Kinetic theory of a normal quantum fluid: Weak-coupling approximation*

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In the linear-response regime, a normal Bose or Fermi fluid can be described by an exact kinetic equation whose kernel is nonlocal in space and time. We derive a general expression for this kernel and evaluate it explicitly to second order in the interparticle potential. The result is a wave-vector- and frequency-dependent generalization of the linear Uehling-Uhlenbeck kernel with the Born-approximation cross section. Our theory is developed in terms of a second-quantized form of the Wigner representation. Convenient expressions are obtained for the commutators and anticommutators of the phase-space density operators, and the equilibrium averages of these operators are analyzed in terms of momentum-dependent generalizations of the classical pair distribution function $h(k)$ and direct correlation function $c(k)$. The central quantity in this study is a two-particle equilibrium correlation function, the phase-space density-density anticommutator, whose Fourier transform $S(k\omega pp')$ gives the symmetrized scattering function $S(k\omega)$ by integration over the momenta. The kinetic equation is obtained by a formal closure of the quantum BBGKY hierarchy, with the nonlocal kernel expressed in terms of correlation functions involving two, three, and four particles. We show that our method for approximating the kernel and initial condition by a second-order expansion preserves all the sum rules of $S(k\omega pp')$ to the same order and that the result satisfies the appropriate positivity and symmetry conditions.

I. INTRODUCTION

The classical Boltzmann equation occupies a uniquely successful position in nonequilibrium statistical mechanics. Intermediary between the microscopic and macroscopic worlds, it serves as a model whose intuitive appeal and wide applicability have not been equaled by any other method of description.[1] Although it is limited to dilute gases, similar equations have been developed for denser fluids. A Boltzmann-like kinetic theory for quantum-mechanical fluids has long been an attractive possibility, but the problems involved in its formulation are much more severe than in the classical case. A quantum kinetic equation known as the Uehling-Uhlenbeck equation can be obtained from the Boltzmann equation by the substitution of the quantum-mechanical cross section for the classical one and the insertion of statistical factors to reproduce the Bose or Fermi ideal-gas distributions at equilibrium.[2,3] Several derivations have been given for the Uehling-Uhlenbeck equation,[4-6] and corrections to it have been suggested,[7-9] but systematic attempts to improve upon it have generally not gone beyond the formal stage.[10,11] An exception is the transport equation derived by Kadanoff and Baym for systems slowly varying in space and time,[12] but, as discussed below, there remains a need for a quantum kinetic theory valid on all scales of length and time.

Our approach to this problem was stimulated by certain recent developments in classical kinetic theory. It has been known for some time that a liquid or dense gas can truly be described by a Boltzmann-like kinetic equation only if the kinetic kernel is made nonlocal in time and space to account for the duration and spatial extent of the collision process.[13,14] Indeed, the first attempt to treat such effects was made by Enskog more than 50 years ago.[15] Although nonlocal kernels have subsequently appeared in many derivations, it is only in the last few years that systematic and explicit approximations have been obtained. For small deviations from equilibrium, a classical fluid can be described by an exact linear kinetic equation, whose kernel has now been evaluated to second order in the interparticle potential,[16,17] and to first[18,19] and second[20] order in a density expansion. Other techniques have been used for the special case of a hard-sphere gas.[21,22] There have also been several methods proposed for obtaining a kernel valid at liquid densities.[23-28]

An important aspect of these new kinetic theories is that they are derived without reference to any length or time scale, and therefore may be useful for the full range of fluid phenomena from the molecular to the hydrodynamic regime. Support for this notion comes from an analysis of the weak-coupling equation by Forster and Martin,[17] who showed that it gives consistent predictions of the sum rules and the transport coefficients, reflecting a balanced treatment of the short-time and the long-time behavior. A similar result holds for the low-density equations.[20,29] From the work of Forster and Martin and of Résibois,[30,31] it fol-

lows that to determine the transport coefficients exactly, the kinetic kernel must be correct through second order in space and time derivatives. The quantum transport equation derived by Kadanoff and Baym was explicitly limited to the long-time large-distance regime by the omission of terms higher than first order in the gradients, and it is therefore insufficient even for a complete description of linear hydrodynamics.

Our object in this paper is to demonstrate that a linear kinetic theory for normal quantum fluids with Bose or Fermi statistics can be formulated with the same conceptual simplicity and consistency as has now been attained in the classical case, and, specifically, to calculate the nonlocal kernel to second order in a potential expansion. We show that in the classical limit this second-order kernel properly reduces to the classical one, and that in the limit of large times and distances it reduces to the linear Uehling-Uhlenbeck kernel with the Born-approximation cross section.

