Relativistic distribution functions and applications to electron beams

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Equilibrium and monoenergetic distribution functions in the laboratory frame are derived from first principles. Thermodynamically consistent macroscopic properties of relativistic beam systems are defined and discussed as a foundation for specific applications. Characteristics of superpinched relativistic electron beams such as isotropy in the lab frame, rms cone angle of the electrons with respect to the beam axis, and the energy flux across a unit surface at an arbitrary angle to the beam axis are obtained. These properties provide important information on the expected symmetry of irradiation of fusion targets by focused electron beams. A linearized kinetic theory is outlined which leads to an integral expression for the general form of the relativistic plasma dispersion function.

I. INTRODUCTION

The continuing development of high-current relativistic electron accelerators[1] and their fusion applications to plasma heating in confinement devices[2] and inertial confinement systems[3] has provided additional motivation for work involving relativistic distribution functions. Electron beams produced by such accelerators can display relativistic effects in either their net drift motion or random thermal motion, or both. In some cases, as we shall discuss below, the beams may even approach the characteristic of a hot relativistic gas. Although the high currents give rise to important self-field effects,[4] much can be learned about properties of these systems without a detailed treatment of self-consistent fields, merely by examining certain moments of the distribution function.

The relativistic equilibrium has been given previously in the rest frame[5-9] (zero net drift) and in the laboratory frame.[7,10] It is possible to develop relativistic fluid equations by taking moments of this distribution.[10-13] Much of the previous work has used manifestly covariant four-dimensional notation. This approach is elegant in its ease of formal manipulation, but it can sometimes obscure physical quantities which one would like to compare directly with experiment. For this reason the authors will remain in three-vector notation throughout. In addition, we emphasize the evaluation of lab-frame quantities which are of direct physical interest to experimenters. Analysis of lab-frame integrals is presented which permits calculation of some quantities which apparently cannot be evaluated analytically by transformation to the rest frame.

Many laboratory relativistic electron beams are neither cold enough to use cold-beam approximations,[14-17] nor are they sufficiently cool to satisfy the paraxial or two-mass approximation.[17-19] Therefore, we concentrate on the general form of the relativistic equilibrium distribution function and also derive from it a monoenergetic lab-frame distribution function for comparison of results.

In Sec. II, we provide a thermodynamically consistent derivation of the equilibrium distribution function from first principles. Macroscopic quantities such as temperature, internal (random kinetic) energy, and macroscopic kinetic energy in the laboratory frame, and their properties under Lorentz transformation, are presented. For nonequilibrium systems such as relativistic electron beams produced in high-current diodes, we derive a monoenergetic form of a distribution function in Sec. III, and discuss some of its macroscopic properties as compared to the equilibrium distribution. Sections II and III provide the foundation for fluid models which will be presented in a separate publication.

Applications to characteristic properties of superpinched electron beams and symmetry considerations for fusion pellet targets are presented in Sec. IV. Leaving the realm of fluidlike properties of relativistic beams, Sec. V takes a look at relativistic kinetic theory and defines a single integral representation of the general form of the relativistic plasma-dispersion function in the lab frame. Appendices A–C contain some of the formalism used to obtain the results given in the bulk of the paper.

II. RELATIVISTIC EQUILIBRIUM DISTRIBUTION FUNCTION AND MACROSCOPIC PROPERTIES

Here we derive the general form of the relativistic equilibrium distribution function in an arbitrary Lorentz frame. The starting point taken here follows earlier work,[18] and is presented for completeness. However, the development is carried further here in a thermodynamically consistent definition of macroscopic properties.

Moreover, the procedure by which integrals over the distribution function can be performed directly in the laboratory frame opens up new applications of the relativistic distribution function, some of which will be given in Secs. IV and V.

Our choice of notation is as follows: The number of particles (dN) within a phase-space volume between $(\vec{x}, \vec{w})$ and $(\vec{x}+d\vec{x}, \vec{w}+d\vec{w})$ will be given by

$$dN = f(\vec{x}, \vec{w})\, d^3x\, d^3w\ . \tag{1}$$

We have scaled the momentum by the rest mass times the speed of light, $\vec{w} = \vec{p}/mc$. For a given species volume element, the average particle density, momentum density, total energy density, and scaled fluid velocity are given by

$$n = \int f(\vec{x}, \vec{w})\, d^3w\ , \tag{2}$$

$$\vec{P} = mc \int \vec{w} f(\vec{x}, \vec{w})\, d^3w\ , \tag{3}$$

$$\epsilon = mc^2 \int \gamma f(\vec{x}, \vec{w})\, d^3w\ , \tag{4}$$

$$\vec{\beta} = \frac{1}{n}\int \frac{\vec{w}}{\gamma} f(x, w)\, d^3w\ , \tag{5}$$

where

$$\gamma = (1+w^2)^{1/2} = 1/(1-v^2)^{1/2}\ , \tag{6}$$

and v is the dimensionless particle velocity.

We will define the rest frame as that frame in which $\vec{\beta} = 0$; any other frame characterized by $\vec{\beta} \neq 0$ and $\Gamma = (1-\beta^2)^{-1/2}$ will be referred to as a laboratory frame. For convenience and later usage, we list the relations between some laboratory (L) and rest (R) frame[20] quantities,

$$\vec{v}_R = \left[\frac{\vec{v}_L}{\Gamma} + \left(\frac{(\Gamma-1)}{\Gamma\beta^2}\vec{\beta}\cdot\vec{v}_L - 1\right)\vec{\beta}\right]\frac{1}{1-\vec{\beta}\cdot\vec{v}_L}, \tag{7}$$

$$\vec{w}_R = \vec{w}_L + \beta^{-2}\left[(\Gamma-1)\vec{\beta}\cdot\vec{w}_L - \Gamma\beta^2\gamma_L\right]\vec{\beta}, \tag{8}$$

$$\gamma_R = \Gamma(\gamma_L - \vec{\beta}\cdot\vec{w}_L) = \Gamma\gamma_L(1-\vec{\beta}\cdot\vec{v}_L)\ . \tag{9}$$

Note that the reverse transformations are obtained simply by switching the subscripts L and R and replacing $\vec{\beta}$ by $-\vec{\beta}$ in the above expressions.

The equilibrium distribution function is defined to be that distribution function which maximizes the specific entropy (s), consistent with fixed values for $\vec{P}$ and ϵ in a given laboratory frame.[18] The specific entropy is defined in terms of the distribution function through the Boltzmann H function,[21]

$$s = -k_B H = -k_B \int f(\vec{x}, \vec{w}) \ln f(\vec{x}, \vec{w})\, d^3w\ , \tag{10}$$

where k_B is the Boltzmann constant. Introducing the Lagrange multipliers $\vec{\delta}$ and α for $\vec{P}/mc$ and ϵ/mc^2, respectively, we use standard techniques[22] from the calculus of variations to find the form of $f(x, w)$ which maximizes s:

$$f(\vec{x}, \vec{w}) = Ae^{-\alpha\gamma + \vec{\delta}\cdot\vec{w}}\ . \tag{11}$$

Evaluation of A, α, $\vec{\delta}$ in terms of physical quantities may be accomplished by substituting Eq. (10) into Eqs. (2) and (5) giving (cf. Appendix A)

$$n_L = [4\pi K_2(\lambda\alpha)/\alpha\lambda^2]A \tag{12}$$

and

$$\vec{\beta} = \frac{\alpha\lambda^2}{4\pi K_2(\alpha\lambda)}\frac{\partial}{\partial\vec{\delta}}\int e^{-\alpha\gamma+\vec{\delta}\cdot\vec{w}}\frac{d^3w}{\gamma}, \tag{13}$$

where we have defined

$$\lambda^2 \equiv 1 - \vec{\delta}\cdot\vec{\delta}/\alpha^2$$

and $K_2(\alpha\lambda)$ is a modified Bessel function. The integral in Eq. (13) is performed using the same technique as given in Appendix A, and the derivative is performed by making use of the recursion relation [8.485.15],[23] so that

$$\vec{\beta} = \frac{\lambda^2}{K_2(\lambda\alpha)}\frac{\partial}{\partial\vec{\delta}}\left(\frac{K_1(\lambda\alpha)}{\lambda}\right) = \frac{\vec{\delta}}{\alpha}\ . \tag{14}$$

