g)\frac{\partial n_p}{\partial p}\\&=-\mathcal{V}\varpi T^{-1}h^{-3}\\&\quad\times\left(\int_0^\infty\tfrac{4}{3}\pi p^3\,dp\,\epsilon_p\frac{\partial n_p}{\partial p}+g\int_0^\infty 4\pi p^2\,dp\,n_p\right)\\&=-\mathcal{V}\varpi T^{-1}h^{-3}\int_0^\infty\tfrac{4}{3}\pi p^3\,dp\,\epsilon_p\frac{\partial n_p}{\partial p}-\frac{Ng}{T}.\end{aligned} \tag{10}$$

The last line is a result of the condition $\mathcal{V}h^{-3}\varpi\times\int d\vec{p}\,n_p=N$. Finally, comparison of Eqs. (8) and (10) yields

$$\overline{P}=(TS+Ng-E)/\mathcal{V}=P, \tag{11}$$

which is the desired result.

For a Boltzmann gas, each of the terms in the first line of Eq. (8) can be evaluated separately. The first term is ($\varpi=1$)

$$\tfrac{1}{3}h^{-3}\int d\vec{p}\,n_p p\frac{\partial\epsilon_p}{\partial p}=\tfrac{1}{3}h^{-3}\int_0^\infty 4\pi p^3\frac{\partial\epsilon_p}{\partial p}n_p\,dp \tag{12}$$

and, from Eq. (6) for Boltzmann gases,

$$\frac{\partial n_p}{\partial p}=-\beta n_p\frac{\partial\epsilon_p}{\partial p}, \tag{13}$$

so that

$$\begin{aligned}\tfrac{1}{3}h^{-3}\int d\vec{p}\,n_p p\frac{\partial\epsilon_p}{\partial p}&=-\tfrac{1}{3}k_B T h^{-3}\int_0^\infty 4\pi p^3\,dp\frac{\partial n_p}{\partial p}\\&=k_B T h^{-3}\int_0^\infty 4\pi p^2\,dp\,n_p\\&=(N/\mathcal{V})k_B T=nk_B T.\end{aligned} \tag{14}$$

It follows immediately that

$$h^{-3}\int d\vec{p}\,n_p\epsilon_p=(E/\mathcal{V})+P-nk_B T, \tag{15}$$

which confirms for an interacting Boltzmann gas, that $E$ and $\sum_p n_p\epsilon_p$ are not identical. Equations (14) and (15) apply only to Boltzmann statistics, whereas Eq. (11) holds for all statistics.

## III. DENSITY EXPANSIONS

The densities of interest are such that the second virial coefficient is important but higher terms in the virial series may be neglected. The energy, pressure, and chemical potential are thus given by

$$E=Nk_B T[\tfrac{3}{2}-nT\partial B/\partial T+O(n^2)], \tag{16}$$

$$P=nk_B T[1+nB+O(n^2)], \tag{17}$$

$$g=k_B T\ln(n\lambda^3)+2nk_B TB+O(n^2), \tag{18}$$

where $\lambda$ is the thermal wavelength and $B$ is the second virial coefficient. It is important to note that Eqs. (16) and (18) may be derived from Eq. (17) by purely thermodynamic arguments. In consequence, these relations remain valid for any approximation to $B$, as long as it is applied self-consistently.

In the limit of zero density, the gas behaves like a free-particle system and the quasiparticles are completely equivalent to the particles. If the quasiparticle energy $\epsilon_p$ is assumed to be an analytic function of density, $\epsilon_p$ must have the form

$$\epsilon_p = p^2/2m + \delta\epsilon_p \tag{19}$$

where, to leading order, $\delta\epsilon_p$ is $O(n)$. The free-particle distribution function is

$$n_p^0 = \exp[\beta(g^0 - p^2/2m)] = n\lambda^3 \exp(-\beta p^2/2m), \tag{20}$$

where $g^0 = k_B T \ln(n\lambda^3)$. It follows that the quasiparticle distribution $n_p$ may be written in terms of the free-particle distribution function according to

$$n_p = n_p^0 + \delta n_p = n_p^0[1 - \beta(\delta\epsilon_p - \delta g) + O(n^2)]. \tag{21}$$

This follows from Eq. (6), where the exponential has been expanded in a power series in density $n$, with $\delta g \equiv g - g^0$. The condition that the particle and quasiparticle numbers are equal, namely

$$\mathcal{V}h^{-3}\int d\vec{p}\,\delta n_p = 0, \tag{22}$$

implies that $\delta g$, and $B$, see Eq. (18), are related to $\delta\epsilon_p$ according to

$$\mathcal{V}h^{-3}\int d\vec{p}\, n_p^0 \delta\epsilon_p = N\delta g + O(n^2)$$
$$= 2Nnk_B TB + O(n^2). \tag{23}$$

The density expansion of $n_p^0$ and $\epsilon_p$ also leads to the relation

$$\mathcal{V}h^{-3}\int d\vec{p}\,\epsilon_p n_p = \mathcal{V}h^{-3}\int d\vec{p}\,(p^2/2m + \delta\epsilon_p)(n_p^0 + \delta n_p) = \tfrac{3}{2}Nk_B T + 2Nnk_B TB + \mathcal{V}h^{-3}\int d\vec{p}\,\delta n_p p^2/2m + O(n^3)$$
$$= \tfrac{3}{2}Nk_B T + Nnk_B T(B - T\,\partial B/\partial T) + O(n^3), \tag{24}$$

where the last line is a combination of Eqs. (15)–(17). As a consequence, the relation

$$\mathcal{V}h^{-3}\int d\vec{p}\,\delta n_p p^2/2m = -Nnk_B T(B + T\,\partial B/\partial T) + O(n^3) \tag{25}$$

is also valid.