The organization of the paper is as follows. In Sec. II we discuss properties of the operators and distribution functions needed for our calculation, and define the quantity for which we obtain the kinetic equation, the two-particle anticommutator correlation function $F(1,1'|t)$. The time evolution of $F(1,1'|t)$ is linked to that of an infinite sequence of higher-order correlation functions by equations of motion analogous to the classical Bogoliubov-Born-Green-Kirkwood-Yvon (BBGKY) hierarchy. In Sec. III we derive a formal closure of the hierarchy as a kinetic equation for $F(1,1'|t)$ and discuss our method of approximation. Section IV contains the calculation of the second-order kernel. We conclude with a brief discussion of our results and their implications for future work.

II. WIGNER REPRESENTATION

A. Phase-space operators

Using the transformation introduced by Wigner,[32] we define a one-particle phase-space density operator by

$$f(rp,t) = (2\pi\hbar)^{-3}\int dr'\, e^{-ip\cdot r'/\hbar}\psi^\dagger(r-\tfrac{1}{2}r',t)\psi(r+\tfrac{1}{2}r',t), \tag{2.1}$$

where $\psi(r,t)$ and $\psi^\dagger(r',t)$ are the Heisenberg field operators satisfying

$$\psi(r)\psi^\dagger(r') - \eta\psi^\dagger(r')\psi(r) = \delta(r-r')$$

and

$$\psi(r)\psi(r') - \eta\psi(r')\psi(r) = 0$$

at equal times. The factor η equals +1 for bosons and −1 for fermions. Throughout the paper, we use the letters k, r, and p to represent vector quantities. In terms of $f(rp,t)$, the ordinary number and current density operators are given by

$$\int dp\, f(rp,t) = \psi^\dagger(r,t)\psi(r,t), \tag{2.2}$$

$$\int dp\, pf(rp,t) = \tfrac{1}{2}i\hbar[(\nabla\psi^\dagger)\psi - \psi^\dagger(\nabla\psi)]. \tag{2.3}$$

It is apparent from these equations that $f(rp,t)$ has a formal similarity to the classical phase-space density

$$f_c(rp,t) = \sum_i \delta(r - r_i(t))\,\delta(p - p_i(t)).$$

Multiparticle density operators are defined by

$$f(12,\ldots,n,t) = (2\pi\hbar)^{-3}\int dr_n'\, e^{-ip_n\cdot r_n'/\hbar}\,\psi^\dagger(r_n - \tfrac{1}{2}r_n',t) \times f(12,\ldots,n-1,t)\,\psi(r_n + \tfrac{1}{2}r_n',t), \tag{2.4}$$

where 1 stands for r_1p_1, and so forth. It should be noted that these operators are Hermitian, and that the $f(1,\ldots,i,\ldots,j,\ldots,n,t)$ are symmetric under permutations (i,j) for both Bose and Fermi statistics. We consider a system of unit-mass point particles interacting through a central potential $v(12) = v(|r_1 - r_2|)$, for which the Hamiltonian can be written

$$\hat{H} = \int d1\, \tfrac{1}{2}p_1^2 f(1) + \tfrac{1}{2}\int d1\, d2\, v(12)f(12). \tag{2.5}$$

In carrying out the perturbation expansion, later, we shall assume that $v(r)$ is bounded and short ranged.

The equal-time commutation relations of the first few Wigner operators can be expressed as

$$[f(1),f(2)] = \delta(1-2)S(1)f(1), \tag{2.6a}$$

$$[f(1),f(23)] = \delta(1-2)S(1)f(13) + \delta(1-3)S(1)f(12), \tag{2.6b}$$

$$\begin{aligned}[f(12),f(34)] = {}& \delta(1-3)S(1)f(124) + \delta(1-4)S(1)f(123)\\ &+ \delta(2-3)S(2)f(124) + \delta(2-4)S(2)f(123)\\ &+ [\delta(1-3)\delta(2-4) + \delta(1-4)\delta(2-3)]\\ &\times S(12)f(12),\end{aligned} \tag{2.6c}$$

where

$$\delta(1-2) = \delta(r_1 - r_2)\delta(p_1 - p_2)$$

and

$$S(1) = 2i\sin[\tfrac{1}{2}\hbar D(1)],$$

$$S(12) = 2i \sin[\tfrac{1}{2}\hbar D(1) + \tfrac{1}{2}\hbar D(2)],$$

$$D(1) = \overleftarrow{\nabla}_{r_1} \cdot \overrightarrow{\nabla}_{p_1} - \overleftarrow{\nabla}_{p_1} \cdot \overrightarrow{\nabla}_{r_1}.$$

The gradients in $D(1)$ act to the left or right as indicated by the arrows. The anticommutation relations are given by expressions, beginning with

$$\tfrac{1}{2}\{f(1), f(2)\} = f(12) + \delta(1-2)\cos[\tfrac{1}{2}\hbar D(1)]f(1), \tag{2.7}$$

that are similar to (2.6) except for the presence of higher-order operators on the right-hand side. The relations (2.6) and (2.7), which we believe are new, are a convenient restatement of the commutation and anticommutation relations of pairs $\psi^\dagger(r)\psi(r')$ of the field operators. The intermediate formula

$$(2\pi\hbar)^{-6} \int dr_1' \int dr_2' \, e^{-i(p_1 \cdot r_1' + p_2 \cdot r_2')/\hbar} \{\delta(r_1 + \tfrac{1}{2}r_1' - r_2 + \tfrac{1}{2}r_2')\psi^\dagger(r_1 - \tfrac{1}{2}r_1')\psi(r_2 + \tfrac{1}{2}r_2')\} = \delta(1-2)e^{i\hbar D(1)/2} f(1) \tag{2.8}$$

is helpful for verifying them. It should be emphasized that both (2.6) and (2.7) are true for both Bose and Fermi statistics.

