Onefold photoelectron counting statistics for non-Gaussian light: Scattering from polydispersive suspensions

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Onefold photoelectron counting distributions are evaluated for coherent light scattered by an ensemble of particles. The ensemble contains, in general, a random number of nonidentical particles. Expressions are obtained for the probability density function and the moments of the scattering intensity. These results reduce to those of Pusey, Schaefer, and Koppel when the particles are identical. Photocount probabilities are also evaluated. The variance, coefficient of skewness, and coefficient of excess are evaluated when the probability density of the intensity contributed by each scatterer is specified by a narrow Gaussian distribution centered about the deterministic mean.

I. INTRODUCTION

Photoelectron counting provides a common tool for studying the light scattered by a wide variety of systems. When the system under study contains a large number of independent scatterers, the amplitudes of the scattered field are Gaussian, and in the limit of short counting times the time-integrated intensity is described by an exponential probability density. Probabilities $P(m, T)$ of counting m photoelectrons in a time T are then given by the Bose-Einstein distribution. When the system contains a smaller number of scatterers, however, the field amplitudes are no longer Gaussian. Consequently the integrated intensity and the $P(m, T)$ depart from the usual behavior. This paper presents an analysis of the onefold photoelectron counting statistics under these conditions. Two cases are considered: systems with precisely N scatterers and systems in which the number of scatterers is allowed to fluctuate randomly.

The problem of non-Gaussian light has received considerable attention. A detailed list of relevant references is given by Chu.[1] The papers of Schaefer and Pusey[2] and of Pusey, Schaefer, and Koppel[3] are especially pertinent since they deal with the probability density $W(\Omega)$ of the integrated intensity Ω. The analysis presented here depends upon a somewhat different analytical approach suggested by the treatment of non-Gaussian statistics of laser speckle patterns given by Barakat.[4]

After reviewing the necessary general theory, we derive expressions for the photoelectron statistics for fixed and random numbers of scatterers. We next examine the moments of the time-integrated intensity, and treat as an example the case in which the probability density of light contributed by each scatterer is Gaussian. Finally we examine the coefficients of skewness and excess for W, and the factorial moments of $P(m, T)$.

II. SUMMARY OF PERTINENT THEORY

Assuming the existence of the P representation for the density operator of the electromagnetic field, then the probability of obtaining m photoelectrons in a time interval T is given by[5]

$$P(m, T) = \int_0^\infty W(\Omega, T) \frac{\Omega^m e^{-\Omega}}{m!} d\Omega , \qquad (1)$$

where $W(\Omega, T)$ is the inverse Laplace transform of the onefold generating function

$$Q(\lambda, T) = \int_0^\infty W(\Omega, T) e^{-\lambda\Omega} d\Omega . \qquad (2)$$

Under these conditions, the variable Ω can be *interpreted* as the time-integrated intensity of the c-number representation of the field,

$$\Omega = s \int_0^T |\hat{\mathcal{E}}(t)|^2 dt . \qquad (3)$$

Here s is a sensitivity factor which reflects the quantum efficiency of the detector and the spatial coherence of the light. Glauber has shown that $W(\Omega, T)$ is a non-negative function of both Ω and T. Furthermore, its integral with respect to Ω over $(0, \infty)$ is finite for all T, so that $W(\Omega, T)$ satisfies the necessary and sufficient condition that it be a continuous probability density.

The probability density function $W(\Omega, T)$ has been evaluated for short counting times by Barakat[4] under the assumption that the light is scattered by N independent (but not necessary identical) particles,

$$W(\Omega | N) = \frac{1}{2} \int_0^\infty \phi_N(t) J_0(\Omega^{1/2} t) t \, dt , \qquad (4)$$

where

$$\phi_N(t) = \prod_{n=1}^{N} \int_0^\infty J_0(t a_n) f_{a_n}(a_n) \, da_n . \qquad (5)$$

Here a_n is the contribution of the nth particle to the amplitude of the scattered light; specifically, if there is only one scatterer in the system, then

$$a_1 \equiv \langle \Omega_1 \rangle^{1/2} ; \qquad (6)$$

$f_{a_n}(a_n)$ is the probability density function of a_n.

In the situation envisaged, all of the scattering particles will have the same probability density, so that Eq. (5) reduces to

$$\phi_N(t) = \left(\int_0^\infty f_a(a) J_0(ta)\, da \right)^N . \qquad (7)$$

If $f_a(a)$ obeys

$$f_a(a) = \delta(a - \langle \Omega_1 \rangle^{1/2}) \qquad (8)$$

(in other words, all of the particles are of the same fixed size and thus scatter light equally), then

$$\phi_N(t) = [J_0(\langle \Omega_1 \rangle^{1/2} t)]^N \qquad (9)$$

and Eq. (4) reduces to the case previously studied. Since the observation time T is taken to be smaller than the reciprocal of the characteristic linewidth of the spectrum of the scattered light, the shape of the spectrum of the scattered light does not enter into the analysis and the two relevant parameters are the time T and the average count rate w. In other words, we are looking at a single-mode scattering problem.

In an actual experiment, it is very difficult to keep the number of scatterers fixed in the scattering volume and it is more realistic to consider that the number of scatterers is itself a random variable. We take N to be a discrete random variable having a Poisson distribution. Thus $\langle N \rangle$, the average number of independent scatterers in V, characterizes the random scatterers since $\langle N \rangle$ is also the variance. Any higher moments of the Poisson distribution can be expressed in terms of $\langle N \rangle$.

For convenience in typography we now drop the explicit time dependence in P, W, and Q. Since the solution for a random (Poisson) number of scatterers depends on the solution for a fixed number of scatterers, we begin with the latter case in Sec. III.