Recall that $\epsilon_p$ is defined as the functional derivative of $E$ with respect to the *quasiparticle* distribution function. This provides motivation for writing $E$ as a functional of $n_p$ in the form

$$E = \mathcal{V}h^{-3}\int d\vec{p}\,(p^2/2m)n_p + \tfrac{1}{2}\mathcal{V}h^{-6}\int d\vec{p}_1\, d\vec{p}_2\, F_{\vec{p}_1\vec{p}_2} n_{p_1} n_{p_2} + \cdots, \tag{26}$$

where $F_{\vec{p}_1\vec{p}_2}$ is initially an unknown function of $\vec{p}_1$ and $\vec{p}_2$. To the order in density of present interest, Eq. (26) may be rewritten as

$$E = \mathcal{V}h^{-3}\int d\vec{p}\,(p^2/2m)(n_p^0 + \delta n_p) + \tfrac{1}{2}\mathcal{V}h^{-6}\int d\vec{p}_1\, d\vec{p}_2\, F_{\vec{p}_1\vec{p}_2} n_{p_1}^0 n_{p_2}^0 + \cdots. \tag{27}$$

It then follows from Eqs. (16) and (25) that

$$\mathcal{V}h^{-6}\int d\vec{p}_1\, d\vec{p}_2\, F_{\vec{p}_1\vec{p}_2} n_{p_1}^0 n_{p_2}^0 = 2Nnk_B TB + O(n^3). \tag{28}$$

Equation (28) is a *necessary* condition on $F_{\vec{p}_1\vec{p}_2}$ for a Landau-like quasiparticle representation. It is now shown that a set of conditions on $F_{\vec{p}_1\vec{p}_2}$ *sufficient* to generate a Landau-like quasiparticle representation is: (1) that $F_{\vec{p}_1\vec{p}_2}$ obey Eq. (28); (2) be symmetric in $\vec{p}_1$ and $\vec{p}_2$; and (3) be independent of temperature.

To prove that these conditions on $F_{\vec{p}_1\vec{p}_2}$ are sufficient, the quantity $\overline{E}$ is constructed as follows:

$$\overline{E} \equiv \mathcal{V}h^{-3}\int d\vec{p}\,(p^2/2m)n_p + \tfrac{1}{2}\mathcal{V}h^{-6}\int d\vec{p}_1\, d\vec{p}_2\, F_{\vec{p}_1\vec{p}_2} n_{p_1} n_{p_2}, \tag{29}$$

with the understanding that $n_p$ is given in terms of $\epsilon_p$ by Eq. (6) for a Boltzmann gas and where $\epsilon_{p_1}$ is chosen to be

$$\epsilon_{p_1} = \frac{\delta\overline{E}}{\delta n_{p_1}} = \frac{p_1^2}{2m} + h^{-3}\int d\vec{p}_2\, F_{\vec{p}_1\vec{p}_2} n_{p_2}. \tag{30}$$

In order to show that this is consistent with Landau's quasiparticle picture, it is necessary only to prove that $\overline{E} = E + O(n^3)$.

With the energy-momentum relation, Eq. (30), Eq. (29) may be written as

$$\overline{E} = \mathcal{V}h^{-3}\int d\vec{p}\,(p^2/2m)n_p^0\left(1 + \beta\delta g - \beta h^{-3}\int d\vec{p}'\, F_{\vec{p}\vec{p}'} n_{p'}^0\right) + \tfrac{1}{2}\mathcal{V}h^{-6}\int d\vec{p}\, d\vec{p}'\, F_{\vec{p}\vec{p}'} n_p^0 n_{p'}^0 + O(n^3)$$
$$= \tfrac{3}{2}Nk_B T + 4Nnk_B TB - \tfrac{1}{2}\beta\mathcal{V}h^{-6}\int d\vec{p}\, d\vec{p}'\left(\frac{p^2}{2m} + \frac{p'^2}{2m}\right) n_p^0 n_{p'}^0 F_{\vec{p}\vec{p}'} + O(n^3), \tag{31}$$

where the assumed symmetry of $F$ has been used, and $\delta g$ is given by Eq. (23). Keeping in mind that $F$ is also assumed to be independent of $T$, Eqs. (20) and (28) imply that

$$-2Nnk_B\frac{d(TB)}{d\beta}=2Nnk_B^2T^2\left(B+T\frac{dB}{dT}\right)=\mathcal{V}h^{-6}\int\int d\vec{p}\,d\vec{p}'\left(\frac{p^2}{2m}+\frac{p'^2}{2m}\right)n_p^0n_{p'}^0F_{\vec{p}\,\vec{p}'}-6Nn(k_BT)^2B+O(n^3). \tag{32}$$

Finally, on comparing Eqs. (28), (31), and (32), the desired result,

$$\overline{E}=\tfrac{3}{2}Nk_BT-Nnk_BT^2\,dB/dT+O(n^3)$$
$$=E+O(n^3) \tag{33}$$

is obtained. Moreover, the quasiparticle distribution function necessarily satisfies the number conservation law, Eq. (22), because $\delta g$ is related to $B$ as given in Eq. (23), or Eq. (18).

The function $F_{\vec{p}\,\vec{p}'}$ is, to leading order in density, the Boltzmann analog of the familiar Landau $f$ function,[1] the functional derivative of the quasiparticle energy with respect to the quasiparticle occupation number,

$$F_{\vec{p}\,\vec{p}'}=\mathcal{V}\frac{\delta\epsilon_p}{\delta n_{p'}}=\mathcal{V}\frac{\delta^2E}{\delta n_p\delta n_{p'}}. \tag{34}$$

This is obviously symmetric in $\vec{p}$ and $\vec{p}'$. The present expression is, in fact, merely the zero-density limit of Landau's $f$ function. It is likely that explicitly temperature-dependent terms are present in higher-density corrections for $F$ as defined by Eq. (34).