With the aid of Eqs. (2.6), one can work out the commutators in the equations of motion $\partial f/\partial t = (i\hbar)^{-1}[f, \hat{H}]$ by straightforward integration. The result is the coupled system of equations

$$\left(\frac{\partial}{\partial t} + iL(1)\right) f(1,t) = -\int d2 \, iL_1(12) f(12,t), \tag{2.9a}$$

$$\left(\frac{\partial}{\partial t} + iL(12)\right) f(12,t) = -\int d3 \, [iL_1(13) + iL_1(23)] \times f(123,t), \tag{2.9b}$$

and so forth, where

$$L(1) = L_0(1) = -ip_1 \cdot \frac{\partial}{\partial r_1}, \tag{2.10a}$$

$$L(12) = L_0(1) + L_0(2) + iv(12)(2/\hbar) \times \sin(\tfrac{1}{2}\hbar \overleftarrow{\nabla}_{r_1} \cdot \overrightarrow{\nabla}_{p_1} + \tfrac{1}{2}\hbar \overleftarrow{\nabla}_{r_2} \cdot \overrightarrow{\nabla}_{p_2}), \tag{2.10b}$$

$$L_1(12) = iv(12)(2/\hbar) \sin(\tfrac{1}{2}\hbar \overleftarrow{\nabla}_{r_1} \cdot \overrightarrow{\nabla}_{p_1}). \tag{2.10c}$$

Except for the definition of the interaction operators in (2.10b) and (2.10c), this system has the same form as the classical BBGKY hierarchy.[33]

The occurrence of operators like $\sin[\frac{1}{2}\hbar D(1)]$ is typical of formulas involving the Wigner distribution.[33-37] Such operators can be expanded in powers of $\hbar$ to give quantum corrections to classical results. The classical expression corresponding to the first term in the expansion of (2.6a), for example, is the Poisson-bracket formula

$$[f_c(1), f_c(2)]_{PB} = \delta(1-2)(\overleftarrow{\nabla}_{r_1} \cdot \overrightarrow{\nabla}_{p_1} - \overleftarrow{\nabla}_{p_1} \cdot \overrightarrow{\nabla}_{r_1}) f_c(1).$$

These operators can be employed in another way, however. After a Fourier transformation with respect to r_1, an operator like $e^{-i\hbar \overleftarrow{\nabla}_{r_1} \cdot \overrightarrow{\nabla}_{p_1}}$ acts on the functions to its right to produce a displacement in momentum:

$$\exp\left(\hbar k \cdot \frac{\partial}{\partial p}\right) f(p) = f(p + \hbar k).$$

It is primarily in this form that we use Eqs. (2.6)–(2.10).

B. Distribution functions

Equilibrium averages of the density operators will be denoted by

$$n(1) = \langle f(1) \rangle\!\rangle, \quad n(12) = \langle f(12) \rangle\!\rangle, \tag{2.11}$$

where the angular brackets indicate an average in the grand canonical ensemble with inverse temperature β and chemical potential μ. For $\hbar \to 0$, these functions reduce to the classical one- and two-particle phase-space distribution functions. Because of translational invariance in our ensemble, the one-particle function is simply the momentum distribution $n(p)$, which is normalized to the density by $\int dp \, n(p) = n$. For free particles, it is the Bose or Fermi distribution

$$n_0(p) = (2\pi\hbar)^{-3}(e^{\beta(p^2/2 - \mu)} - \eta)^{-1}. \tag{2.12}$$

Our normalization ensures that the $\hbar \to 0$ limit of $n(p)$ is $n\phi(p)$, where

$$\phi(p) = (2\pi/\beta)^{-3/2} e^{-\beta p^2/2} \tag{2.13}$$

is the Maxwellian. Similarly, the classical limit of $n(12)$ is

$$\lim_{\hbar \to 0} n(12) = n^2 g_c(r_1 - r_2)\phi(p_1)\phi(p_2), \tag{2.14}$$

where $g_c(r)$ is the classical pair-distribution function. As will be seen below, the momentum and position variables in the quantum $n(12)$ do not separate in this way, but the pair distribution $g(r)$ is nevertheless given by

$$n^2 g(r_1 - r_2) = \int dp_1 \, dp_2 \, n(12), \tag{2.15}$$

as in the classical case. Though $n(12)$ is real, it is not necessarily non-negative for all values of its variables. This is characteristic of the Wigner representation. The function $n(12)$ is best regarded as a particular off-diagonal Fourier transform of the two-particle density matrix. Since an integration over all momenta as in (2.15) gives the diag-

onal part, the pair distribution $g(r)$ is non-negative, as it must be.