III. PHOTOELECTRON STATISTICS FOR FIXED NUMBER OF SCATTERERS

For $N=0$ (no scatterers), we have

$$W(\Omega \,|\, 0) = \frac{1}{2} \int_0^\infty J_0(\Omega^{1/2} t) t\, dt = \delta(\Omega) . \qquad (10)$$

Consequently,

$$P(m \,|\, 0) = \begin{cases} 1, & m = 0 \\ 0, & m \neq 0 . \end{cases} \qquad (11)$$

This is exactly what we expect, namely, that the probability of obtaining no photoelectrons is unity.

For $N \geq 1$, we follow Ref. 4. We express $W(\Omega \,|\, N)$ in the form of a Fourier-Bessel series whose coefficients are sampled values of the characteristic function $\phi_N(t)$ of $W(\Omega \,|\, N)$. We now assume that

$$f_a(a) \equiv 0, \quad a > \alpha, \qquad (12)$$

where α is finite. Obviously

$$W(\Omega \,|\, N) \equiv 0, \quad \Omega > N^2 \alpha^2 . \qquad (13)$$

Consequently $\phi_N(t)$ is a band-limited function since its Fourier-Bessel transform $W(\Omega \,|\, N)$ vanishes identically outside a compact region. It can be shown that (see details in Ref. 4)

$$W(\Omega \,|\, N) = \begin{cases} \displaystyle\sum_{n=1}^\infty \frac{[\phi_1(\gamma_n / N\alpha)]^N}{N^2 \alpha^2 [J_1(\gamma_n)]^2} J_0\!\left(\frac{\gamma_n \Omega^{1/2}}{N\alpha} \right), & 0 \leq \Omega \leq N^2 \alpha^2 \\ 0, & \text{elsewhere,} \end{cases} \qquad (14)$$

where $\gamma_1, \gamma_2, \ldots$ are the positive roots of J_0 [i.e., $J_0(\gamma_n) = 0$]. Since this is a Fourier-Bessel series, its convergence is governed by the smoothness (continuity) of $W(\Omega \,|\, N)$. The smoother $W(\Omega \,|\, N)$, the more rapid the convergence of its series expansion.

In the special case where all of the particles are of the same fixed size, it is possible to obtain closed-form solutions of $W(\Omega \,|\, N)$ for $N = 1, 2, 3$. These expressions are listed in Appendix A.

As N becomes very large, $W(\Omega \,|\, N)$ approaches a negative exponential probability density

$$W(\Omega \,|\, N) \sim (1/\langle \Omega \rangle) e^{-\Omega / \langle \Omega \rangle} , \qquad (15)$$

so that the underlying statistics of the *field amplitudes* are Gaussian. The corresponding expression for $P(m \,|\, N)$ is the Bose-Einstein distribution

$$P(m \,|\, N) \sim \langle \Omega \rangle^m / (1 + \langle \Omega \rangle)^{m+1} . \qquad (16)$$

The expressions in Eqs. (15) and (16) are the leading terms in the asymptotic series in powers of N^{-1} (see Appendix B for details).

In spite of the fact that $W(\Omega \,|\, N)$ varies drastically for small N, the resultant photoelectron counting distributions are not strongly dependent on N. This is simply a consequence of the smoothing action of the Poisson term in Eq. (1). The factor Ω^m effectively damps out any irregular behavior of $W(\Omega \,|\, N)$ for $\Omega < 1$, while the negative exponential accomplishes the same result for $\Omega > 1$. It is for this reason that we have confined our *numerical* calculations to the case of identical particles. Even

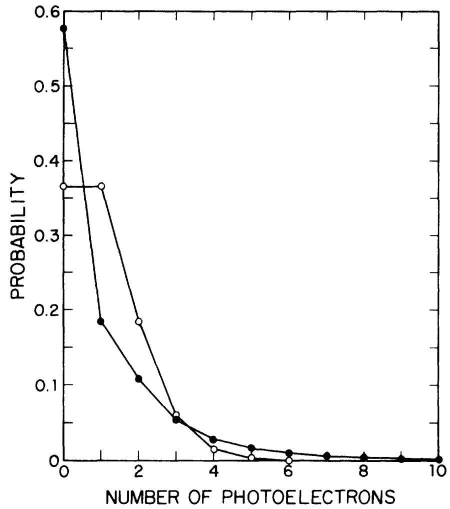

FIG. 1. Photoelectron counting distributions $P(m|1)$, open circles; $P(m|\langle 1\rangle)$, solid circles.

though $W(\Omega|N)$ will be somewhat different according to the situation considered (e.g., particles with different fixed sizes, particles with range of different sizes, etc.), the smoothing action of the integrand in Eq. (1) is the determining factor. The results for $N=1, 2, 3, 8$ are shown in Figs. 1–4 (see curves with *open* circles), in all cases $\langle\Omega_1\rangle=1$. Note that the maxima of these distributions are at $m=0$, as we would expect.

We have chosen to work with $\langle\Omega\rangle$, but it is also just as easy to work with the average total count rate w,

$$w=\langle\Omega\rangle/T=N\langle\Omega_1\rangle/T\,. \tag{17}$$

The question naturally arises as to how large N must be in order that $P(m|N)$ approximate the Bose-Einstein distribution. Answering this question is not a simple matter, but we can offer the following remarks: Based on test calculations (not reproduced here), it appears that for practical purposes $N\geqslant 5$ gives a Bose-Einstein distribution. Of course, this is not strictly true. The factorial moments of $P(m|N)$ are evaluated in Sec. V and compared to those of the Bose-Einstein distribution; one can see how they differ as a function of N. Nevertheless an experimenter would be hard put to dintinguish $P(m|5)$ from $P(m|100)$, say, purely on the basis of a *onefold* counting experiment. The probability of obtaining no photoelectrons is a useful statistic, and its behavior is depicted in Fig. 5.