## IV. EXISTENCE AND UNIQUENESS OF THE LANDAU FUNCTION

The conditions on $F$ are now examined to determine whether a solution for $F$ exists and, if so, whether the solution is unique. Since only the first-order density dependence of $\epsilon$ is considered here, $F$ should depend only on binary interactions. It should therefore be a function of the magnitude of the relative momentum $\vec{p}$,

$$\vec{p}\equiv\tfrac{1}{2}(\vec{p}_2-\vec{p}_1). \tag{35}$$

Note that this condition does *not* hold for degenerate Fermi systems.[11,15]

The variables of integration in Eq. (28) may be changed to $\vec{p}$ and the total momentum $\vec{P}=\vec{p}_1+\vec{p}_2$. The integral over $\vec{P}$ can be evaluated explicitly, with the result

$$\int F(p)\exp(-\beta p^2/2\mu)\,d\vec{p}=2(2\pi\mu)^{3/2}\beta^{-5/2}B. \tag{36}$$

Here $\mu=\frac{1}{2}m$ is the reduced mass. Alternatively, Eq. (36) may be expressed in terms of the relative kinetic energy $\mathcal{E}\equiv p^2/2\mu$:

$$\int_0^\infty \mathcal{E}^{1/2}F(\mathcal{E})e^{-\beta\mathcal{E}}\,d\mathcal{E}=\pi^{1/2}\beta^{-5/2}B(\beta). \tag{37}$$

Since $F$ is by hypothesis independent of $\beta$, it is seen that $\mathcal{E}^{1/2}F(\mathcal{E})$, as a function of $\mathcal{E}$, is the inverse Laplace transform of $\pi^{1/2}\beta^{-5/2}B(\beta)$ with respect to the conjugate variable $\beta$. Therefore, if null functions are disallowed, $F$ is unique.

As long as $F(\mathcal{E})$ is required to be a regular function of $\mathcal{E}$, a solution does not exist unless

$$\lim_{\beta\to 0}\beta^{-5/2}B(\beta)=0. \tag{38}$$

Thus for classical gases, a solution cannot exist if $V(r)$ has an attractive part, as is now shown. The classical expression for $B$ is

$$B=-2\pi\int_0^\infty r^2\,dr\,(e^{-\beta V(r)}-1). \tag{39}$$

Suppose that $V(r)<-V_0$, where $V_0$ is a positive number, in the interval $a\leq r\leq b$. The contribution $B_{ab}$ to the virial coefficient from this interval satisfies the inequality

$$|B_{ab}|>\tfrac{2}{3}\pi(b^3-a^3)(e^{\beta V_0}-1). \tag{40}$$

Thus $\beta^{-5/2}|B_{ab}|$, and hence $\beta^{-5/2}|B|$, increases without bound as $\beta\to 0$.

The quantum expression for the second virial coefficient of a Boltzmann gas is[16]

$$B=\tfrac{1}{2}h^3(2\pi\mu k_BT)^{-3/2}\,\mathrm{Tr}\,(e^{-\beta K}-e^{-\beta H}), \tag{41}$$

where the trace is over the relative coordinates of the two-particle system. $K$ and $H$ are respectively the kinetic-energy operator and total Hamiltonian in relative coordinates. If the potential $H$ has bound states with binding energy $|\mathcal{E}_i|$, then terms are present in $B$ proportional to $e^{-\beta\mathcal{E}_i}>1$. It follows that $\beta^{-5/2}B$ does not approach zero as $\beta\to 0$. Consequently, $F$ may exist only if $V(r)$ does not support bound states. This condition on $V(r)$ is less restrictive than in the classical case, since an attractive part of $V(r)$ may not support bound states if it is sufficiently shallow. A counterpart to this condition occurs in Fermi-fluid theory, where it is known that the Landau quasiparticle picture is not valid for the extraordinary phases of liquid $^3$He below 2 mK, believed to be due to a modified form of Cooper pairing.[17]

Since $\mathcal{E}^{1/2}F(\mathcal{E})$ is an inverse Laplace transform, explicit formulas for $F$ may be easily calculated for certain elementary model potentials. For example, for the classical finite barrier potential, i.e.,

$$V(r)=V_0>0,\quad r<a$$
$$=0,\qquad r>a \tag{42}$$

the second virial coefficient is

$$B=\tfrac{2}{3}\pi a^3(1-e^{-\beta V_0}) \tag{43}$$

and the corresponding $F$ function is given by

$$F(p)=\tfrac{4}{9}\pi a^3p^2\mu^{-1},\quad p<(2\mu V_0)^{1/2}$$
$$=\tfrac{4}{9}\pi a^3p^2\mu^{-1}[1-(1-2\mu V_0/p^2)^{3/2}],\quad p>(2\mu V_0)^{1/2}. \tag{44}$$

The result for the rigid-sphere potential is obtained from this by setting $V_0=\infty$.