To investigate $n(12)$ in more detail, we can employ perturbation theory.[38] The notation we introduce here will also be needed later. It will be convenient to use a Fourier transform of $f(12)$,

$$f(k_1p_1,k_2p_2) = \int dr_1\, dr_2\, e^{-ik_1\cdot r_1}\, e^{-ik_2\cdot r_2} f(r_1p_1,r_2p_2), \tag{2.16}$$

which is given in terms of the operators

$$\varphi(\hbar k) = \int dr\, e^{-ik\cdot r}\, \psi(r)$$

by

$$f(k_1p_1,k_2p_2) = (2\pi\hbar)^{-6}\varphi^\dagger(1-)\varphi^\dagger(2-)\varphi(2+)\varphi(1+), \tag{2.17}$$

where $1\pm = p_1 \pm \frac{1}{2}\hbar k_1$, $2\pm = p_2 \pm \frac{1}{2}\hbar k_2$. Now $n(12)$ is given by the sum of imaginary-time-ordered momentum-space diagrams indicated in Fig. 1, where the lines in the first two diagrams represent the fully interacting one-particle propagator. For infinitesimal time differences the propagator is

$$G(p,\tau) = \begin{cases} \gamma n(p) & \text{for } \tau = 0- \\ \tilde{n}(p) & \text{for } \tau = 0+ \end{cases}, \tag{2.18}$$

where $\gamma = (2\pi\hbar)^3\eta$ and $\tilde{n}(p) = 1 + \gamma n(p)$. The creation and annihilation points of the diagram are assigned infinitesimal times corresponding to the order of the operators in (2.17). The evaluation of these diagrams follows standard rules,[38] except for the convention that there is a momentum-conserving factor $(2\pi\hbar)^3\delta(p_a - p_b)$ for an uninterrupted one-particle line connecting endpoints labeled p_a and p_b. For an m-particle diagram representing $n(12,\ldots,m)$ there is an over-all factor of γ^{-m}. Thus the uncorrelated diagram, Fig. 1(a), gives

$$(2\pi)^6\delta(k_1)\delta(k_2)n(p_1)n(p_2),$$

while the exchange diagram, Fig. 1(b), gives

$$(2\pi)^3\delta(k_1+k_2)\delta(p_1-p_2)\gamma n(1+)n(1-).$$

There is an extra factor of η for the crossing of external particle lines in Fig. 1(b), corresponding to a permutation of field operators. The contribution of the connected part of $n(12)$, Fig. 1(c), is represented by

$$(2\pi)^3\delta(k_1+k_2)H(k_1p_1p_2),$$

where the nonsingular function $H(kpp')$ is a special form, indicated below, of the sum of all two-particle connected diagrams, Fig. 2. In summary, $n(12)$ is given by

$$n(12) = n(p_1)n(p_2) + \int \frac{dk}{(2\pi)^3}\, e^{ik\cdot(r_1-r_2)}\left[\delta(p_1-p_2)\gamma n(p_1-\tfrac{1}{2}\hbar k)n(p_1+\tfrac{1}{2}\hbar k) + H(kp_1p_2)\right]. \tag{2.19}$$

The term containing $\delta(p_1-p_2)$ vanishes in the classical limit, while the function H reduces to

$$\lim_{\hbar\to 0} H(kpp') = h_c(k)n\phi(p)n\phi(p'), \tag{2.20}$$

where $h_c(k)$ is the Fourier transform of $g_c(r)-1$.

It will be useful to define a quantum generalization $h(kpp')$ of $h_c(k)$ by

$$H(kpp') = h(kpp')N(kp)N(kp'), \tag{2.21}$$

where

$$N(kp) \equiv \tfrac{1}{2}n(p+\tfrac{1}{2}\hbar k)\tilde{n}(p-\tfrac{1}{2}\hbar k) + \tfrac{1}{2}n(p-\tfrac{1}{2}\hbar k)\tilde{n}(p+\tfrac{1}{2}\hbar k). \tag{2.22}$$

Obviously there are many functions besides $N(kp)$ which reduce to $n\phi(p)$ in the classical limit; the reason for this particular definition of $h(kpp')$ will be apparent in Sec. IIC. Like $H(kpp')$, $h(kpp')$ is real, even under $k \to -k$, and symmetric in p and p'. It vanishes for an ideal gas. Our notation should not be interpreted to suggest too close an analogy with the classical distribution functions, however. It should be noted, for example, that the quantum $g(r)-1$ is *not* equal to the Fourier transform of $\int dp\, dp'\, h(kpp')$, as we do not include the exchange term in our definition of $h(kpp')$.