This identifies $\vec{\delta}$ as being proportional to $\vec{\beta}$ and determines $\lambda = \Gamma^{-1} = (1-\beta^2)^{1/2}$. We now express the distribution function in the form

$$f = \frac{n\alpha\lambda^2}{4\pi K_2(\alpha\lambda)}e^{-\alpha(\gamma - \vec{\beta}\cdot\vec{w})}$$

$$= \frac{n_L\xi}{4\pi\Gamma K_2(\xi)}e^{-\xi\Gamma(\gamma_L - \vec{\beta}\cdot\vec{w}_L)} \tag{15a}$$

$$= \frac{n_R\xi}{4\pi K_2(\xi)}e^{-\xi\gamma_R}\ . \tag{15b}$$

The invariant parameter $\xi = \alpha/\Gamma$ has been used in the last two expressions, which give the distribution function in laboratory-frame and rest-frame variables. These expressions show its Lorentz invariance[24] clearly.

Before a discussion of the parameter ξ is attempted, a remark is in order concerning relativistic thermodynamics. Throughout this paper the authors take as an assumption the form invariance of the equations of nonrelativistic thermodynamics. By this we mean that all the equations of nonrelativistic thermodynamics are assumed to hold in any Lorentz frame unchanged in form. This assumption then allows one to uniquely specify the transformation properties of all thermodynamic quantities.[25]

Proceeding now to the identification of ξ, we use the thermodynamic definition of the temperature as the derivative of the internal energy with respect to the entropy at constant volume.[26] By using Eq. (15b) in Eq. (10) and recognizing that the total energy density in the rest frame is just internal plus rest mass ($\epsilon_R = U_R + n_R mc^2$), we obtain

$$s_R = \frac{k_B \xi}{mc^2} U_R - k_B n_R \ln\left(\frac{n_R \xi e^{-\xi}}{4\pi K_2(\xi)}\right), \tag{16}$$

$$\xi = mc^2/k_B T_R . \tag{17}$$

Therefore, ξ^{-1} is the rest-frame temperature scaled by the rest-mass energy.

We now give expressions for some moments which we will use later. The lab-frame expressions can be obtained either by direct evaluation as developed here, or by transforming to the rest frame,[12] using the proper transformation of the momentum volume element[27]

$$d^3w_L = (\gamma_L/\gamma_R)\, d^3w_R = \Gamma(1 + \vec{\beta}\cdot\vec{w}_R/\gamma_R)\, d^3w_R . \tag{18}$$

The total energy density in the rest frame is obtained by evaluating Eq. (4):

$$\epsilon_R = n_R mc^2 + U_R = n_R mc^2[1 + \mu(\xi)/\xi], \tag{19}$$

with

$$\mu(\xi) = -\xi\left(1 + \frac{d}{d\xi}\ln\frac{K_2(\xi)}{\xi}\right).$$

Here, $\mu(\xi)$ is a monotonic function which increases from a value of $\frac{3}{2}$ for nonrelativistic temperatures ($\xi >> 1$), to a value of 3 for ultrarelativistic temperatures ($\xi << 1$). The internal energy density in the rest frame is defined as the random kinetic energy density of the particles,

$$U_R = \mu(\xi) n_R k_B T_R . \tag{20}$$

The internal state of the system is determined by U_R, which satisfies the virial theorem only in the nonrelativistic temperature limit.

The average momentum [Eq. (3)] vanishes in the rest frame, since f_R is isotropic. The momentum flux tensor $\overleftrightarrow{P}$ is related to the pressure tensor $\overleftrightarrow{\pi}$ (random momentum flux) by

$$\overleftrightarrow{\pi} \equiv c\int (mc\vec{w} - \vec{P}/n)(\vec{w}/\gamma) f\, d^3w = \overleftrightarrow{P} - c\vec{\beta}\,\vec{P} . \tag{21}$$

This relation reduces to the proper nonrelativistic expression for the pressure tensor, and is the same form which is obtained in deriving the relativistic fluid equations by taking moments of a kinetic equation. Whereas the momentum flux tensor gives the total momentum flux in a given direction, the pressure tensor gives only the flux of the random component of momentum in a given direction. This difference vanishes in the rest frame where the isotropic equilibrium distribution gives

$$\overleftrightarrow{\pi}_R = \overleftrightarrow{P}_R = p_R \overleftrightarrow{1}, \tag{22}$$

$$p_R \equiv \frac{mc^2}{\beta^2}\int (\vec{\beta}\cdot\vec{w}_R)^2 f \frac{d^3w_R}{\gamma_R} \tag{23a}$$

$$= n_R k_B T_R , \tag{23b}$$

with $\overleftrightarrow{1}$ the unit tensor, and where we have used Eq. (15b) and [8.432.9][23] to evaluate the integral. The scalar pressure p_R obeys the ideal-gas law, which can also be obtained by applying the appropriate thermodynamic Maxwell relation[26] to the expression for the rest-frame entropy density, Eq. (10).

We now determine the form of the corresponding laboratory-frame expressions for these quantities. The expressions may be obtained by direct evaluation or transformation of the integrands back to the rest frame. The total energy and momentum densities in the lab frame are found to be

$$\epsilon_L = \Gamma n_L mc^2[1 + \beta^2/\xi + \mu(\xi)/\xi] \tag{24a}$$

$$= \Gamma^2(\epsilon_R + \beta^2 p_R) \tag{24b}$$

$$= n_L mc^2 + U_L + K_L, \tag{24c}$$

$$\vec{P}_L = \Gamma n_L mc[1 + 1/\xi + \mu(\xi)/\xi]\vec{\beta} \tag{25a}$$

$$= \Gamma^2(\epsilon_R + p_R)\vec{\beta}/c . \tag{25b}$$

The total energy density in the lab frame now contains a macroscopic kinetic energy in addition to the rest-mass energy and internal (random kinetic) energy densities.

We will need explicit expressions for U_L and K_L for a calculation in Sec. IV. Based on the assumption of form invariance of thermodynamic equations, it can be shown that $U_L = U_R$. This can also be explained physically by the fact that the internal energy density characterizes an internal state of the system which is independent of relative motion. The expressions for U_L and K_L are found to be

$$U_L = mc^2\int (\gamma_L - \vec{\beta}\cdot\vec{w}_L - 1/\Gamma) f\, d^3w_L \tag{26a}$$

$$= n_L mc^2 \mu(\xi)/\xi\Gamma \tag{26b}$$

$$= \epsilon_L - c\vec{\beta}\cdot\vec{P}_L - n_L mc^2/\Gamma, \tag{26c}$$

$$K_L = c\vec{\beta}\cdot\vec{P}_L - (\Gamma - 1)n_L mc^2/\Gamma . \tag{27}$$

The lab-frame entropy density can now be expressed in terms of the internal energy density

$$s_L = \frac{\Gamma k_B \xi}{mc^2} U_L - k_B n_L \ln\left(\frac{n_L \xi e^{-\xi}}{4\pi\Gamma K_2(\xi)}\right), \tag{28}$$

so that the thermodynamic definition of the lab-frame temperature gives $T_R = \Gamma T_L$.[20] Note that the entropy density transforms as $s_L = \Gamma s_R$.