The fact that $F$ is only a function of the magnitude $p$ enables the expression, Eq. (30), for the quasiparticle energy to be simplified. Straightforward transformations of the integral lead to the equation

$$\epsilon_{p_1}=\frac{p_1^2}{2m}+4np_1^{-1}(2\pi mk_BT)^{-1/2}\int_0^\infty F(p)p\,dp\left[\exp\left(\frac{-\beta(2p-p_1)^2}{2m}\right)-\exp\left(-\frac{\beta(2p+p_1)^2}{2m}\right)\right] \tag{45}$$

for $\epsilon_{p_1}$. This may be compared with the results of Allen and Nicolis[7] for small $p_1$. After expanding the integrand in a Taylor series in $p_1$ and comparing with Eq. (36), it is found that

$$\epsilon_{p_1}=n\beta^{-1}B(2\beta)+\frac{p_1^2}{2m}\left[1+\frac{2n}{3}\left(B(2\beta)-\beta\frac{\partial B(2\beta)}{\partial\beta}\right)\right]+O(p_1^4)$$
$$\equiv E^*+\tfrac{1}{2}m^*v_1^2+O(v_1^4). \tag{46}$$

Here the velocity $v_1$ is $p_1/m$, while the "self-consistent energy" is $E^*\equiv n\beta^{-1}B(2\beta)$ and the "effective mass" $m^*$ is

$$m^*\equiv m\left[1+\frac{2n}{3}\left(B(2\beta)-\beta\frac{\partial B(2\beta)}{\partial\beta}\right)\right]. \tag{47}$$

The above expressions agree with those of Allen and Nicolis[7] for the classical gas and those of Colinet[8] for the quantum gas, except that their $E^*$ contains an additional term $[-n\beta^{-1}B(\beta)]=-\tfrac{1}{2}\delta g$. However, this term may be absorbed into the normalization constant for the momentum distribution function.

For the classical hard-sphere gas, Eqs. (44) and (45) give

$$\epsilon_p=\tfrac{2}{3}\pi na^3k_BT+(p^2/2m)[1+\tfrac{4}{9}\pi na^3], \tag{48}$$

which is exact for all $p$.

## V. $F(p)$ IN TERMS OF PHASE SHIFTS

The continuum contribution to the second virial coefficient of a quantum gas is expressible in terms of scattering phase shifts according to the well-known Beth-Uhlenbeck formula[18]

$$B=-\pi^{-1}h^3(2\pi\mu k_BT)^{-3/2}\times{\sum_l}'(2l+1)\int_0^\infty dp\,\frac{d\eta_l}{dp}\exp\left(-\frac{\beta p^2}{2\mu}\right). \tag{49}$$

Here $\eta_l(p)$ is the phase shift of the $l$th partial wave for two-body scattering with relative momentum $p$. The sum $\sum_l'$ is restricted to odd $l$ for Fermi systems and even $l$ for Bose systems. For a Boltzmann gas, the sum may be taken over all $l$ with a compensating factor of $\tfrac{1}{2}$. When no bound states are allowed, an integration of Eq. (49) by parts with the help of Levinson's theorem[19] [$\eta_l(0)=0$ for no bound states] gives

$$B=-h^3(2\pi\mu)^{-5/2}\beta^{5/2}\times\sum_l(2l+1)\int_0^\infty p\,dp\,\eta_l(p)\exp\left(-\frac{\beta p^2}{2\mu}\right). \tag{50}$$

Comparison with Eq. (36) immediately yields an expression for $F(p)$,

$$F(p)=-h^3(4\pi^2\mu)^{-1}\sum_l(2l+1)\eta_l(p)/p. \tag{51}$$

Equation (51) is formally identical to the expression for the Landau interaction energy of a degenerate Fermi fluid according to Pethick and Carneiro.[11] The meaning of the phase shift in the degenerate Fermi case is not identical to that for isolated two-body scattering, since in the Fermi fluid the Landau interaction energy is not an isotropic function of the relative momentum. Pethick and Carneiro therefore do not use Eq. (51) directly, but rather utilize the fact that the same relationships exist between the Landau interaction energy, the scattering $T$ matrix, and the Heitler $K$ matrix for Fermi fluids as for isolated two-body scattering. An expression formally equivalent to Eq. (51) has also been derived by Balian and De Dominicis[10] in their study of impurity systems in Fermi fluids.

The form for $F(p)$ given by Eq. (51) can be identified explicitly as an effective, momentum-dependent interaction between pairs of particles after the potential has been turned on adiabatically. This is seen by the following arguments: A single pair of particles is considered with their relative coordinates constrained to lie inside a spherical enclosure of radius $R$ and volume $\mathcal{V}$. If the particles are noninteracting, the relative coordinate eigenstates have the form

$$\psi(\vec{r}) \propto j_l(kr) Y_{lm}(\theta, \phi) \tag{52}$$

with boundary condition

$$j_l(kR) = 0 . \tag{53}$$

In addition, the energy of state $\{k, l\}$ is $\hbar^2 k^2/2\mu$. Since the number of states between $k$ and $k+dk$ (summed over all $l$) is, to a good approximation,[20] $(\mathcal{V}/2\pi^2)k^2\,dk$, the normalized occupation probability of a state $\{k, l\}$ is

$$P(k, l) = P(k) = h^3 \mathcal{V}^{-1}(2\pi\mu k_B T)^{-3/2} \exp(-\hbar^2 k^2/2\mu k_B T). \tag{54}$$

After starting with the noninteracting particle pair, the interaction is turned on adiabatically so that $P(k)$ remains unchanged. The energy of each state $\{k, l\}$ shifts by an amount[21]

$$\delta\mathcal{E}(k, l) = -\hbar^2 k \eta_l(k)/\mu R . \tag{55}$$

The expectation value of the energy shift for the pair of particles is

$$\langle \delta\mathcal{E} \rangle = \sum_{k,l} \delta\mathcal{E}(k,l) P(k,l), \tag{56}$$

the sum being over all states. Since appreciable phase shifts occur only if $l \ll kR$, the number of states[20] per unit $k$ for a given value of $l$ is effectively $R/\pi$. Each $\{k,l\}$ state has a degeneracy $(2l+1)$, so that

$$\begin{aligned}\langle \delta\mathcal{E} \rangle &= R\pi^{-1} \sum_l (2l+1) \int_0^\infty dk\, \delta\mathcal{E}(k,l) P(k,l) \\ &= -\hbar^2(\mu\pi)^{-1} h^3 \mathcal{V}^{-1}(2\pi\mu k_B T)^{-3/2} \\ &\quad \times \sum_l (2l+1) \int_0^\infty k\, dk\, \eta_l(k) \exp\left(-\frac{\beta\hbar^2 k^2}{2\mu}\right).\end{aligned} \tag{57}$$