To illustrate the relationship of $h(kpp')$ to its classical limit, we give here the first term in its perturbation expansion. For the calculations of

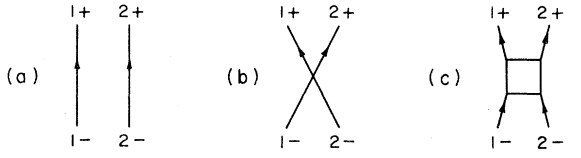

FIG. 1. Diagrams for $n(12)$. The lines in (a) and (b) represent the fully interacting propagator. Part (c) represents the sum of all two-particle connected diagrams.

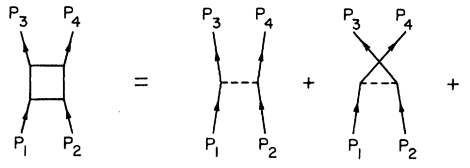

FIG. 2. First-order terms of the two-particle connected diagram.

Sec. IV, we need the sum of the two-particle connected diagrams with the momentum labeling of Fig. 2, which we write as

$$(2\pi\hbar)^3\delta(p_1+p_2-p_3-p_4)\mathcal{K}(p_1,p_2,p_3,p_4). \quad (2.23)$$

A short calculation of the two diagrams on the right-hand side of Fig. 2 gives the first-order result

$$\mathcal{K}_1(p_1,p_2,p_3,p_4) = \beta[\hat{v}(p_1-p_3)+\eta\hat{v}(p_1-p_4)]A_0(p_1,p_2,p_3,p_4) \times \frac{\tanh[\frac{1}{2}\beta(p_1^2+p_2^2-p_3^2-p_4^2)]}{\frac{1}{2}\beta(p_1^2+p_2^2-p_3^2-p_4^2)}, \quad (2.24)$$

where

$$A_0(p_1,p_2,p_3,p_4) = \tfrac{1}{2}n_0(p_1)n_0(p_2)\tilde{n}_0(p_3)\tilde{n}_0(p_4) + \tfrac{1}{2}\tilde{n}_0(p_1)\tilde{n}_0(p_2)n_0(p_3)n_0(p_4) \quad (2.25)$$

and

$$\hat{v}(\hbar k) = v(k) = \int dr\, e^{-ik\cdot r}v(r).$$

For the special choice of variables that defines $H(kpp')$,

$$H(kpp') = \mathcal{K}(p-\tfrac{1}{2}\hbar k, p'+\tfrac{1}{2}\hbar k, p+\tfrac{1}{2}\hbar k, p'-\tfrac{1}{2}\hbar k), \quad (2.26)$$

we can use the identity

$$n_0(p-\tfrac{1}{2}\hbar k)\tilde{n}_0(p+\tfrac{1}{2}\hbar k) = n_0(p+\tfrac{1}{2}\hbar k)\tilde{n}_0(p-\tfrac{1}{2}\hbar k)e^{\beta\hbar k\cdot p} \quad (2.27)$$

to extract a factor $N_0(kp)N_0(kp')$ from A_0, leaving a final expression for the first-order $h_1(kpp')$ in the form

$$h_1(kpp') = -\beta[v(k)+\eta\hat{v}(p-p')] \times \frac{\tanh(\frac{1}{2}\beta\hbar k\cdot p)-\tanh(\frac{1}{2}\beta\hbar k\cdot p')}{\frac{1}{2}\beta\hbar k\cdot(p-p')}. \quad (2.28)$$

In the limit $\hbar\to 0$ the momentum dependence vanishes and we recover the first-order classical $h_c(k) = -\beta v(k)$.

C. Correlation functions

Traditional kinetic theories describe the time evolution of the nonequilibrium average of $f(rp,t)$. For small deviations from equilibrium produced by an external potential $u(rp)$, introduced adiabatically in the distant past and turned off at $t=0$, this average is given by an expression of the form

$$\langle f(rp,t)\rangle_u - \langle f(rp)\rangle = \beta\int dr'\,dp'\,L(rp,r'p'|t)u(r'p') + O(u^2). \quad (2.29)$$

For sufficiently small disturbances, therefore, $\langle f(rp,t)\rangle_u - \langle f(rp)\rangle$ obeys the same kinetic equation as the linear-response function $L(rp,r'p'|t)$, which is an equilibrium two-particle correlation function. As is well known, the linear-response regime is sufficient to account for many of the important properties of a fluid system, including its full neutron-scattering and light-scattering spectra as well as its transport coefficients.[39-41]

Closely related to L is the anticommutator correlation function defined by

$$F(1,1'|t-t') = \langle\tfrac{1}{2}\{f(1,t),f(1',t')\}\rangle - \langle f(1)\rangle\langle f(1')\rangle, \quad (2.30)$$

which also obeys a kinetic equation and which contains equivalent information. We prefer to formulate our theory in terms of F rather than L. Although the formal development of a kinetic equation for L would be identical to that given in Sec. III for F, the perturbation calculations it would require are somewhat more difficult. In particular, obtaining the initial value of L requires an analysis of time- or frequency-dependent diagrams rather than equal-time diagrams as in the case of F. The connection between L and F is given in terms of the Fourier transform

$$S(k\omega pp') = \int d(r-r')\,e^{-ik\cdot(r-r')} \times \int_{-\infty}^{\infty} dt\, e^{i\omega t}F(rp,r'p'|t) \quad (2.31)$$

by

$$L(k\omega pp') = S(k\omega pp')\tau(\beta\hbar\omega), \quad (2.32)$$

where $\tau(x) = (\frac{1}{2}x)^{-1}\tanh(\frac{1}{2}x)$. It can be seen that the distinction between them vanishes in the classical limit.