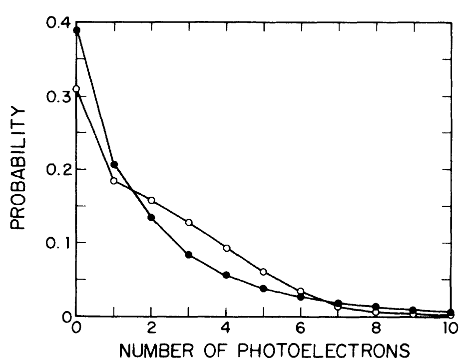

FIG. 2. Photoelectron counting distributions $P(m|2)$, open circles; $P(m|\langle 2\rangle)$, solid circles.

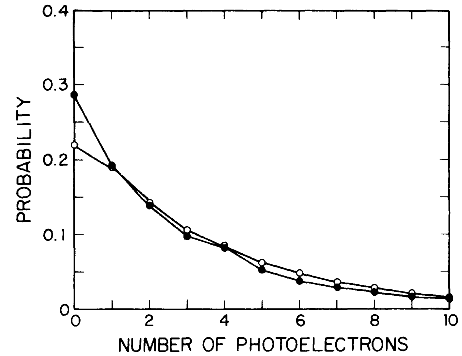

FIG. 3. Photoelectron counting distributions $P(m|3)$, open circles; $P(m|\langle 3\rangle)$, solid circles.

IV. PHOTOELECTRON STATISTICS FOR RANDOM NUMBER OF SCATTERERS

We have just examined the case of N fixed, so we can interpret these results as holding for a canonical ensemble in the language of statistical mechanics. The case of random N leads to an interpretation as a grand canonical ensemble.

It is a simple exercise in probability theory to prove that if N is distributed according to a Poisson distribution having a mean value $\langle N\rangle$, then

$$W(\Omega|\langle N\rangle)=\sum_{N=0}^{\infty} W(\Omega|N)\frac{\langle N\rangle^N e^{-\langle N\rangle}}{N!}, \tag{18}$$

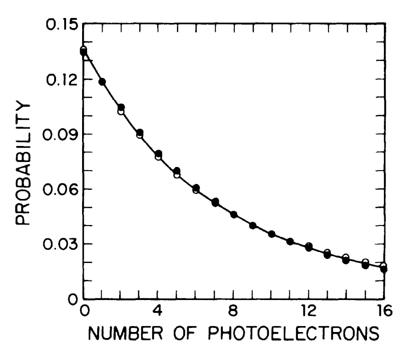

FIG. 4. Photoelectron counting distributions $P(m|8)$, open circles; $P(m|\langle 8\rangle)$, solid circles.

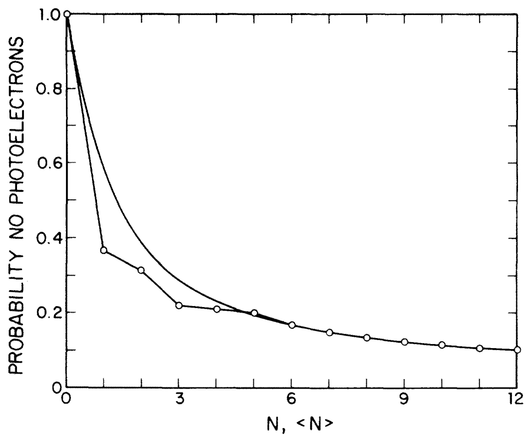

FIG. 5. Photoelectron counting distributions $P(0|N)$, open circles; $P(0|\langle N\rangle)$, solid line.

$$P(m|\langle N\rangle)=\sum_{N=0}^{\infty} P(m|N)\frac{\langle N\rangle^N e^{-\langle N\rangle}}{N!}. \tag{19}$$

As $\langle N\rangle$ is made to increase, the Poisson distribution becomes very peaked at $\langle N\rangle = N$ and acts somewhat like a Dirac δ function centered at $\langle N\rangle = N$. Consequently,

$$W(\Omega|\langle N\rangle)\sim W(\Omega|N), \quad P(m|\langle N\rangle)\sim P(m|N) \tag{20}$$

for large $\langle N\rangle$.

We can easily plot W as a function of $\langle N\rangle$, but it hardly seems worthwhile to do so, and we pass directly to the photoelectron statistics. Numerical values of $P(m|\langle N\rangle)$ for $\langle N\rangle = 1, 2, 3, 8$ are plotted in Figs. 1–4 (see solid circles). As we would expect, the photoelectron counting distributions for the deterministic and stochastic situations are measurably different for one and two scatterers. However, even for three scatterers, the two situations are not very different. For more than three scatterers, the two situations yield practically the same result, as witness $N = \langle N\rangle = 8$ (Fig. 4).