For a gas of $N$ particles, where only binary interactions are considered, the total energy shift $\delta E_{\text{tot}}$ is simply $\langle \delta\mathcal{E} \rangle$ multiplied by the number of distinct pairs of particles, $N(N-1)/2 \simeq N^2/2$, namely

$$\begin{aligned}\delta E_{\text{tot}} &= -Nn\hbar^2(2\pi\mu)^{-1} h^3 (2\pi\mu k_B T)^{-3/2} \\ &\quad \times \sum_l (2l+1) \int_0^\infty k\, dk\, \eta_l(k) \exp\left(-\frac{\beta\hbar^2 k^2}{2\mu}\right).\end{aligned} \tag{58}$$

Upon noting that $\hbar k = p$ is the relative momentum, Eq. (58) may be reexpressed as an integral over the coordinates of particles 1 and 2 separately,

$$\delta E_{\text{tot}} = \tfrac{1}{2}\mathcal{V}^2 h^{-6} \int d\vec{p}_1\, d\vec{p}_2\, n^0_{p_1} n^0_{p_2} \mathcal{V}^{-1} F(p). \tag{59}$$

$F(p)$ is again given by Eq. (51).

Equation (59) demonstrates, in a particularly transparent manner, the physical meaning of $F(p)$. Specifically, $\mathcal{V}^{-1}F(p)$ is an effective momentum-dependent interaction energy which, when summed over all pairs of particles, gives (to leading order in density) the total energy shift of the gas when the potential is switched on adiabatically.

It is important to note that $\delta E_{\text{tot}}$ is not equal to the first density correction to the energy at constant temperature as given by Eq. (16); that is, $\delta E_{\text{tot}}$ does *not* equal $-Nnk_B T^2\, dB/dT$. Instead, $\delta E_{\text{tot}}$ equals $Nnk_B TB$, as can easily be seen by a simple thermodynamic argument. Essentially, the adiabatic shift, as calculated above, is at constant entropy. Consequently, to leading order in the density,

$$\delta E_{\text{tot}} = n\left[\frac{\partial(E - E_{\text{id}})}{\partial n}\right]_{N,S} = -\mathcal{V}\left[\frac{\partial(E - E_{\text{id}})}{\partial \mathcal{V}}\right]_{N,S} = \mathcal{V}(P - P_{\text{id}}). \tag{60}$$

The subscript "id" refers to the ideal-gas value. In order to find the energy shift at constant temperature, the probability $P(k,l)$ must be modified for the interacting system. In particular, the factor $\delta\mathcal{E}(k,l)P(k,l)$ in Eq. (56) must be changed to

$$[\delta\mathcal{E}(k,l) + \hbar^2 k^2/2\mu]\, P'(k,l) - (\hbar^2 k^2/2\mu) P(k,l),$$

where $P'(k,l)$ is proportional to $\exp\{-\beta[\delta\mathcal{E}(k,l) + \hbar^2 k^2/2\mu]\}$ and is suitably normalized. The result for the energy change at constant temperature then agrees with Eq. (16).

## VI. $F(p)$ IN TERMS OF MØLLER OPERATORS

An alternate expression for $F(p)$ which does not require a partial-wave decomposition of the two-body scattering functions is now presented. The starting point is Eq. (41), the quantum-mechanical expression for the second virial coefficient.

The trace in Eq. (41) can be written in position representation and then transformed by means of an integration by parts according to

$$\begin{aligned}\operatorname{Tr}(e^{-\beta K} - e^{-\beta H}) &= \int d\vec{r}\, \langle \vec{r} | e^{-\beta K} - e^{-\beta H} | \vec{r} \rangle = -\frac{1}{3}\int d\vec{r}\, \vec{r} \cdot \frac{\partial}{\partial \vec{r}} \langle \vec{r} | e^{-\beta K} - e^{-\beta H} | \vec{r} \rangle \\ &= -\frac{1}{3}\int d\vec{r}\, d\vec{p}\, d\vec{p}'\, \vec{r} \cdot \frac{\partial}{\partial \vec{r}} \langle \vec{r} | \vec{p} \rangle \langle \vec{p} | e^{-\beta K} - e^{-\beta H} | \vec{p}' \rangle \langle \vec{p}' | \vec{r} \rangle \\ &= -\tfrac{1}{3} i\hbar^{-1} \int d\vec{r}\, d\vec{p}\, d\vec{p}'\, \langle \vec{r} | \vec{r}_{\text{op}} | \vec{p} \rangle \cdot \langle \vec{p} | [\vec{p}_{\text{op}}, e^{-\beta K} - e^{-\beta H}]_- | \vec{p}' \rangle \langle \vec{p}' | \vec{r} \rangle \\ &= \tfrac{1}{3} i\hbar^{-1} \operatorname{Tr} \vec{r}_{\text{op}} \cdot [\vec{p}_{\text{op}}, e^{-\beta H}]_- = \tfrac{1}{3} \operatorname{Tr} \vec{r}_{\text{op}} \cdot \frac{\partial e^{-\beta H}}{\partial \vec{r}_{\text{op}}}.\end{aligned} \tag{61}$$