It will be convenient to use a function of complex argument z, defined by

$$F(kzpp') = -\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\frac{S(k\omega pp')}{\omega-z}, \quad (2.33)$$

which for $\mathrm{Im}z>0$ is the Laplace transform of $F(kpp'|t)$, with the convention

$$F(z) = -i\int_0^{\infty} dt\, e^{izt}F(t). \quad (2.34)$$

We will also use the r-space form $F(1,1'|z)$. From the definitions (2.30) and (2.31), it can be shown that $S(k\omega pp')$ is real, and that its symmetric integrals over an arbitrary function of the momentum are non-negative:

$$\int dp\,dp'\,g^*(p)S(k\omega pp')g(p') \geq 0. \quad (2.35)$$

In particular, this guarantees the positivity of the symmetrized scattering function $S(k\omega)$,[42] which is

given by

$$S(k\omega) = \int dp\, dp'\, S(k\omega pp'). \tag{2.36}$$

Furthermore, $S(k\omega pp')$ is even under $k,\omega \leftrightarrow -k,-\omega$ and symmetric in p and p'. In terms of $F(kzpp')$ these properties may be summarized as

$$F(k,z^*;pp') = [F(k,z;pp')]^*, \tag{2.37a}$$

$$\lim_{\epsilon\to 0^+} \mathrm{Im} \int dp\, dp'\, g^*(p) F(k,\omega+i\epsilon;pp') g(p') \leq 0, \tag{2.37b}$$

$$F(-k,-z;pp') = -F(k,z;pp'), \tag{2.37c}$$

$$F(k,z;pp') = F(k,z;p'p). \tag{2.37d}$$

The importance of preserving these relations in an approximate theory for $F(kzpp')$ has been indicated in the classical case.[17,43]

We turn now to the initial value $F(1,1') = F(1,1'|t=0)$. Using the anticommutator identity (2.7), we can express it in terms of $n(11')$ and $n(1)$ by

$$F(1,1') = n(11') - n(1)n(1') + \delta(1-1')\cos(\tfrac{1}{2}\hbar\overleftarrow{\nabla}_{r_1}\cdot\vec{\nabla}_{p_1})n(1). \tag{2.38}$$

After a Fourier transformation with respect to $r_1 - r_1'$, the last term becomes

$$\delta(p-p')\cosh\left(\frac{\hbar}{2}k\cdot\frac{\partial}{\partial p}\right)n(p) = \delta(p-p')[\tfrac{1}{2}n(p+\tfrac{1}{2}\hbar k) + \tfrac{1}{2}n(p-\tfrac{1}{2}\hbar k)]. \tag{2.39}$$

Combining this with expressions (2.19)–(2.22) for $n(11')$, we obtain the important result

$$F(kpp') = \delta(p-p')N(kp) + h(kpp')N(kp)N(kp'). \tag{2.40}$$

A comparison with the classical formula

$$\lim_{\hbar\to 0} F(kpp') = \delta(p-p')n\phi(p) + h_c(k)n^2\phi(p)\phi(p') \tag{2.41}$$

emphasizes that $N(kp)$ is indeed the proper quantum generalization of $n\phi(p)$ in this context.

To obtain an explicit formula for the static inverse F^{-1} satisfying

$$\int d\bar{p}\, F(kp\bar{p})F^{-1}(k\bar{p}p') = \delta(p-p'), \tag{2.42}$$

we define a generalization $c(kpp')$ of the classical direct correlation function $c_c(k)$ by the Ornstein-Zernike-like equation

$$h(kpp') = c(kpp') + \int d\bar{p}\, h(kp\bar{p})N(k\bar{p})c(k\bar{p}p'). \tag{2.43}$$

It can readily be verified that the static inverse F^{-1} is given by

$$F^{-1}(kpp') = \delta(p-p')/N(kp) - c(kpp'). \tag{2.44}$$

Like $h(kpp')$, the function $c(kpp')$ vanishes for free particles. The first-order term $c_1(kpp')$ is identical to $h_1(kpp')$, given by (2.28).

III. QUANTUM KINETIC EQUATION

In the first part of this section we derive a kinetic equation for $F(1,1'|z)$ by a formal closure of the BBGKY hierarchy, with the kinetic kernel defined in terms of correlation functions involving two, three, and four particles. Our scheme for approximating the kernel in a way that preserves its symmetries is discussed in Sec. III B. We show that this scheme also preserves all the sum rules of $S(k\omega pp')$ to the same order of approximation.