We can also derive integral representations for both $W(\Omega|\langle N\rangle)$ and $P(m|\langle N\rangle)$. If we substitute Eq. (19) into Eq. (4) and sum the series, we obtain

$$W(\Omega|\langle N\rangle)=\frac{1}{2}\int_0^\infty e^{-\langle N\rangle[1-\phi_1(t)]}J_0(\Omega^{1/2}t)t\,dt. \tag{21}$$

When Eq. (21) is substituted into Eq. (1), we can prove that

$$P(m|\langle N\rangle)=\frac{1}{2}\int_0^\infty e^{-\langle N\rangle[1-\phi_1(t)]}e^{-t^2/4}L_m(\tfrac{1}{4}t^2)t\,dt \tag{22}$$

$$=\frac{1}{2}\int_0^\infty e^{-y}e^{-\langle N\rangle[1-\phi_1(2y^{1/2})]}L_m(y)\,dy. \tag{23}$$

In its second form, the integral can be evaluated via Gauss Laguerre quadrature if desired.[6]

The probability distributions for fixed and random numbers of scatterers, $P(m|N)$ and $P(m|\langle N\rangle)$, resemble each other closely, as we have seen. To distinguish between these two cases under the constraint of onefold counting it is perhaps easiest to examine the probability of obtaining no photoelectrons during the measurement. These distributions are shown in Fig. 5. A clear distinction can be seen for $N = \langle N\rangle = 5$, but the distributions merge for larger numbers of scatterers.

V. MOMENTS OF $W(\Omega|N)$ AND $W(\Omega|\langle N\rangle)$

In order to calculate the moments (more precisely, the conditional moments about the origin), we introduce the generating function

$$Q(\lambda|N)\equiv\int_0^\infty e^{-\lambda\Omega}W(\Omega|N)\,d\Omega \tag{24}$$

from which the moments can be obtained by differentiation,

$$\langle\Omega^k|N\rangle=(-1)^k\frac{\partial^k}{\partial\lambda^k}Q(\lambda|N)\bigg|_{\lambda=0}, \quad k=1,2,\ldots. \tag{25}$$

$Q(\lambda|N)$ can be expressed in the form

$$Q(\lambda|N)=\frac{1}{4\lambda}\int_0^\infty e^{-t^2/4\lambda}\phi_N(t)t\,dt \tag{26}$$

by substituting Eq. (4) into Eq. (24) and employing

$$\int_0^\infty e^{-ax^2}J_0(tx)x\,dx=\frac{1}{2a}e^{-t^2/4a}. \tag{27}$$

It is convenient to express Q as a power series in λ; the differentiations are then trivial. To this end, we express $\phi_N(t)$ as a power series in t^2 by expanding $J_0(ta)$ in a power series and then integrating termwise; we have

$$\phi_N(t)=\left(\sum_{n=0}^{\infty}A_n t^{2n}\right)^N, \tag{28}$$

where

$$A_0=1, \quad A_2=-\tfrac{1}{4}\langle a^2\rangle, \quad A_3=\tfrac{1}{64}\langle a^4\rangle, \ldots, \tag{29}$$

and

$$\langle a^n\rangle=\int_0^\infty f_a(a)a^n\,da. \tag{30}$$

We now rewrite $\phi_N(t)$ in the form

$$\phi_N(t)=\sum_{n=0}^{\infty}B_n t^{2n}. \tag{31}$$

The B coefficients can be calculated from the A coefficients via the recurrence relation[7]

$$B_n = \frac{1}{nA_0} \sum_{k=1}^{n} [k(N+1) - n] A_k B_{n-k}, \quad n \geq 1. \tag{32}$$

When Eq. (31) is substituted into Eq. (26), the resulting integration is trivial and the final result is that $Q(\lambda | N)$ is a power series in λ. The resulting differentiations can now be performed, and the first four moments about the origin are

$$\begin{aligned}
\langle \Omega | N \rangle &= N\langle a^2 \rangle, \\
\langle \Omega^2 | N \rangle &= 2N(N-1)\langle a^2 \rangle^2 + N\langle a^4 \rangle, \\
\langle \Omega^3 | N \rangle &= 6N(N-1)(N-2)\langle a^2 \rangle^3 \\
&\quad + 9N(N-1)\langle a^2 \rangle\langle a^4 \rangle + N\langle a^6 \rangle, \\
\langle \Omega^4 | N \rangle &= 24N(N-1)(N-2)(N-3)\langle a^2 \rangle^4 \\
&\quad + 72N(N-1)(N-2)\langle a^2 \rangle^2\langle a^4 \rangle \\
&\quad + 16N(N-1)\langle a^2 \rangle\langle a^6 \rangle + N\langle a^8 \rangle.
\end{aligned} \tag{33}$$

When all the particles are of fixed size then these expressions reduce to those given in Ref. 3.

The limiting values of $\langle \Omega^k | N \rangle$, when $N \gg 1$, are

$$\langle \Omega^k | N \rangle \sim k! N^k \langle a^2 \rangle^k, \tag{34}$$

characteristic of a negative exponential probability density function, Eq. (15), as expected. Thus the *rate* at which the moments approach their limiting values depends not only on N (as in the case of identical particles) but also on the spread of the probability density function $f(a)$ as measured by the ratio of the various moments of a.

Since $W(\Omega | N)$ is a true probability distribution as regards our problem, then $P(m | N)$ is the probability distribution of a mixed Poisson process.[8] It is a well-known property of mixed Poisson processes that the kth *factorial* moment defined by

$$\langle m^{(k)} | N \rangle \equiv \langle m(m-1) \cdots (m-k+1) | N \rangle, \quad k = 1, 2, \ldots, \tag{35}$$

is related to the kth moment of $W(\Omega | N)$ by

$$\langle m^{(k)} | N \rangle = \langle \Omega^k | N \rangle. \tag{36}$$

Thus we can list the factorial moments of $P(m | N)$ once we have the moments of $W(\Omega | N)$.