The pressure tensor for this relativistic ideal gas in the lab frame can be evaluated from Eq. (19), with the result $\overleftrightarrow{\pi}_L = p_R \overleftrightarrow{1} = \overleftrightarrow{\pi}_R$. The pressure tensor is isotropic since it gives the momentum flux in the fluid frame (random momentum flux in lab frame), and we are using a distribution function which is isotropic in the fluid (rest) frame.

There are many physical systems where it is difficult to justify the use of the equilibrium distribution function; so we next consider a specific nonequilibrium distribution.

III. MONOENERGETIC DISTRIBUTION FUNCTIONS

Under certain conditions in relativistic electron-beam diodes (such as neglect of beam scattering, time-dependent space charge, and voltage effects) the energy of the beam particles can be considered to be a function only of their position across the gap. Thus at a given position in the lab frame, the beam distribution can be taken as monoenergetic,[13]

$$f_L = \delta(w_L - w_0) G(\Omega_L) ,$$

where Ω_L is the solid angle in the lab-frame momentum space.

We are guided in our selection of $G(\Omega_L)$ by recognizing that f_L should have the following properties:

(i) $\vec{\beta} \equiv \int f_L(\vec{w}_L/\gamma_L)\, d^3w_L \neq 0$;

(ii) $f_L \geq 0$;

(iii) f_L should possess a parameter which characterizes randomness in momentum space (for the equilibrium distribution this parameter is the temperature).

Our investigation of possible functions satisfying (i)–(iii) has resulted in the selection $G(\Omega_L) \propto e^{\vec{a}\cdot \vec{w}_L}$ as being the simplest analytical form. This result may be obtained by multiplying Eq. (15a) by $\delta(w_L - w_0)$, a procedure which has been used previously[28] to obtain monoenergetic distribution functions from other distributions. Renormalizing to the lab-frame density [Eq. (2)] gives the form

$$f_a = \frac{n_L a \delta(w_L - w_0)}{4\pi w_0^2 \sinh a} e^{\vec{a}\cdot \vec{w}_L/w_0} , \tag{29}$$

where $\vec{a}$ is related to the drift velocity $\vec{\beta}$ by Eq. (5), which gives

$$\vec{\beta} = \frac{w_0}{\gamma_0}\left(\coth a - \frac{1}{a}\right)\frac{\vec{a}}{a}. \tag{30}$$

For convenience we define the ratio of the drift velocity to the particle velocity by

$$b \equiv \beta\gamma_0/w_0 = \coth a - 1/a . \tag{31}$$

Note that not only is f_a anisotropic in the lab frame (for $b \neq 0$), *but it is also anisotropic in every other Lorentz frame*, including the rest frame. If, for some reason, it is desired to have an isotropic monoenergetic distribution function, then it must be chosen monoenergetic in the rest frame. However, a monoenergetic form in the lab frame has a stronger physical basis for our purposes here.

The lab-frame expressions for the total energy density [Eq. (4)], internal energy density [Eq. (26a)], average momentum [Eq. (3)], and pressure tensor [Eq. (21)] for the distribution f_a are

$$\epsilon_0 = n_L m c^2 \gamma_0 , \tag{32}$$

$$U_0 = n_L m c^2(\gamma_0 - \Gamma)/\Gamma , \tag{33}$$

$$\vec{P}_0 = n_L m c \gamma_0 \vec{\beta} , \tag{34}$$

$$\overleftrightarrow{\pi}_0 = p_0[\overleftrightarrow{1} - g(b)\hat{e}_\beta \hat{e}_\beta] , \tag{35}$$

with $\hat{e}_\beta = \vec{\beta}/\beta$, $p_0 = n_L m c^2 w_0 \beta/a$, and $g(b) = 3 - (1 - b^2)a/b$. Note that these quantities defined by averages over a nonequilibrium distribution do not have the same thermodynamic implications as the corresponding quantities in Sec. II.

For a given value of b, a is determined from Eq. (31), and therefore, the anisotropic part of the pressure tensor depends only on the ratio of the dirft speed to the particle speed. The anisotropy factor g is always slightly larger than b^2, becoming equal to b^2 for a cold beam ($b = 1$). In this limit, Eq. (31) requires a to approach infinity so that the isotropic term vanishes, as it should. In the opposite limit of a slowly drifting beam ($b < 0.1$), we find $a \approx 3b(1 + \frac{3}{5}b^2)$, so that $p_0 \approx n_L m c^2 w_0^2/3\gamma_0$, and $g \approx \frac{6}{5}b^2$. It should be stressed that b is the ratio of the drift velocity to the particle velocity, which can be greater than the ratio of either of these velocities to the speed of light.

Note that, for the equilibrium distribution function, $\vec{\beta}$ is simply a Lorentz-frame label, whereas here $\vec{\beta}$ designates the specific lab frame in which the distribution f_a is chosen to be monoenergetic, as distinguished from other arbitrary Lorentz frames.

IV. SOME APPLICATIONS TO RELATIVISTIC ELECTRON BEAMS

In this section we investigate some applications of the previously developed formalism and expressions to problems of electron-beam irradiation of fusion pellets.[29] The first important concept to be discussed is that of a superpinched electron beam,[4] which is necessary to concentrate the electron-beam energy on a small target with dimensions on the order of a millimeter. To achieve a reasonable degree of symmetric loading, it is desirable to have the superpinched beam behave as nearly as possible like a hot electron gas. At the very least, its internal energy in the lab frame should exceed its drift kinetic energy. In terms of the quantities defined in Sec. II, the ratio $R = U_L/K_L$ for the equilibrium and monoenergetic distributions considered previously is given by

$$R_{\rm eq}=\frac{\mu(\xi)}{(\Gamma-1)[(\Gamma+1)(1+\mu)+\xi\Gamma]}, \tag{36a}$$

$$R_0=\frac{1-\Gamma/\gamma_0}{(\Gamma-1)(\Gamma+1-\Gamma/\gamma_0)}. \tag{36b}$$

From these expressions it is clear that there is some critical drift speed, with $\Gamma_c<2$, above which the beam will always have more drift kinetic energy than internal energy in the lab frame. From Eq. (36b) we see that

$$\Gamma_c=\frac{(1+2\gamma_0^2-2\gamma_0)^{1/2}-1}{\gamma_0-1}\leqslant\sqrt{2}\ .$$

From Eq. (36a), for a relatively cold beam ($\xi\gg 1$), we find $\Gamma_c\simeq 1+7/\xi$, and for a hot beam $\Gamma_c=1.26$. In both PIC simulations and relativistic fluid-envelope calculations,[4] superpinched electron beams seem to have drift velocities which satisfy $\Gamma<\Gamma_c$, and thus have more energy in random thermal motion than in axial-drift motion.

It may seem surprising at first that, regardless of the thermal-energy content of the beam, it can have characteristics of a cold beam for moderate values of Γ. Moreover, this result does not qualitatively depend on the form chosen for the distribution, but in fact arises from relativistic mechanics. To illustrate this, we consider a single particle which has a velocity component v in the axial direction in the lab frame. We define a "particle frame" as that frame translating axially with speed v. In this particle frame, the particle velocity is strictly perpendicular to the x axis, say of magnitude u along the y axis. From Eq. (2) we have the lab-frame velocity components $V_\parallel=v$, $V_\perp=u/\Gamma$. Defining a transverse kinetic energy content of the particle in the lab frame by $E_\perp=mc^2[(1-V_\perp^2/c^2)^{-1/2}-1]$, and a longitudinal kinetic energy content by $E_\parallel=mc^2[(1-V_\parallel^2/c^2)^{-1/2}-1]$, we find that their ratio lies in the range

$$0\leqslant\frac{E_\perp}{E_\parallel}\leqslant\frac{1-\beta}{\beta(\Gamma-1)}\ ,$$

with $\beta=v/c$, $\Gamma=(1-\beta^2)^{-1/2}$. Therefore for $\Gamma\beta>1$ ($\Gamma>\sqrt{2}$) there is more energy content in the axial motion in the lab frame than in the transverse motion, regardless of the size of $u\leqslant c$. We will find this sensitivity of lab frame averages to the drift velocity occurs in subsequent calculations also.