A. Nonlocal kernel

From the operator equations of motion (2.9), we obtain a hierarchy of correlation-function equations beginning with

$$[z - L(1)]F(1,1'|z) = F(1,1') + \int d2\, L_1(12)F(12,1'|z), \tag{3.1a}$$

$$[z - L(12)]F(12,1'|z) = F(12,1') + \int d3\,[L_1(13)+L_1(23)]\times F(123,1'|z), \tag{3.1b}$$

together with the complementary sequence

$$[z + L(1')]F(1,1'|z) = F(1,1') - \int d2'\, L_1(1'2')\times F(1,1'2'|z), \tag{3.2a}$$

$$[z + L(1')]F(12,1'|z) = F(12,1') - \int d2'\, L_1(1'2')\times F(12,1'2'|z), \tag{3.2b}$$

where the higher-order correlation functions analogous to $F(1,1'|t)$ are defined by

$$F(1,\ldots,n,1',\ldots,n'|t-t') = \langle \tfrac{1}{2}\{f(1,\ldots,n,t), f(1',\ldots,n',t')\}\rangle - \langle f(1,\ldots,n)\rangle\langle f(1',\ldots,n')\rangle. \tag{3.3}$$

Our goal in this section is a closed kinetic equation for $F(1,1'|z)$ in the form

$$[z - L(1)]F(1,1'|z) = F(1,1') + \int d2\, \Sigma(1,2|z)F(2,1'|z). \tag{3.4}$$

A comparison of (3.1a) and (3.4) shows that the kinetic kernel or memory function $\Sigma(1,1'|z)$ must satisfy

$$\int d2\, \Sigma(1,2|z)F(2,1'|z) = \int d2\, L_1(12)F(12,1'|z). \tag{3.5}$$

Equation (3.5) defines $\Sigma(1,1'|z)$ uniquely provided there exists a z-dependent inverse $F^{-1}(kzpp')$ satisfying

$$\int d\bar{p}\, F(kzp\bar{p})F^{-1}(kz\bar{p}p') = \delta(p - p') \tag{3.6}$$

for all k and z. Although there has been no explicit proof, the existence of such an inverse seems well established in the classical case, and we are confident that this is also true in the case of a normal quantum fluid.[44] For a noninteracting quantum fluid the inverse is given by

$$F_0^{-1}(kzpp') = (z - k\cdot p)\delta(p - p')/N_0(kp); \tag{3.7}$$

for an interacting system the inverse can be formally obtained from its large-z expansion

$$F^{-1}(1,1'|z) = zF^{-1}(1,1') - i\int d2\, d2'\, F^{-1}(1,2)\dot{F}(2,2')F^{-1}(2',1') + O(z^{-1}), \tag{3.8}$$

where $\dot{F}(1,1')$ is the time derivative of $F(1,1'|t)$ at $t = 0$. Note that $F^{-1}(1,1')$ is the static inverse, Eq. (2.44).

The existence of an inverse for the anticommutator function $F(kzpp')$ is a special property not shared by every correlation function of interest: a counterexample is provided by the commutator function

$$\chi(1,1'|t - t') = (2\hbar)^{-1}\langle[f(1,t), f(1',t')]\rangle.$$

From Eq. (2.6a) it can be seen that the initial value is

$$\chi(kpp') = (2\hbar)^{-1}\delta(p - p')[n(p - \tfrac{1}{2}\hbar k) - n(p + \tfrac{1}{2}\hbar k)], \tag{3.9}$$

and therefore vanishes for $k = 0$. Consequently, χ does not have a well-behaved static inverse and the derivation given below for F cannot be applied to χ. One can readily verify in the classical limit that χ does not, in fact, satisfy a kinetic equation of the form (3.4), by combining Eq. (3.4) for F with the fluctuation-dissipation theorem

$$\chi_c(1,1'|t) = \tfrac{1}{2}i\beta\frac{\partial}{\partial t}F_c(1,1'|t).$$

We now proceed to derive an expression for the kinetic kernel in a more useful form than (3.5).[45] It will be convenient to work with the function

$$K(kzpp') = \int d\bar{p}\, \Sigma(kzp\bar{p})F(k\bar{p}p'). \tag{3.10}$$

Applying $[z + L(1')]$ to (3.5) and using (3.2), we have

$$K(1,1'|z) - \int d2'\, d3'\, \Sigma(1,3'|z)L_1(1'2')F(3',1'2'|z) = \int d2\, L_1(12)\left(F(12,1') - \int d2'\, L_1(1'2')F(12,1'2'|z)\right). \tag{3.11}$$