The same procedure can be employed to determine $\langle \Omega^k | \langle N \rangle \rangle$. We define the generating function

$$Q(\lambda | \langle N \rangle) \equiv \int_0^\infty e^{-\lambda \Omega} W(\Omega | \langle N \rangle) \, d\Omega. \tag{37}$$

Upon substituting Eq. (21) into the integral and subsequently employing Eq. (27), we can show that

$$Q(\lambda | \langle N \rangle) = \frac{1}{4\lambda} \int_0^\infty e^{-\langle N \rangle [1 - \phi_1(t)]} e^{-t^2/4\lambda} t \, dt. \tag{38}$$

In order to obtain the moments, we expand the first term in the integrand in a power series in t; thus

$$e^{-\langle N \rangle [1 - \phi_1(t)]} = \sum_{n=0}^{\infty} C_n t^{2n}, \tag{39}$$

where

$$\begin{aligned}
&C_0 = 1, \quad C_2 = -\tfrac{1}{4} \langle N \rangle \langle a^2 \rangle, \\
&C_3 = \tfrac{1}{32} \langle N \rangle^2 \langle a^2 \rangle^2 + \tfrac{1}{64} \langle N \rangle \langle a^4 \rangle, \ldots.
\end{aligned} \tag{40}$$

Proceeding as before, we obtain $Q(\lambda | \langle N \rangle)$ as a power series in λ. Differentiation then yields the moments about the origin, of which the first four are

$$\begin{aligned}
\langle \Omega | \langle N \rangle \rangle &= \langle N \rangle \langle a^2 \rangle, \\
\langle \Omega^2 | \langle N \rangle \rangle &= 2\langle N \rangle^2 \langle a^2 \rangle^2 + \langle N \rangle \langle a^4 \rangle, \\
\langle \Omega^3 | \langle N \rangle \rangle &= 6\langle N \rangle^3 \langle a^2 \rangle^3 + 9\langle N \rangle^2 \langle a^2 \rangle \langle a^4 \rangle + \langle N \rangle \langle a^6 \rangle, \\
\langle \Omega^4 | \langle N \rangle \rangle &= 24\langle N \rangle^4 \langle a^2 \rangle^4 + 72\langle N \rangle^3 \langle a^2 \rangle^2 \langle a^4 \rangle \\
&\quad + 34\langle N \rangle^2 \langle a^2 \rangle \langle a^6 \rangle + \langle N \rangle \langle a^8 \rangle.
\end{aligned} \tag{41}$$

When all the particles are of same fixed size, then these expressions also reduce to those given Ref. 3.

The limiting values of $\langle \Omega^k | \langle N \rangle \rangle$ for large $\langle N \rangle$ are again given by Eq. (34), with N replaced by $\langle N \rangle$. However, specifying $\langle N \rangle$ instead of N permits the number of particles in the scattering volume to fluctuate. This introduces additional fluctuations in the intensity of the scattered light. Accordingly the moments for the random case will approach their limiting values more slowly than the corresponding moments in the deterministic case.

$P(m | \langle N \rangle)$ is also the probability distribution of a mixed Poisson process because $W\langle \Omega | \langle N \rangle \rangle$ is a probability density; consequently,

$$\langle m^{(k)} | \langle N \rangle \rangle = \langle \Omega^k | \langle N \rangle \rangle. \tag{42}$$

VI. $f_a(f)$ IS NARROW GAUSSIAN

It is useful to examine the case where $f_a(a)$ is sharply peaked around its mean value, because this provides an approximation to a slightly polydisperse suspension of particles. For computational simplicity, we let

$$f_a(a) = (1/\sigma_a \sqrt{2\pi}) \exp[-(a - \langle \Omega_1 \rangle^{1/2})^2 / 2\sigma_a^2], \tag{43}$$

where $\langle \Omega_1 \rangle \gg \sigma_a^2$. The moments of a are

$$\langle a^k \rangle = \int_0^\infty f_a(a) a^k \, da. \tag{44}$$

If we form the dimensionless parameter $\delta \equiv \sigma_a^2 / \langle \Omega_1 \rangle$, then for $\delta \ll 1$, the negative contribution of the probability density function in Eq. (43) is negligible and we can safely replace the limits $(0, \infty)$ by $(-\infty, \infty)$ in Eq. (44). Consequently, we obtain

$$\langle a^2\rangle = (1+\delta)\langle\Omega_1\rangle\,,$$
$$\langle a^4\rangle = (1+6\delta+3\delta^2)\langle\Omega_1\rangle^2\,,$$
$$\langle a^6\rangle = (1+15\delta+45\delta^2+15\delta^3)\langle\Omega_1\rangle^3\,,$$
$$\langle a^8\rangle = (1+28\delta+210\delta^2+420\delta^3+105\delta^4)\langle\Omega_1\rangle^4\,. \quad (45)$$

In order to utilize these explicit formulas, let us consider, for example, the variance of Ω:

$$\mathrm{var}(\Omega|N) = N(N-2)\langle a^2\rangle^2 + N\langle a^4\rangle\,, \quad (46)$$

$$\mathrm{var}(\Omega|\langle N\rangle) = \langle N\rangle^2\langle a^2\rangle^2 + \langle N\rangle\langle a^4\rangle\,; \quad (47)$$

then

$$V(N,\delta) \equiv \frac{\mathrm{var}(\Omega|N)}{N^2\langle a^2\rangle^2} = \left(1+\frac{1}{2N}\right) + \frac{4}{N}(\delta - 3\delta^2 + 2\delta^3 - \cdots)\,, \quad (48)$$

$$V(\langle N\rangle,\delta) \equiv \frac{\mathrm{var}(\Omega|\langle N\rangle)}{\langle N\rangle^2\langle a^2\rangle^2} = \left(1+\frac{1}{\langle N\rangle}\right) + \frac{4}{\langle N\rangle}(\delta - 3\delta^2 + 2\delta^3 - \cdots)\,. \quad (49)$$

The first terms (in large parentheses) are the contributions when all of the scatterers are identical. The second term represents the influence of the width of $f_a(a)$ and arises from the varying amounts of light scattered by different-sized particles. The second term is the same for both situations. Some typical numerical results are shown in Fig. 6 and are self-explanatory. Other moments can be handled in much the same manner.