Turning now to the computation of another important relativistic electron-beam parameter, we define the rms cone half-angle of beam electrons in the lab frame with respect to the beam axis by

$$\sin^2\bar{\theta}=\langle p_\perp^2\rangle/\langle p^2\rangle, \tag{38}$$

where the brackets indicate averages over the appropriate distribution function. This quantity gives a measure of the rms angle of incidence of beam electrons on a plane surface normal to the beam axis, which can be compared with particle simulations, and is important input for Monte Carlo energy-deposition calculations.

For the equilibrium distribution, it is possible to perform the integrations in the lab frame by the methods of Appendix A, or by transformation to the rest frame. However, it is easier to use Eq. (11) to express Eq. (38) in terms of derivatives of the normalization integral:

$$\sin^2\bar{\theta}=\frac{\partial^2}{\partial\vec{\delta}_\perp^2}\left(\frac{K_2(z)}{z}\right)\Big/\frac{\partial^2}{\partial\vec{\delta}^2}\left(\frac{K_2(z)}{z}\right), \tag{39}$$

with $z=\alpha\lambda=(\alpha^2-\delta^2)^{1/2}$. The differentiations are readily performed, and α and δ are given their physical meanings to yield

$$\sin^2\bar{\theta}_{\rm eq}=\tfrac{2}{3}[1+\tfrac{1}{3}\beta^2\Gamma^2\xi K_4(\xi)/K_3(\xi)]^{-1}. \tag{40a}$$

Using the same procedure for the monoenergetic distribution, we obtain

$$\sin^2\bar{\theta}_0=2b/a, \tag{40b}$$

with b and a defined by Eq. (31). There are similarities between Eqs. (40a) and (40b). For a completely stagnated beam ($b\to 0$), we find from both of these expressions that the rms cone half-angle of an electron in a system with zero net drift is 54.7° with respect to an arbitrarily chosen axis. The factor involving ξ in Eq. (40a) increases monotonically from a value of 6 for $\xi\lesssim 1$ (hot beam), to a value equal to ξ for $\xi\gg 1$. Thus, for hot beams, the rms angle $\bar{\theta}_{\rm eq}$ becomes independent of the beam temperature, allowing us to fix an upper bound on $\bar{\theta}_{\rm eq}$ given by

$$\sin^2\bar{\theta}_{\rm eq}\leqslant 2/3(2\Gamma^2-1), \tag{41}$$

where we have used the identity $\Gamma^2=1+\beta^2\Gamma^2$. Using this upper bound, we see that $\bar{\theta}_{\rm eq}$ drops rapidly from its isotropic value as the beam drift velocity increases, falling to less than 11° for $\Gamma=3$. Similar behavior is obtained from Eq. (40b).

A distribution of electron angles of incidence near the beam axis at the anode plane was obtained from a particle simulation[30] of a superpinched beam. The averages over this distribution gave $\theta_{\rm rms}=32.5°$ and a beam drift $\Gamma=1.46$. Using this value of Γ in Eq. (41), we find $\bar{\theta}_{\rm eq}\leqslant 29°$, and using the value $\gamma_0=2.2$ (also obtained from the simulation), Eq. (40b) gives $\bar{\theta}_0=31.5°$. This agreement is quite good, considering the crudeness of the distribution generated from the simulation.

Although the results obtained thus far indicate that superpinched beams display some tendencies toward hot gas behavior, it is important to quantify the results further by considering the energy flux carried across a surface with unit normal $\hat{e}_s$ which is at an angle θ with respect to the beam axis. Thus we can consider the symmetry of the energy flux across the surface of an arbitrarily shaped target.

The one-way energy flux across the surface is defined by

$$\Psi = -mc^2 \int_{\Omega'} \hat{e}_s \cdot \vec{v}_L (\gamma_L - 1) f \, d^3 w_L$$
$$= -mc^3 \hat{e}_s \cdot \int_{\Omega'} \vec{w}_L (1 - 1/\gamma_L) f \, d^3 w_L ,$$

where the momentum integration is over the subspace Ω' defined by $\hat{e}_s \cdot \vec{w}_L < 0$, corresponding to a flux into the surface. Choosing the $\hat{e}_s$ axis as the polar axis in momentum space, the expression for the flux can be written as

$$\Psi = -mc^3 \int_0^\infty w^2 \, dw \int_0^{2\pi} d\phi \int_{\pi/2}^{\pi} \sin\theta_w \, d\theta_w (1 - 1/\gamma) \times [w \cos\theta_w f(\vec{r}, \vec{w})] , \tag{42}$$

where we have dropped the laboratory subscripts. This expression must be evaluated in the lab frame, since a transformation to the rest frame in this case hopelessly distorts the momentum subspace Ω', making a closed-form evaluation impossible. However, even the lab-frame evaluation is sufficiently abstruse that we have given the details in Appendix B. Designating the component of the drift along $\hat{e}_s$ by $\hat{e}_s \cdot \beta = -\beta \cos\theta$, and defining the quantities $\nu = \beta\Gamma \cos\theta$, $\mu^2 = 1 + \nu^2$, $\eta = \ln(\mu + \nu)$, we find the energy flux into the surface for the drifting equilibrium distribution to be

$$\Psi = \frac{n_L m c^3}{2K_2(\xi)} \left[\left(\sum_{l=0}^{5} c_l (\mu\xi)^l \right) \frac{e^{-\mu\xi}}{(\mu\xi)^3} + \nu \left(K_3(\xi) + K_3(\eta, \xi) - \frac{K_2(\xi)}{\Gamma} - \frac{K_2(\eta, \xi)}{\Gamma} \right) \right] , \tag{43}$$

with

$$c_0 = 4\mu^2(\mu^2 + \nu^2) - 1 , \quad c_1 = c_0 - \mu(\mu^2 + \nu^2)/\Gamma , \quad c_2 = 1 - \mu(\mu^2 + \nu^2)/\Gamma + \nu^2(4\mu^2 + 1) ,$$
$$c_3 = \nu^2[(3\mu^2 + \nu^2)/3 - \mu/\Gamma] , \quad c_4 = \tfrac{1}{3}\nu^4(1 - 1/\mu\Gamma) , \quad c_5 = \nu^6/15\mu^2 ,$$

where $K_3(\xi)$ is a modified Bessel function and $K_3(\eta, \xi)$ is its associated incomplete Bessel function.[31] For a stagnated beam, $\nu = 0$ ($\mu = 1$), and Eq. (43) reduces to

$$\Psi(\mu = 1) = \frac{mc^3}{2K_2(\xi)} n_L (3 + 2\xi) \frac{e^{-\xi}}{\xi^3} , \tag{44}$$

so that the only anisotropy possible would arise from a nonuniform beam density ($n_L \neq$ constant). For a relatively hot beam ($\xi \lesssim 0.1$), we obtain

$$\Psi(\xi \lesssim 0.1) = n_L (c k_B T_R / 4\mu^3) \times [4\mu^2(\mu^2 + \nu^2) - 1 + 8\nu\mu^3 - (\mu^2 + \nu^2 + 2\nu\mu)\mu^2 \xi/\Gamma] , \tag{45}$$

whereas in the exact cold-beam limit ($\xi >> 1$) we find the expected result

$$\Psi(\xi >> 1) = n_L m c^3 \nu (1 - 1/\Gamma)$$
$$= n_L (mc^2\Gamma)(c\beta)(\Gamma - 1)\cos\theta .$$