Now $\partial H/\partial \vec{r}_{op} = \partial V/\partial \vec{r}_{op}$, but $\partial V/\partial \vec{r}_{op}$ does not commute with $H$. The gradient of $e^{-\beta H}$ thus involves the Kubo transform[22] of $\partial V/\partial \vec{r}_{op}$, i.e.,

$$\frac{\partial e^{-\beta H}}{\partial \vec{r}_{op}} = -\beta \int_0^1 d\alpha\, e^{-(1-\alpha)\beta H} \frac{\partial V}{\partial \vec{r}_{op}} e^{-\alpha \beta H} . \qquad (62)$$

For the sake of simplicity, consider first an approximation for Eq. (62), namely the case in which $\partial V/\partial \vec{r}_{op}$ commutes with $H$. In this approximation, the second virial coefficient becomes

$$B \simeq -\tfrac{1}{6}\beta h^3 (2\pi\mu k_B T)^{-3/2} \operatorname{Tr} \vec{r}_{op} \cdot \frac{\partial V}{\partial \vec{r}_{op}} e^{-\beta H} . \qquad (63)$$

This form was also obtained by Thomas and Snider[14] in their calculation at equilibrium of the interaction-dependent part of the pressure tensor.

In order to construct $F(p)$, it is convenient to rewrite the trace of Eq. (63) in terms of $e^{-\beta K}$ instead of $e^{-\beta H}$. This may be accomplished with the help of the Møller wave operator $\Omega$, which satisfies the intertwining relation

$$H\Omega = \Omega K \qquad (64)$$

and is unitary, i.e., $\Omega\Omega^\dagger = 1$, *provided that $V$ does not support bound states*. If no bound states are present, the trace may be written

$$\begin{aligned} \operatorname{Tr} \vec{r}_{op} \cdot \frac{\partial V}{\partial \vec{r}_{op}} e^{-\beta H} &= \operatorname{Tr} \vec{r}_{op} \cdot \frac{\partial V}{\partial \vec{r}_{op}} e^{-\beta H}\Omega\Omega^\dagger \\ &= \operatorname{Tr} \vec{r}_{op} \cdot \frac{\partial V}{\partial \vec{r}_{op}} \Omega e^{-\beta K}\Omega^\dagger \\ &= \int d\vec{p} \langle \vec{p} | \Omega^\dagger \vec{r}_{op} \cdot \frac{\partial V}{\partial \vec{r}_{op}} \Omega | \vec{p} \rangle \\ &\quad \times \exp(-\beta p^2/2\mu) , \end{aligned} \qquad (65)$$

where the cyclic property of the trace has been used. On comparison with Eq. (36), Eqs. (63) and (65) imply that

$$F(p) = -\tfrac{1}{3} h^3 \langle \vec{p} | \Omega^\dagger \vec{r}_{op} \cdot \frac{\partial V}{\partial \vec{r}_{op}} \Omega | \vec{p} \rangle . \qquad (66)$$

Equation (66) is an approximation which may be used for any short-range potential which does not support any bound states. This approximation does reduce in the classical limit ($h \to 0$) to the correct classical $F(p)$ function. The full quantum expression for $F$ is somewhat more complicated; a form for this is derived in the Appendix.

## VII. COMMENTS ON THE WEAK POTENTIAL LIMIT

The results derived thus far for the Landau interaction energy [Eqs. (51) and (66)] will now be compared with some previous treatments in which "dressed particles" are employed. First there is the well-known Hartree approximation,[12] in which the potential is assumed to be weak and therefore two-particle correlations are ignored. [Specifically, the Hartree approximation corresponds to the case $\beta V(r) \ll 1$ for all $r$, and where thermodynamic functions are expanded in a power series in $\beta V(r)$ with only the leading term retained.] A gas behaves in the Hartree approximation like a gas of Landau-like quasiparticles with an interaction energy

$$F(p) = \int V(r)\, d\vec{r} \qquad (67)$$

and a single-quasiparticle energy function

$$\epsilon_p = p^2/2m + n \int V(r)\, d\vec{r} . \qquad (68)$$

The shift in energy $\delta\epsilon_p$ is thus independent of momentum. To show that Eq. (66) is consistent with the Hartree approximation, $\Omega$ and $\Omega^\dagger$ are both set equal to 1, which is the leading term in $\Omega$ as a power series in $V$, or, equivalently, corresponds to neglect of two-particle correlations. In this approximation,

$$\begin{aligned} \langle \vec{p} | \Omega^\dagger \vec{r}_{op} \cdot \frac{\partial V}{\partial \vec{r}_{op}} \Omega | \vec{p} \rangle \to \langle \vec{p} | \vec{r}_{op} \cdot \frac{\partial V}{\partial \vec{r}_{op}} | \vec{p} \rangle &= \int d\vec{r} \langle \vec{p} | \vec{r} \rangle \langle \vec{r} | \vec{r}_{op} \cdot \frac{\partial V}{\partial \vec{r}_{op}} | \vec{p} \rangle \\ &= \int d\vec{r}\, r \frac{\partial V}{\partial r} |\langle \vec{p} | \vec{r} \rangle|^2 \\ &= -3h^{-3} \int d\vec{r}\, V(r) , \end{aligned} \qquad (69)$$

where the last line follows after integration by parts. The Hartree approximation, Eq. (67), then follows directly from Eq. (66). It can also be shown that the complete quantum expression for $F(p)$, Eq. (A5), reduces to the Hartree result when $\Omega = 1$.