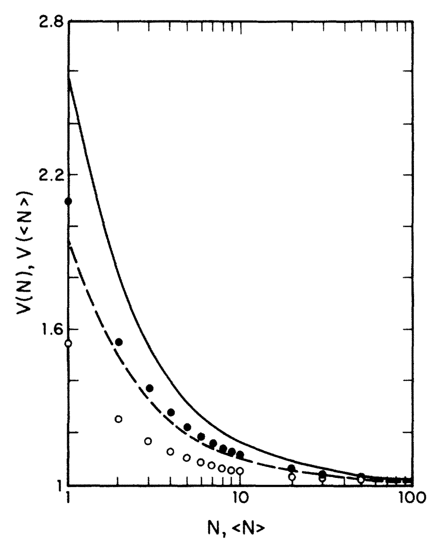

FIG. 6. $V(N,\delta)$ and $V_2(\langle N\rangle,\delta)$ as functions of N, $\langle N\rangle$ for fixed δ. $V_1(N,0)$, open circles; $V_1(N,0.2)$, solid circles; $V(\langle N\rangle,0)$, dotted line; $V_2(\langle N\rangle,0.2)$, solid line.

VII. COEFFICIENTS OF SKEWNESS AND EXCESS FOR W

Since both $W(\Omega|N)$ and $W(\Omega|\langle N\rangle)$ are far different from their limiting values for small N and $\langle N\rangle$, it is important to evaluate their global shape parameters, their coefficient of skewness, and coefficient of excess.

The coefficient of skewness γ_1 is defined as

$$\gamma_1 = \langle(\Omega-\langle\Omega\rangle)^3\rangle / \langle(\Omega-\langle\Omega\rangle)^2\rangle^{3/2}\,. \quad (50)$$

The skewness is a normalized measure of the mode minus the mean and serves as one indicator of the length of the "tail" of the density function. If γ_1 is positive (negative), then the corresponding probability density function is skewed to the right-hand side (left-hand side) of the mode. The larger $|\gamma_1|$, the longer the resultant tail. If $\gamma_1=0$, then the density function is symmetric about the mean. In the special case where all of the scatters are of the same fixed size, the explicit formulas are

$$\gamma_1(N) = \frac{(2N-4)}{(N^2-N)^{1/2}} \simeq 2 - \frac{3}{N} + O(N^{-2}) \quad (51)$$

and

$$\gamma_1(\langle N\rangle) = \frac{2\langle N\rangle^3 + 6\langle N\rangle^2 + \langle N\rangle}{(\langle N\rangle^2+\langle N\rangle)^{3/2}} \simeq 2 + \frac{3}{\langle N\rangle} + O(\langle N\rangle^{-2})\,. \quad (52)$$

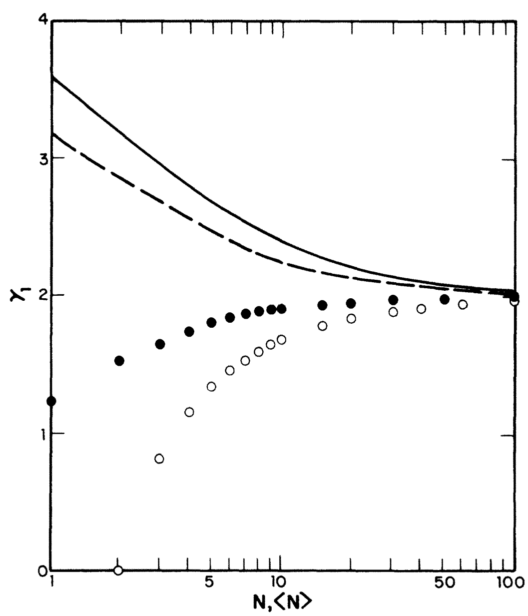

FIG. 7. Coefficient of skewness, γ_1, as a function of N and $\langle N\rangle$ for fixed δ. $\gamma_1(N,0)$, open circles; $\gamma_1(N,0.2)$, solid circles; $\gamma_1(\langle N\rangle,0)$, dotted line; $\gamma_1(\langle N\rangle,0.2)$, solid line.

Both approach the limiting value $\gamma_1 = 2$ characteristic of a negative exponential probability density. We have calculated $\gamma_1(N)$ and $\gamma_1(\langle N\rangle)$ for $\delta = 0, 0.2$; the results are shown in Fig. 7. For fixed N, the result of allowing $\delta > 0$ is to hasten the rate at which $\gamma_1(N)$ tends to 2. When N is allowed to fluctuate, $\gamma_1(\langle N\rangle)$ is larger than the limiting value, thereby indicating that the tail dies off more slowly than for the corresponding fixed case. This is exactly what we would expect.