Since our previous results imply that isotropy can only be approached in the lab frame for $\beta\Gamma \lesssim 1$, we obtain the expression for the energy flux correct to first order in $\beta\Gamma$ by setting $\mu = 1$ in Eq. (43) and keeping only the term linear in ν [$K_3(\eta, \xi) \sim \nu^7$]:

$$\Psi(\beta\Gamma << 1) = \frac{n_L m c^3}{2K_2(\xi)} \{(3 + 2\xi) e^{-\xi}/\xi^3 + \beta[K_3(\xi) - K_2(\xi)] \cos\theta\} . \tag{46}$$

The relative degree of anisotropy due to net beam drift motion (ignoring beam-profile effects) is obtained by taking the ratio of Eqs. (46) and (44) to obtain

$$Q(\beta, \theta) = 1 + F(\xi)\beta \cos\theta , \tag{47}$$

with

$$F(\xi) = \frac{\xi^3 e^{\xi}[K_3(\xi) - K_2(\xi)]}{(2\xi + 3)} \geq 2.67 .$$

Therefore, we see that a 10% requirement in symmetry over one hemisphere of a target pellet for a uniform beam requires $\beta < 0.0375$ ($V_{\text{drift}} \lesssim 10^9$ cm/sec). This appears to put severe restrictions on the degree of beam stagnation. However, we have neglected the possible isotropizing effect of the large surface scattering of high-energy obliquely incident electrons on a high-atomic-number target. More than half of the incident electrons may be scattered and subsequently returned to the target by the diode electric field and beam self-fields. However, if further studies of this scattering effect do not appreciably relax the constraint obtained above, then we can infer a high degree of beam stagnation from experiments[29] which show behavior characteristic of symmetric loading.

V. APPLICATION TO RELATIVISTIC KINETIC THEORY

Although research into relativistic kinetic theory has been performed for some time[7–9, 32–40] the continuing development of high-current relativistic electron beams is providing a further stimulus for studying relativistic beam-plasma interactions which concern beam propagation or plasma heating. The basic approach taken in most of the work to date is based on either a cold beam where the beam-particle velocities are all parallel, or on the paraxial or two-mass approximation. The latter case applies to a cool beam, where the transverse momentum of beam particles is small compared to their total momentum. The exact form of the relativistic distribution, Eq. (15), can then be simplified by expansion of $\gamma=(1+w^2)^{1/2}$ and renormalized to a bi-Maxwellian distribution with a parallel mass $m_\parallel=\Gamma^3 m$ and a perpendicular mass $m_\perp=\Gamma m$.

However, many high-current beams generated in the laboratory may not satisfy either of these simplifying assumptions. Therefore, it is necessary to develop the theory using the general form of Eq. (15). The purpose of this investigation (to provide a foundation for exact relativistic kinetic theory) is identical to that of recent work,[39] but the approach is different. Rather than performing the analysis in manifestly covariant four-vector notation, we follow the philosophy of the other sections of this paper in remaining in three-vector space, where our expressions may be directly related to physical quantities. We will find an integral form for the plasma-dispersion function in the lab frame below which can be reduced to an expression obtained elsewhere in the rest frame of a species.

Since we will be concerned with laboratory-frame quantities throughout this section, the subscript on laboratory variables will be suppressed for convenience. For later comparison of results, we will need to make use of the Lorentz-invariant quantity[20]

$$\gamma\omega - c\vec{k}\cdot\vec{w} = \gamma_R\omega_R - c\vec{k}_R\cdot\vec{w}_R\,, \tag{48}$$

where ω is the radian frequency and $\vec{k}$ is the wave vector in the lab frame, which are related by the dispersion characteristics of the wave.

To begin the analysis, we consider a multicomponent plasma with species designated by α. Not all species need to be relativistic, but we assume that elastic and inelastic collisions are negligible so that the number of particles in a given species remains constant and the evolution of the particle distributions is governed by the collisionless Vlasov equation:

$$\frac{\partial f_\alpha}{\partial t} + \frac{c\vec{w}}{\gamma}\cdot\frac{\partial f_\alpha}{\partial\vec{x}} + \frac{q_\alpha}{m_\alpha}\left(\vec{E}+\frac{\vec{w}}{\gamma}\times\vec{B}\right)\cdot\frac{\partial f_\alpha}{\partial\vec{w}}=0\,, \tag{49}$$

with q_α the charge and m_α the rest mass of species α. The electric and magnetic fields arise from self-fields as well as externally imposed fields, and they satisfy Maxwell's equations. The net current due to drift motion of the species, and the net charge density are

$$\vec{j}=\sum_\alpha n_\alpha q_\alpha c\vec{\beta}_\alpha\,, \tag{50}$$

$$\sigma=\sum_\alpha n_\alpha q_\alpha\,, \tag{51}$$

where n and $\vec{\beta}$ are defined in Eqs. (2) and (5).

Defining a current-neutral, charge-neutral, external-field free-plasma equilibrium allows us to look for small deviations from such an equilibrium by assuming that the distribution functions deviate by a small amount from their equilibrium form. Solving the linearized Vlasov-Maxwell equations by a standard procedure,[40] we obtain the Fourier-Laplace-transformed self-consistent plasma electric-field response to initial perturbations ($\vec{a}$),

$$\overleftrightarrow{R}\cdot\vec{E}_{\vec{k},\omega}=\vec{a}\,. \tag{52}$$

The normal modes of the system are determined from the determinant of the tensor[40]

$$\overleftrightarrow{R}=(c^2k^2-\omega^2)\overleftrightarrow{1}-c^2\vec{k}\vec{k}-\sum_\alpha\frac{4\pi q_\alpha^2}{m\alpha}\int\left[\omega\frac{\partial f_\alpha}{\partial\vec{w}}+\frac{c\vec{k}}{\gamma}\times\left(\vec{w}\times\frac{\partial f_\alpha}{\partial\vec{w}}\right)\right]\frac{\vec{w}\,d^3w}{\gamma\omega-c\vec{k}\cdot\vec{w}}\,. \tag{53}$$

For nonrelativistic species at rest in the lab frame, using the Maxwell-Boltzmann distribution reduces the final term to the familiar form ($\vec{k}=k\hat{e}_3$)

$$\omega\omega_\alpha^2\overleftrightarrow{Q}_\alpha^{nr}=\frac{\omega_\alpha^2\omega}{u_\alpha^2(2\pi u_\alpha^2)^{3/2}}\int\frac{\vec{v}\vec{v}e^{-v^2/2u_\alpha^2}}{\omega-\vec{k}\cdot\vec{v}}d^3v \tag{54a}$$

$$=-\omega_\alpha^2\zeta_\alpha[Z(\zeta_\alpha)(\hat{e}_1\hat{e}_1+\hat{e}_2\hat{e}_2)-\zeta_\alpha Z'(\zeta_\alpha)\hat{e}_3\hat{e}_3]\,, \tag{54b}$$

where the plasma frequency is $\omega_\alpha=(4\pi n_\alpha e^2/m_\alpha)^{1/2}$, the thermal velocity is $u_\alpha=(k_BT_\alpha/m_\alpha)^{1/2}$, $\zeta_\alpha=2^{-1/2}\omega/ku_0$, and Z, Z' are the nonrelativistic plasma-dispersion function[41] and its first derivative.