The interacting Boltzmann gas has also been analyzed from the Landau-theory point of view by Grossmann.[4] His expression for the interaction energy is

$$\begin{aligned} F(p) &= h^3 \operatorname{Re} \langle \vec{p} | t | \vec{p} \rangle \\ &= \tfrac{1}{2} h^3 \langle \vec{p} | V\Omega + \Omega^\dagger V | \vec{p} \rangle . \end{aligned} \qquad (70)$$

Grossmann's expression also reduces to the Hartree result in the limit $\Omega = 1$, or equivalently, in the Born approximation, $t \simeq V$. However, in general, his $F$ is clearly not equivalent to Eq. (66) nor Eq. (A5) and in fact, as pointed out by Baerwinkel and Grossmann,[5] it has been derived microscopically only to first order in a power series in $t$. Furthermore, Baerwinkel[6] finds it necessary, when considering higher orders in $t$, to introduce "correction" terms to Landau's equation of state, Eq. (8). We have shown, however, that for a properly formulated quasiparticle picture, Landau's equation of state is correct as it stands. Therefore Grossmann's effective single-particle energy does not truly correspond to a Landau-like quasiparticle energy.

Finally, the result for $F(p)$ in term of scattering phase shifts, Eq. (51), may be examined in the weak-potential limit. A weak interaction implies that the phase shifts are small, so that $\eta_l(p)$ may be approximated by $\sin\eta_l(p)\cos\eta_l(p)$, in which case

$$F(p) \approx -h^3(4\pi\mu p)^{-1}\operatorname{Re}\sum_l (2l+1)\sin\eta_l(p)e^{i\eta_l(p)}$$
$$= h^3\operatorname{Re}\langle\vec{p}\,|t|\,\vec{p}\rangle \ , \qquad (71)$$

which agrees with Grossmann's result.

## VIII. CONCLUSIONS

Landau's representation of an interacting fluid as a gas of "quasiparticles" with a renormalized energy-momentum relation is usually applied only to degenerate Fermi systems. However, it is equally applicable to a moderately dense interacting Boltzmann gas. The thermodynamic properties of the Boltzmann gas may be derived self-consistently by picturing the system as composed of quasiparticles having energy $\epsilon_p$, Eq. (30), where the Landau $f$ function $F_{\vec{p}_1\vec{p}_2} = F(p)$ is essentially the inverse Laplace transform of "the second virial coefficient multiplied by $\beta^{-5/2}$." The theory is valid only if the "bare particle" interaction potential $V(r)$ is purely repulsive in the classical case and does not support bound states in the quantum case.

Explicit representations for $F(p)$, and hence the quasiparticle energy spectrum, have been derived in terms of phase shifts and the Møller scattering operator. These expressions are consistent with the well-known Hartree approximation in the weak-potential limit, yet retain their validity for arbitrarily strong potentials.

We have seen from a survey of previous articles that the choice of an effective momentum-dependent dressed-particle energy is by no means unique. Uniqueness is, however, obtained if the theory is to be "Landau-like"; in particular, that the total number of quasiparticles is identical to the total number of particles and the single-quasiparticle energy is to be the functional derivative of the total energy with respect to the self-consistent quasiparticle distribution function.

## APPENDIX

A fully quantum-mechanical Landau quasiparticle interaction energy is derived. The starting point is Eqs. (41) and (61) for the quantum second virial coefficient.

By means of the intertwining relation, Eq. (64), the quantum virial coefficient $B$ may be written as

$$-6k_B T h^{-3}(2\pi\mu k_B T)^{3/2}B = \int_0^1 d\alpha\,\operatorname{Tr}\vec{r}_{op}\cdot\Omega e^{-(1-\alpha)\beta K}\Omega^\dagger\frac{\partial V}{\partial\vec{r}_{op}}\Omega e^{-\alpha\beta K}\Omega^\dagger$$
$$= \int_0^1 d\alpha\,\operatorname{Tr}\Omega^\dagger\vec{r}_{op}\cdot\Omega e^{-(1-\alpha)\beta K}\Omega^\dagger\frac{\partial V}{\partial\vec{r}_{op}}\Omega e^{-\alpha\beta K}$$
$$= \int\int d\vec{p}'\,d\vec{p}''\langle\vec{p}'|\Omega^\dagger\vec{r}_{op}\Omega|\vec{p}''\rangle\cdot\langle\vec{p}''|\Omega^\dagger\frac{\partial V}{\partial\vec{r}_{op}}\Omega|\vec{p}'\rangle$$
$$\times\exp\left(-\frac{\beta p''^2}{2\mu}\right)\int_0^1 d\alpha\,\exp\left(\frac{\alpha\beta(p''^2-p'^2)}{2\mu}\right)$$
$$= \int\int d\vec{p}'\,d\vec{p}''\langle\vec{p}'|\Omega^\dagger\vec{r}_{op}\Omega|\vec{p}''\rangle\cdot\langle\vec{p}''|\Omega^\dagger\frac{\partial V}{\partial\vec{r}_{op}}\Omega|\vec{p}'\rangle$$
$$\times\left[\exp\left(-\frac{\beta p'^2}{2\mu}\right)-\exp\left(-\frac{\beta p''^2}{2\mu}\right)\right](p''^2-p'^2)^{-1}2\mu \ . \qquad (A1)$$