The coefficient of excess γ_2 is defined as

$$\gamma_2 = \frac{\langle(\Omega - \langle\Omega\rangle)^4\rangle - 3\langle(\Omega\langle\Omega\rangle)^2\rangle^2}{\langle(\Omega - \langle\Omega\rangle)^2\rangle^2} \tag{53}$$

and is a measure of the peakedness of the corresponding density function about the mode. $\gamma_2 = 0$ for a Gaussian, and $\gamma_2 = 6$ for a negative exponential. Explicit formulas for γ_2, when all of the scatterers are identical, are

$$\gamma_2(N) = \frac{6N^3 - 36N^2 + 45N - 15}{N^3 - N} \simeq 6 - 24/N + O(N^{-2}) \tag{54}$$

and

$$\gamma_2(\langle N\rangle) = \frac{6\langle N\rangle^3 + 36\langle N\rangle^2 + 39\langle N\rangle + 1}{\langle N\rangle^3 + 2\langle N\rangle^2 + \langle N\rangle} \simeq 6 + 24/\langle N\rangle + O(\langle N\rangle^{-2}). \tag{55}$$

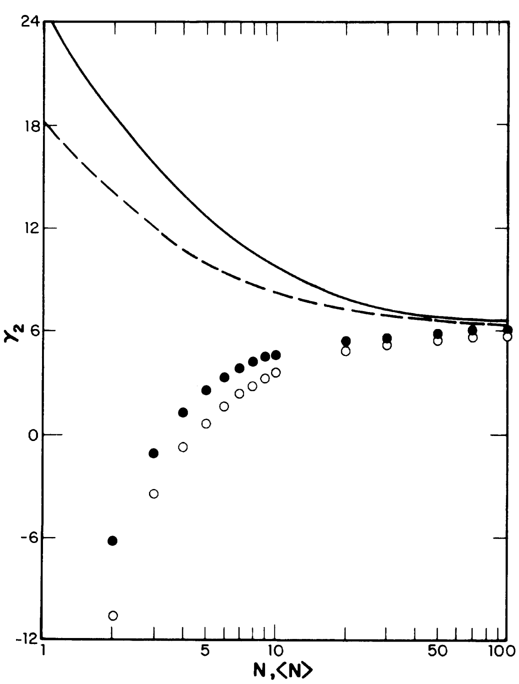

FIG. 8. Coefficient of excess, γ_2, as a function of N and $\langle N\rangle$ for fixed δ. $\gamma_2(N, 0)$, open circles; $\gamma_2(N, 0.2)$, solid circles; $\gamma_2(\langle N\rangle, 0)$, dotted line; $\gamma_2(\langle N\rangle, 0.2)$, solid line.

The limiting value $\gamma_2 = 6$ characteristic of a negative exponential is approached in both cases. $\gamma_2(N)$ is always smaller than 6, while $\gamma_2(\langle N\rangle)$ is always larger than 6. Again, numerical calculations were performed for $\delta = 0, 0.2$, and the results are summarized in Fig. 8.

VIII. FACTORIAL MOMENTS OF P

We can list the factorial moments of $P(m|N)$ and $P(m|\langle N\rangle)$ by virtue of Eqs. (36) and (42). However, we content ourselves with deriving the second factorial moment (variance). Thus

$$\text{var}(m|N) = N(N-2)\langle a^2\rangle^2 + N\langle a^4\rangle + N\langle a^2\rangle, \tag{56}$$

$$\text{var}(m|\langle N\rangle) = \langle N\rangle^2\langle a^2\rangle^2 + N\langle a^4\rangle + \langle N\rangle\langle a^2\rangle. \tag{57}$$

When $f_a(a)$ is the Gaussian discussed in Sec. VI, then we can show that

$$\text{var}(m|N) = [(N^2 - N)\langle\Omega_1\rangle^2 + N\langle\Omega_1\rangle] + (2\delta + \delta^2)N^2\langle\Omega_1\rangle^2 + \delta N\langle\Omega_1\rangle, \tag{58}$$

$$\text{var}(m|\langle N\rangle) = [(\langle N\rangle^2 + \langle N\rangle)\langle\Omega_1\rangle^2 + \langle N\rangle\langle\Omega_1\rangle] + (2\delta + \delta^2)(\langle N\rangle^2 + 2\langle N\rangle)\langle\Omega_1\rangle^2 + \delta N\langle\Omega_1\rangle. \tag{59}$$

When all particles are identical,

$$\text{var}(m|N) = (N^2 - N)\langle\Omega_1\rangle^2 + N\langle\Omega_1\rangle, \tag{60}$$

$$\text{var}(m|\langle N\rangle) = (\langle N^2\rangle + \langle N\rangle)\langle\Omega_1\rangle^2 + \langle N\rangle\langle\Omega_1\rangle. \tag{61}$$

The variance for the stochastic situation is larger than that for the deterministic situation, as expected; both approach the same limiting value.

APPENDIX A

For $N = 1$, we have

$$W(\Omega|1) = \frac{1}{2}\int_0^\infty J_0(\langle\Omega_1\rangle^{1/2}t)J_0(\Omega^{1/2}t)t\,dt = \delta(\Omega - \langle\Omega_1\rangle) \tag{A1}$$

(i.e., there are no fluctuations in the light scattered by a single particle). Thus the system is in a coherent state,[5] and the photoelectron counting distribution is Poisson with mean value $\langle\Omega_1\rangle$,

$$P(m|1) = (1/m!)\langle\Omega_1\rangle^m e^{-\langle\Omega_1\rangle}. \tag{A2}$$

For $N = 2$, Rayleigh[9] evaluated the equivalent of $W(\Omega|2)$; his result, translated into the language of the present problem, is

$$W(\Omega|2) = \begin{cases} 1/\pi[\Omega(4\langle\Omega_1\rangle - \Omega)]^{1/2}, & 0 \le \Omega \le 4\langle\Omega_1\rangle \\ 0, & \text{elsewhere}. \end{cases} \tag{A3}$$

Note the square-root singularities at $\Omega = 0, 4\langle\Omega_1\rangle$. The corresponding photoelectron counting distribution is

$$P(m|2) = \frac{1}{\pi m!}\int_0^{4\langle\Omega_1\rangle} \frac{\Omega^m e^{-\Omega}\, d\Omega}{[\Omega(4\langle\Omega_1\rangle - \Omega)]^{1/2}} . \qquad \text{(A4)}$$

Although this integral cannot be evaluated analytically, it is easily handled by Chebyschev quadrature,[6] which automatically takes account of the square-root singularities at the end points of the integration.