For relativistic species (dropping the species subscripts), we use Eq. (15a) to evaluate the expression in the brackets of Eq. (53)

$$\omega\frac{\partial f}{\partial\vec{w}}+\frac{c\vec{k}}{\gamma}\times\left(\vec{w}\times\frac{\partial f}{\partial\vec{w}}\right)$$

$$=\xi\Gamma\left[(\omega-c\vec{k}\cdot\vec{\beta})\frac{\vec{w}}{\gamma}-\left(\omega-c\vec{k}\cdot\frac{\vec{w}}{\gamma}\right)\vec{\beta}\right]f\,. \tag{55}$$

The contribution of a relativistic species to the last term in Eq. (53) becomes

$$-\omega_b^2[\xi\Gamma^5\vec{\beta}\vec{\beta} - \Gamma(\omega - \vec{\kappa}\cdot\vec{\beta})\overleftrightarrow{Q}^r], \tag{56}$$

with

$$\overleftrightarrow{Q}^r = \frac{\xi^2\Gamma^3}{4\pi K_2(\xi)}\int \frac{d^3w}{\gamma}\frac{\vec{w}\vec{w}}{\gamma\omega - \vec{\kappa}\cdot\vec{w}}e^{-\xi\Gamma(\gamma - \vec{\beta}\cdot\vec{w})}, \tag{57}$$

with $\vec{\kappa} = c\vec{k}$, and the beam plasma frequency measured in the lab frame is $\omega_b^2 = 4\pi n e^2/m\Gamma^3$. Note that the factor multiplying $\overleftrightarrow{Q}^r$ in Eq. (56) is, by Eq. (48), just the wave frequency transformed to the rest frame

$$\omega_R = \Gamma(\omega - \vec{\kappa}\cdot\vec{\beta}). \tag{58}$$

The corresponding transformation of the wave vector can be obtained using Eqs. (8), (9), (48), and (58)

$$\vec{\kappa}_R = \vec{\kappa} + \beta^{-2}[(\Gamma - 1)\vec{\kappa}\cdot\vec{\beta} - \Gamma\beta^2\omega]\vec{\beta}. \tag{59}$$

Comparing Eq. (57) with Eq. (54), we would expect that it is possible to define a relativistic dispersion function by

$$W(\xi, \omega/k, \vec{\beta}) = \frac{-\kappa\Gamma}{4\pi K_2(\xi)}\int \frac{d^3w\, e^{-\xi\Gamma(\gamma - \vec{\beta}\cdot\vec{w})}}{\gamma(\gamma\omega - \vec{\kappa}\cdot\vec{w})}, \tag{60}$$

so that

$$\overleftrightarrow{Q}^r = -\frac{1}{\kappa}\frac{\partial}{\partial\vec{\beta}}\frac{\partial W}{\partial\vec{\beta}}, \tag{61}$$

where the relativistic drift factor Γ is considered to be constant in performing the differentiations. Two of the momentum-space integrals in Eq. (60) can be performed (Appendix C) by choosing the coordinate axes so that the wave vector and drift velocity are specified by $\vec{\kappa} = \kappa e_3$, $\vec{\beta} = \beta_\perp\hat{e}_\perp + \beta_\parallel\hat{e}_3$, with the result

$$W(\xi, \omega/\kappa, \vec{\beta}) = \frac{1}{2\xi K_2(\xi)}\int_{-1}^{1}\frac{dv\,\gamma}{v - \omega/\kappa}\frac{\exp\{-\xi\Gamma[(1-\beta_\parallel v)^2\gamma^2 - \beta_\perp^2]^{1/2}\}}{[(1-\beta_\parallel v)^2\gamma^2 - \beta_\perp^2]^{1/2}}, \tag{62}$$

where v is the scaled physical velocity component along $\vec{\kappa}$, and $\gamma = (1 - v^2)^{-1/2}$. The components of $\overleftrightarrow{Q}^r$ are found to be $(Q_{12} = Q_{21} = Q_{23} = Q_{32} = 0)$

$$Q_{11} = \frac{(2\pi)^{-1/2}}{\xi K_2(\xi)}\int_{-1}^{1}\frac{dv\,\gamma}{\omega - \kappa v}\left(\frac{K_{3/2}(z)}{z^{3/2}} + (\beta_\perp\Gamma\xi)^2\frac{K_{5/2}(z)}{z^{5/2}}\right), \tag{63a}$$

$$Q_{22} = \frac{(2\pi)^{-1/2}}{\xi K_2(\xi)}\int_{-1}^{1}\frac{dv\,\gamma}{\omega - \kappa v}\left(\frac{K_{3/2}(z)}{z^{3/2}}\right), \tag{63b}$$

$$Q_{33} = \frac{(2\pi)^{-1/2}}{\xi K_2(\xi)}\int_{-1}^{1}\frac{dv\,\gamma^3 v^2}{\omega - \kappa v}\left((\xi\Gamma\gamma)^2(1-\beta_\parallel v)^2\frac{K_{5/2}(z)}{z^{5/2}} - \frac{K_{3/2}(z)}{z^{3/2}}\right), \tag{63c}$$

$$Q_{13} = Q_{31} = \frac{(2\pi)^{-1/2}}{K_2(\xi)}\,\xi\Gamma^2\beta_\perp\int_{-1}^{1}\frac{dv\,\gamma^3}{\omega - \kappa v}v(1-\beta v)\left(\frac{K_{5/2}(z)}{z^{5/2}}\right), \tag{63d}$$

with $z = \xi\Gamma[(1-\beta_\parallel v)^2\gamma^2 - \beta_\perp^2]^{1/2}$, and the terms involving the modified Bessel functions of odd half-integer order can be expressed in terms of a finite inverse power series in z multiplied by e^{-z}.[42]

It is possible to relate the relativistic dispersion function, Eq. (60), to a function obtained earlier by Godfrey, Newberger, and Taggert,[39]

$$T(z, \xi) = \int_{-1}^{1}\frac{\exp[-\xi(1-x^2)^{-1/2}]}{x - z}dx. \tag{64}$$

This function is explicitly the relativistic dispersion function for $\vec{\beta} = 0$. But more generally, by transforming to the rest frame using Eqs. (7), (9), (18), (48), (58), and (59), we obtain

$$W(\xi, \omega/\kappa, \vec{\beta}) = \frac{\Gamma\kappa/\kappa_R}{2\xi K_2(\xi)}T(\xi, \omega_R/\kappa_R). \tag{65}$$

The T function has been analyzed and evaluated elsewhere.[39]

VI. SUMMARY AND CONCLUSIONS

We have presented a derivation of the relativistic-equilibrium-distribution function and have evaluated certain averages over that distribution in the laboratory frame. Some of these expressions, such as the energy-flux calculations in Appendix B, cannot be obtained analytically by the usual transformation of the integrand to rest-frame variables. A monoenergetic lab-frame distribution was introduced to compare moment calculations with the equilibrium distribution. The qualitative results obtained are fairly insensitive to the assumed form of the distribution function.

The pressure tensor in the lab frame for the monoenergetic distribution is genuinely anisotropic, with an anisotropy factor larger than the square of the ratio of the drift speed to the particle speed.

Expressions for the root-mean-square cone half-angle of the beam electron with respect to the beam axis were obtained, and compared with electron angles of incidence on a plane normal to the beam axis. The ratio of beam internal energy to drift kinetic energy was calculated in the lab frame, as well as the energy flux across a unit area surface at an arbitrary angle to the beam axis. These quantities provide information on the characteristics of superpinched electron beams, and deposition symmetry on fusion targets; and they depend strongly on the beam drift velocity. We have shown that, independent of the beam temperature, and for only moderately relativistic drifts, the beam exhibits relatively cold-beam behavior in the lab frame. In the absence of symmetrizing effects such as scattering, the beam must exhibit a high degree of stagnation for a reasonably symmetric irradiation of a hemispherical surface.