Now define new variables of integration $x \equiv p'^2/2\mu$, $y \equiv p''^2/2\mu$. With $d\omega'$ and $d\omega''$ as the differential solid angles of $\vec{p}'$ and $\vec{p}''$ respectively, the above equation becomes

$$B = -\tfrac{1}{3}\beta h^3\left(\frac{\mu}{2\pi k_B T}\right)^{3/2}\int d\omega'\int d\omega''\int_0^\infty dx\int_0^\infty dy\, x^{1/2}y^{1/2}(y-x)^{-1}$$
$$\times\langle\vec{p}'|\Omega^\dagger\vec{r}_{op}\Omega|\vec{p}''\rangle\cdot\langle\vec{p}''|\Omega^\dagger\frac{\partial V}{\partial\vec{r}_{op}}\Omega|\vec{p}'\rangle\int_x^y dz\,e^{-\beta z} \ . \qquad (A2)$$

This integrand is symmetric on interchange of $x$ and $y$, so the upper limit on the $x$ integration may be changed from $\infty$ to $y$ with the introduction of a compensating factor of 2. A rearrangement of the order of the $x$, $y$, and $z$ integrations gives the result

$$B=-\tfrac{2}{3}\beta h^3\left(\frac{\mu}{2\pi k_B T}\right)^{3/2}\int d\omega'\int d\omega''\int_0^\infty dz\, e^{-\beta z}\int_0^z dx\int_z^\infty dy\, x^{1/2}y^{1/2}(y-x)^{-1}$$
$$\times\langle\vec{p}'\,|\,\Omega^\dagger\vec{r}_{op}\Omega\,|\,\vec{p}''\rangle\cdot\langle\vec{p}''\,|\,\Omega^\dagger\frac{\partial V}{\partial\vec{r}_{op}}\Omega\,|\,\vec{p}'\rangle\,. \qquad \text{(A3)}$$

At this point $p'$ and $p''$ are reintroduced as the integration variables. A new momentum variable $p=(2\mu z)^{1/2}$ is defined with the result

$$B=-\tfrac{1}{6}\beta h^3\pi^{-1}(2\pi\mu k_B T)^{-3/2}$$
$$\times\int d\vec{p}\exp(-\beta p^2/2\mu)p^{-1}\int_0^p p'^2dp'\int d\omega'\int_p^\infty p''^2dp''\int d\omega''(p''^2-p'^2)^{-1}$$
$$\times\langle\vec{p}'|\,\Omega^\dagger\vec{r}_{op}\Omega\,|\,\vec{p}''\rangle\cdot\langle\vec{p}''\,|\,\Omega^\dagger\frac{\partial V}{\partial\vec{r}_{op}}\Omega\,|\,\vec{p}'\rangle\,. \qquad \text{(A4)}$$

From Eqs. (36) and (A4), the quasiparticle interaction energy $F(p)$ may thus be identified as

$$F(p)=-\tfrac{1}{3}h^3(\pi p)^{-1}\int_0^p p'^2dp'\int d\omega'\int_p^\infty p''^2dp''\int d\omega''(p''^2-p'^2)^{-1}$$
$$\times\langle\vec{p}'\,|\,\Omega^\dagger\vec{r}_{op}\Omega\,|\,\vec{p}''\rangle\cdot\langle\vec{p}''\,|\,\Omega^\dagger\frac{\partial V}{\partial\vec{r}_{op}}\Omega\,|\,\vec{p}'\rangle\,. \qquad \text{(A5)}$$

The Landau quasiparticle energy $\epsilon_p$ is then obtained by inserting Eq. (A5) into Eq. (45). The interaction energy obtained here is exact, whereas the simpler expression, Eq. (66), has ignored certain commutation properties.

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[1]L. D. Landau, Z. Eksp. Teor. Fiz. 30, 1058 (1956) [Sov. Phys.-JETP 3, 920 (1957)].
[2]See, for example, P. Nozières, *Theory of Interacting Fermi Systems*, translated by D. Hone (Benjamin, New York, 1964).
[3]R. Balian and C. De Dominicis, Physica 30, 1927 (1964).
[4]S. Grossmann, Z. Phys. 180, 286 (1964); 182, 24 (1964).
[5]K. Baerwinkel and S. Grossmann, Z. Phys. 198, 277 (1967).
[6]K. Baerwinkel, Z. Naturforsch. 24a, 22, 38 (1969).
[7]P. Allen and G. Nicolis, Physica 50, 206 (1970).
[8]E. Colinet, Physica 56, 253 (1971).
[9]P. Clavin and J. Wallenborn, Physica 61, 23 (1972).
[10]R. Balian and C. De Dominicis, Ann. Phys. (N.Y.) 62, 229 (1971).
[11]C. J. Pethick and G. M. Carneiro, Phys. Rev. A 7, 304 (1973).
[12]L. P. Kadanoff and G. Baym, *Quantum Statistical Mechanics* (Benjamin, New York, 1962), Chaps. 3 and 11.
[13]A. Ford, F. Mohling, and J. C. Rainwater, Ann. Phys. (N.Y.) 84, 80 (1974).
[14]M. W. Thomas and R. F. Snider, J. Stat. Phys. 2, 61 (1970).
[15]F. Mohling and J. C. Rainwater, J. Low Temp. Phys. 20, 243 (1975).
[16]See, for example, D. ter Haar, *Elements of Statistical Mechanics* (Rinehart, New York, 1955), p. 188.
[17]A. J. Leggett, Rev. Mod. Phys. 47, 331 (1975).
[18]E. Beth and G. E. Uhlenbeck, Physica 4, 915 (1937).
[19]See, e.g., R. G. Newton, *Scattering Theory of Waves and Particles* (McGraw-Hill, New York, 1966).
[20]R. H. Lambert, Am. J. Phys. 36, 417 (1968).
[21]See, for example, K. Gottfried, *Quantum Mechanics* (Benjamin, New York, 1966), Vol. I, Chap. VII.
[22]R. F. Snider, J. Math Phys. 5, 1580 (1964); R. M. Wilcox, *ibid*. 8, 962 (1967).