Finally, $N=3$ can be expressed in terms of elliptic integrals,[10]

$$W(\Omega|3) = \begin{cases} \dfrac{2K(k)}{\pi^2(\langle\Omega_1\rangle^{1/2}+\Omega^{1/2})[(3\langle\Omega_1\rangle^{1/2}-\Omega^{1/2})(\langle\Omega_1\rangle^{1/2}+\Omega^{1/2})]^{1/2}}, & 0 \le \Omega \le \langle\Omega_1\rangle \\ \dfrac{K(1/k)}{2\pi^2\langle\Omega_1\rangle^{3/4}\Omega^{3/4}}, & \langle\Omega_1\rangle \le \Omega \le 9\langle\Omega_1\rangle \\ 0, & \text{elsewhere}, \end{cases} \qquad \text{(A5)}$$

where $K(k)$ is the complete elliptic integral of the first kind with modulus k given by

$$k \equiv \frac{4\langle\Omega_1\rangle^{3/4}\Omega^{1/4}}{(\langle\Omega_1\rangle^{1/2}+\Omega^{1/2})[(3\langle\Omega_1\rangle^{1/2}-\Omega^{1/2})(\langle\Omega_1\rangle^{1/2}+\Omega^{1/2})]^{1/2}} . \qquad \text{(A6)}$$

The elliptic integral is logarithmically infinite at $k=1$ (i.e., when $\Omega = \langle\Omega_1\rangle$). The logarithmic singularity is very weak and $P(m|3)$ can be evaluated numerically by a trapezoidal rule with a very close mesh.

APPENDIX B

The probability density function of the envelope r of the sum of N independently distributed random sine waves approaches the Rayleigh probability density function as N becomes very large. If we apply the procedure described in Cramer[11] for corrections to the central limit theorem, we can show (details are omitted) that

$$f(r) \sim (2r/N\langle a^2\rangle)e^{-r^2/N\langle a^2\rangle}[1 + D_1 + \cdots], \qquad \text{(B1)}$$

where

$$D_1 \equiv \left(2 - \frac{4r^2}{N\langle a^2\rangle} + \frac{r^4}{N^2\langle a^2\rangle^2}\right) \times (N\langle a^4\rangle - 2N\langle a^2\rangle^2)(4N^2\langle a^2\rangle^2)^{-1} . \qquad \text{(B2)}$$

In the short-counting-time approximation, we can equate the integrated intensity Ω to the square of the envelope r to within a proportionality constant.

Upon transforming to the new independent variable Ω, Eq. (B1) yields

$$W(\Omega|N) \sim (1/\langle\Omega\rangle)\, e^{-\Omega/\langle\Omega\rangle}[1 + C_2 + \cdots], \qquad \text{(B3)}$$

where

$$C_2 = \left(\frac{N\langle a^4\rangle - 2N\langle a^2\rangle^2}{N^2\langle a^2\rangle^2}\right)\left(\frac{2N^2\langle a^2\rangle^2 - 4N\langle a^2\rangle\Omega + \Omega^2}{4N^2\langle a^2\rangle^2}\right). \qquad \text{(B4)}$$

Note that the fourth moment, $\langle a^4\rangle$, enters into the correction term.

In the special case of identical particles $\langle a^4\rangle = \langle a^2\rangle^2$, the correction term reduces to the classical result of Pearson[12] and Rayleigh[9]:

$$C_2 = -\frac{1}{N}\left(\frac{1}{2} - \frac{\Omega}{\langle\Omega\rangle} + \frac{\Omega^2}{4\langle\Omega^2\rangle^2}\right). \qquad \text{(B5)}$$

If the particles are distributed according to the narrow Gaussian, Eq. (38), then

$$C_2 = -\frac{(1 - 2\delta - \delta^2)}{N}\left(\frac{1}{2} - \frac{\Omega}{\langle\Omega\rangle} + \frac{\Omega^2}{4\langle\Omega^2\rangle^2}\right). \qquad \text{(B6)}$$

The asymptotic expression for $P(m|N)$ can be obtained by appropriate integration of Eq. (B3). The final result is

$$P(m|N) \sim B_m(\langle\Omega\rangle) + \left(\frac{N\langle a^4\rangle - 2N\langle a^2\rangle^2}{N^2\langle a^2\rangle^2}\right)\left(\tfrac{1}{2}B_m(\langle\Omega\rangle) - \frac{(m+1)}{\langle\Omega\rangle}B_{m+1}(\langle\Omega\rangle) + \frac{(m+1)(m+2)}{4\langle\Omega\rangle^2}B_{m+2}(\langle\Omega\rangle)\right), \qquad \text{(B7)}$$

where

$$B_m(\langle\Omega\rangle) \equiv \langle\Omega\rangle^m/(1+\langle\Omega\rangle)^{m+1} . \qquad \text{(B8)}$$

The leading terms of Eqs. (B3) and (B7) are the negative exponential probability density function, Eq. (15), and the Bose-Einstein distribution, Eq. (16).

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