A basic development of relativistic kinetic theory given here defines a general form for the relativistic plasma dispersion function. This provides a foundation for further development of a physically motivated study of plasma waves and instabilities in systems with relativistic species.

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APPENDIX A

Here we evaluate the normalization integral given in Eq. (2) by utilizing spherical coordinates in momentum space:

$$n_L = A\int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} \exp\{-\alpha(1+w^2)^{1/2} + w[\sin\theta\,(\delta_1\cos\phi + \delta_2\sin\phi) + \delta_3\cos\theta]\}\, w^2 \sin\theta\, dw\, d\theta\, d\phi$$

$$= A\int_0^{\infty} w^2 e^{-\alpha\gamma}\left(\int_\Omega e^{\vec{\delta}\cdot\vec{w}}\, d\Omega\right) dw\,, \tag{A1}$$

where we have dropped the lab subscripts on the momentum. The angular integrations are carried out by expanding the exponentials in power series. The ϕ integration is performed using [3.621.5][23] and the doubling formula for gamma functions [8.335.1][23] to obtain

$$\int_0^{2\pi} \exp[w\sin\theta(\delta_1\cos\phi + \delta_2\sin\phi)]\, d\phi = 2\pi\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \frac{(\frac{1}{2}\delta_1)^{2k}}{k!}\,\frac{(\frac{1}{2}\delta_2)^{2l}}{l!}\,\frac{(w\sin\theta)^{2(k+l)}}{\Gamma(k+l+1)}\,. \tag{A2}$$

The θ integration is performed in the same manner, yielding

$$\int_\Omega e^{\vec{\delta}\cdot\vec{w}}\, d\Omega = 2\pi^{3/2}\sum_{k,l,m} \frac{(\frac{1}{2}w\delta_1)^{2k}}{k!}\,\frac{(\frac{1}{2}w\delta_2)^{2l}}{l!}\,\frac{(\frac{1}{2}w\delta_3)^{2m}}{m!}\,\frac{1}{\Gamma(k+l+m+\frac{3}{2})}\,. \tag{A3}$$

This result can be put in a more compact form by noticing that the m sum represents a modified Bessel function [8.445],[23] and that the two remaining sums may be collapsed by successive application of the multiplication theorem [9.6.51],[42] with the result

$$\int_\Omega e^{\vec{\delta}\cdot\vec{w}}\, d\Omega = [(2\pi)^{3/2}/(w\delta)^{1/2}]\, I_{1/2}(w\delta) = 4\pi(\sinh w\delta)/w\delta\,, \tag{A4}$$

where we have used [10.2.13][42] and $\delta = (\delta_1^2 + \delta_2^2 + \delta_3^2)^{1/2}$. Note that this is just the result we would have obtained if we had chosen the polar axis along $\vec{\delta}$. The same technique can be used to show that the integral in Eq. (A2) can be expressed as $2\pi I_0(\delta_\perp w\sin\theta)$, with $\delta_\perp^2 = \delta_1^2 + \delta_2^2$.

The remaining integral is performed by substituting Eq. (A3) into Eq. (A1) and using [8.432.9 and 8.486.15][23] to obtain

$$\int \exp[-\alpha(1+w^2)^{1/2}+\vec{\delta}\cdot\vec{w}]\, d^3w = \frac{4\pi}{\alpha}\sum_{k,l,m}\frac{(\delta_1^2/2\alpha)^k}{k!}\frac{(\delta_2^2/2\alpha)^l}{l!}\frac{(\delta_3^2/2\alpha)^m}{m!}K_{k+l+m+2}(\alpha)\,. \tag{A5}$$

Again, recognizing that the triple sum can be collapsed in succession by the multiplication theorem [9.6.51],[42] we obtain

$$\int \exp(-\alpha\gamma+\vec{\delta}\cdot\vec{w})\, d^3w = (4\pi/\alpha\lambda^2)K_2(\alpha\lambda) = (4\pi/\delta)\int_0^\infty w \sinh w\delta \exp[-\alpha(1+w^2)^{1/2}]\, dw\,, \tag{A6}$$

where $\lambda^2 = 1-(\delta/\alpha)^2$, and $\mathrm{Re}(\alpha-\delta)>0$. The last equality in Eq. (A6) gives a definite integral which, to our knowledge has not appeared in standard integral tables. An integral related to Eq. (A6) is given in Ref. 32. Equation (12) follows directly from Eqs. (A1) and (A6).

APPENDIX B

The energy flux into a surface of unit area with unit normal $\hat{e}_s$ is defined in Eq. (42). To evaluate this expression we use Eq. (8) to express Eq. (42) in the form

$$\frac{\Psi}{mc^3} = -A\frac{\partial}{\partial\delta_3}\int_0^\infty (1-1/\gamma)w^2 \exp(-\alpha\gamma)\int_{\Omega'} \exp(\vec{\delta}\cdot\vec{w})\, d\Omega' dw\,, \tag{B1}$$

where we have chosen $\hat{e}_s$ to lie along the 3-axis in momentum space. The flux is defined to be positive entering the surface, so that the θ integration has the limits $\frac{1}{2}\pi$ to π. The ϕ integration is given by Eq. (A2) and the resulting θ integration over half of its range is handled as before, giving

$$\int_{\Omega'} e^{\vec{\delta}\cdot\vec{w}}\, d\Omega = \pi^{3/2}\sum_{k,l,m}\frac{(\frac{1}{2}w\delta_1)^{2k}}{k!}\frac{(\frac{1}{2}w\delta_2)^{2l}}{l!}\left(\frac{(\frac{1}{2}w\delta_3)^{2m}}{m!\,\Gamma(k+l+m+\frac{3}{2})} - \frac{(\frac{1}{2}w\delta_3)^{2m+1}}{\Gamma(m+\frac{3}{2})\Gamma(k+l+m+2)}\right)\,. \tag{B2}$$

The final integral can be performed as in Appendix A, with the result

$$\frac{\Psi}{mc^3} = \frac{-2\pi}{\alpha}A\frac{\partial}{\partial\delta_3}\sum_{k,l,m}\frac{(\delta_1^2/2\alpha)^k}{k!}\frac{(\delta_2^2/2\alpha)^l}{l!}$$
$$\times\left(\frac{(\delta_3^2/2\alpha)^m}{m!}[K_{k+l+m+2}(\alpha)-K_{k+l+m+1}(\alpha)] - \frac{(\delta_3^2/2\alpha)^{m+1/2}}{\Gamma(m+\frac{3}{2})}[K_{k+l+m+5/2}(\alpha)-K_{k+l+m+3/2}(\alpha)]\right)\,. \tag{B3}$$

By defining $\lambda_1^2 = 1-(\delta_1/\alpha)^2$, $\lambda_2^2 = 1-(\delta_2/\alpha\lambda_1)^2$, $\lambda_\perp = \lambda_1\lambda_2$, $\lambda_3^2 = 1-(\delta_3/\alpha\lambda_\perp)^2$, $\lambda = \lambda_1\lambda_2\lambda_3 = (1-\delta^2/\alpha^2)^{1/2}$, all but one sum in Eq. (B3) can be performed using the multiplication theorem [9.6.51][42]

$$\frac{\Psi}{mc^3} = \frac{-2\pi A}{\alpha}\frac{\partial}{\partial\delta_3}\left[\frac{K_2(\lambda\alpha)}{\lambda^2} - \frac{K_1(\lambda\alpha)}{\lambda} + \sum_{m=0}^\infty \frac{(\delta_3^2/2\alpha\lambda_\perp)^{m+1/2}}{\Gamma(m+\frac{3}{2})}\left(\frac{K_{m+5/2}(\alpha\lambda_\perp)}{\lambda_\perp^2} - \frac{K_{m+3/2}(\alpha\lambda_\perp)}{\lambda_\perp}\right)\right]$$
$$= -2\pi A\left[2\left(\frac{1}{\lambda_\perp^2}\frac{d}{d\alpha}+1\right)\sum_{m=0}^\infty \frac{(2m+1)\delta_3^{2m}K_{m+3/2}(\alpha\lambda_\perp)}{\Gamma(m+\frac{3}{2})(2\alpha\lambda_\perp)^{m+3/2}} + \frac{\delta_3}{\alpha^2}\left(\frac{K_3(\alpha\lambda)}{\lambda^3} - \frac{K_2(\alpha\lambda)}{\lambda^2}\right)\right]\,, \tag{B4}$$

where we have performed the δ_3 differentiation and used the recursion relation [8.486.15].[23] Using an integral expression for the modified Bessel function which can be derived from [8.432.9],[23]

$$K_{m+3/2}(\alpha\lambda_\perp) = \frac{\pi^{1/2}}{\Gamma(m+1)}\left(\frac{\alpha}{2\lambda_\perp}\right)^{m+1/2}\frac{1}{\lambda_\perp}\int_0^\infty \exp[-\alpha(x^2+\lambda_\perp^2)^{1/2}]x^{2m+1}\, dx\,, \tag{B5}$$

and using the doubling formula for gamma functions [8.335.1][23] makes it possible to perform the final sum:

$$\frac{\Psi}{mc^3} = -2\pi A\left[\frac{1}{\lambda_\perp^3}\left(\frac{1}{\lambda_\perp^2}\frac{d}{d\alpha}+1\right)\int_0^\infty \frac{x}{\alpha}\cosh(\delta_3 x/\lambda_\perp)\exp[-\alpha(x^2+\lambda_\perp^2)^{1/2}]\, dx + \frac{\delta_3}{\alpha^2}\left(\frac{K_3(\alpha\lambda)}{\lambda^3} - \frac{K_2(\alpha\lambda)}{\lambda^2}\right)\right]\,. \tag{B6}$$

We will now express Eq. (B6) in terms of physical variables by recalling that $\alpha = \Gamma\xi$, $\vec{\delta} = \alpha\vec{\beta}$, $\lambda = \Gamma^{-1}$, $\lambda_\perp = \Gamma_\perp^{-1}$. Also, by choosing the beam drift towards the surface under consideration ($\hat{e}_s \cdot \vec{\beta} < 0$), we have $\delta_3 = -\beta_3\xi\Gamma \equiv -\nu\xi$, with $\nu \geq 0$. We also define $\mu \equiv \Gamma/\Gamma_\perp = (1+\nu^2)^{1/2}$. Carrying out the differentiation in Eq. (B6) gives

$$\frac{2\Psi}{n_L mc^3} = \frac{1}{\mu^3\xi K_2(\xi)} \int_0^\infty \left[\mu\xi(1+t^2)^{1/2} + 1 - \frac{\mu^2\xi}{\Gamma}\right] \exp[-\mu\xi(1+t^2)^{1/2}] t \cosh(\nu\xi t)\, dt + \nu\left(\frac{K_3(\xi)}{K_2(\xi)} - \frac{1}{\Gamma}\right), \tag{B7}$$

where we have changed integration variables from x to $t = \lambda_\perp x$. The integral expression can be simplified considerably by expressing the hyperbolic function in its exponential form and again transforming the integration variable according to

$$t = \mu x \pm \nu(1+x^2)^{1/2}. \tag{B8}$$

This step is the most crucial one in obtaining the closed-form solution, and after some nontrivial manipulation, some of the integrals obtained from the transformation can be integrated exactly and the rest can be expressed in terms of the incomplete modified Bessel function[31]

$$K_0(\eta, \xi) = \int_0^\nu \frac{\exp[-\xi(1+x^2)^{1/2}]}{(1+x^2)^{1/2}} dx = \int_1^\mu \frac{e^{-\xi t}}{(t^2-1)^{1/2}} dt, \tag{B9}$$

with $\eta = \ln(\mu+\nu)$. The result is

$$\begin{aligned}\frac{2\Psi}{n_L mc^3} = &\left(\frac{4\mu^2(\mu^2-1) - \mu(2\mu^2-1)/\Gamma - 1}{\mu\xi} + \frac{(4\mu^2 - \mu/\Gamma)(2\mu^2-1) - 1}{(\mu\xi)^2} + \frac{4\mu^2(2\mu^2-1) - 1}{(\mu\xi)^3}\right)\frac{\exp(-\mu\xi)}{K_2(\xi)} \\ &+ \frac{\nu}{K_2(\xi)}\left\{K_3(\xi) - \Gamma K_2(\xi) - \frac{1}{\mu^2\xi}\left[\xi(4\mu^2-1)\frac{\partial^3}{\partial\xi^3} - 2\left(1 - \frac{\mu^2\xi}{\Gamma}\right)\frac{\partial^2}{\partial\xi^2} - \xi(3\mu^2-1)\frac{\partial}{\partial\xi} + \left(1 - \frac{\mu^2\xi}{\Gamma}\right)\right] K_0(\eta, \xi)\right\}.\end{aligned} \tag{B10}$$

The final form for the flux given in Eq. (43) is obtained by applying recursion formulas[31] for incomplete Bessel functions to eliminate the partial derivatives in Eq. (B10).

We would like to point out again that this flux calculation cannot be performed analytically by transforming the original integrand to the rest frame. This is readily seen from Eq. (42), where the simple integration limits on θ would change to very complicated interdependent limits on each integration variable.

APPENDIX C

Here we indicate the method of reducing Eq. (60) to a single integral expression for the relativistic plasma dispersion function. In order to perform two of the integrals in closed form, we must choose the dimensionless wave vector along the 3-axis and use cylindrical coordinates in velocity space with the drift velocity having the Cartesian components $\vec{\beta} = \beta_\perp\hat{e}_1 + \beta_\parallel\hat{e}_3$. Changing to dimensionless velocity variables in Eq. (60), with $\vec{v} = (v_\perp\cos\phi, v_\perp\sin\phi, v_\parallel)$, and $d^3w = \gamma^5 d^3v$, we find

$$W = \frac{\Gamma}{4\pi K_2(\xi)} \int_{-1}^1 \frac{dv_\parallel}{v_\parallel - w/\kappa} \int_0^{\gamma_\parallel^{-1}} dv_\perp v_\perp \gamma^3 \exp[-\xi\Gamma\gamma(1-\beta_\parallel v_\parallel)] \int_0^{2\pi} d\phi \exp(\xi\Gamma\gamma\beta_\perp v_\perp\cos\phi). \tag{C1}$$

The ϕ integration is just $2\pi I_0(\xi\Gamma\gamma\beta_\perp v_\perp)$ by [8.431.3].[23] Writing the relativistic factor γ in the form $\gamma = (\gamma_\parallel^{-2} - v_\perp^2)^{-1/2}$, and changing variables in the $v_\perp$ integration according to the transformation $t = \gamma/\gamma_\parallel$ puts the second integral in the form

$$\gamma_\parallel \int_1^\infty I_0[\xi\Gamma\beta_\perp(t^2-1)^{1/2}] \exp[-\xi\Gamma(1-\beta_\parallel v_\parallel)\gamma_\parallel t]\, dt = \frac{\gamma_\parallel \exp\{-\xi\Gamma[(1-\beta_\parallel v_\parallel)^2\gamma_\parallel^2 - \beta_\perp^2]^{1/2}\}}{\xi\Gamma[(1-\beta_\parallel v_\parallel)^2\gamma_\parallel^2 - \beta_\perp^2]^{1/2}}, \tag{C2}$$

where we have used a variation of [6.616.2].[23] Substitution of these results into Eq. (C1) gives Eq. (62), with $v = v_\parallel$.

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