Using (3.5) and (3.6) to rewrite $\Sigma(1,3'|z)$ on the left-hand side as

$$\int d2\, d3\, L_1(12)F(12,3|z)F^{-1}(3,3'|z),$$

we obtain the desired expression for $K(1,1'|z)$ in the form

$$K(1,1'|z) = K^{(s)}(1,1') + K^{(d)}(1,1'|z). \tag{3.12a}$$

The static part

$$K^{(s)}(1,1') = \int d2\, L_1(12)F(12,1') \tag{3.12b}$$

is independent of z and represents a mean-field or modified Vlasov contribution to the equation of motion, while the dynamic part

$$K^{(d)}(1,1'|z) = -\int d2\, d2'\, L_1(12)L_1(1'2')G(12,1'2'|z), \tag{3.12c}$$

with

$$G(12,1'2'|z) = F(12,1'2'|z) - \int d3\, d3'F(12,3|z)F^{-1}(3,3'|z) \times F(3',1'2'|z), \tag{3.13}$$

starts at order z^{-1} and describes the Boltzmann-like effect of collisions. The kernel Σ is similarly decomposed into a static part $\Sigma^{(s)}$ and a dynamic part $\Sigma^{(d)}$.

The positivity and symmetry properties (2.37) of $F(kzpp')$ determine corresponding properties of the kernel:

$$K(k,z^*;pp') = [K(k,z;pp')]^*, \tag{3.14a}$$

$$\lim_{\epsilon\to 0^+} \mathrm{Im}\int dp\, dp'\, g^*(p)K(k,\omega + i\epsilon;pp')g(p') \leq 0, \tag{3.14b}$$

$$K(-k,-z;pp') = -K(k,z;pp'), \tag{3.14c}$$

$$K^{(d)}(k,z;pp') = K^{(d)}(k,z;p'p). \tag{3.14d}$$

The static part is real, and odd under $k \to -k$, but by itself it is not symmetric under $p \to p'$; rather, it is the sum of the static part and the streaming term

$$K^{(s)}(kpp') + k\cdot pF(kpp') = K^{(s)}(kp'p) + k\circ p'F(kp'p) \tag{3.14e}$$

which has this property. This combination is simply $i\dot{F}(kpp')$.

The static kernel $K^{(s)}(kpp')$ is closely related to the function $H(kpp')$, as follows. Subtracting the terms of order $1/z$ in the large-z expansions of Eqs. (3.1a) and (3.2a), we find

$$[L(1)+L(1')]F(1,1') = -\int d2\,[L_1(12)F(12,1') + L_1(1'2)F(1'2,1)]$$

or

$$k\cdot(p-p')H(kpp') = -K^{(s)}(kpp') + K^{(s)}(kp'p). \tag{3.15}$$

In the classical limit, $H(kpp')$ becomes even in each momentum variable, while $K^{(s)}(kpp')$ becomes odd in p and even in p'. The sum of (3.15) and its form with $p' \to -p'$ therefore gives

$$\lim_{\hbar\to 0} K^{(s)}(kpp') = -k\cdot pn^2 h_c(k)\phi(p)\phi(p'), \tag{3.16}$$

which is equivalent to the well-known potential-independent expression for the classical $\Sigma^{(s)}$ in terms of the direct correlation function.[46,47] We do not know any analogous expression for $K^{(s)}$ in the quantum-mechanical case.[48]

B. Method of approximation

Equation (3.15) is an example of a relationship between statics and dynamics that should be maintained in any consistent theory, as should the symmetry properties (3.14). In this paper we are concerned with approximating the kinetic kernel by means of a perturbation expansion. To accomplish this in a consistent way, we first write the solution to the kinetic equation (3.4) as

$$F(z) = [z - L_0 - \Sigma(z)]^{-1}F, \tag{3.17}$$

using an abbreviated notation in which $F(z)$ stands for the "matrix" $F(kzpp')$ with indices p and p'. Similarly $(z - L_0)$ is the matrix with components $(z - k\cdot p)\delta(p-p')$. Now we multiply (3.17) by FF^{-1} from the left-hand side and obtain

$$F(z) = F[(z - L_0)F - K(z)]^{-1}F. \tag{3.18}$$

The advantage of writing $F(z)$ in this form is that the term in brackets is symmetric in its momentum indices.[26] Our method of approximation is to truncate F and $K(z)$ in Eq. (3.18) at second order. Thus, writing $\tilde{F}(z)$ for the solution, we have

$$\tilde{F}(z) = F_{(2)}[(z - L_0)F_{(2)} - K_{(2)}(z)]^{-1}F_{(2)}, \tag{3.19}$$

which can be rearranged in the form of an approximate kinetic equation

$$[z - L_0 - \tilde{\Sigma}(z)]\tilde{F}(z) = F_{(2)}, \tag{3.20}$$

where

$$\tilde{\Sigma}(z) = K_{(2)}(z)(F_{(2)})^{-1}. \tag{3.21}$$

It should be noted that the initial condition used in this approximation is the truncation of the exact initial condition.

Equation (3.19) can also be written

$$\tilde{F}(z) = F_{(2)}\{[FF^{-1}(z)F]_{(2)}\}^{-1}F_{(2)}. \tag{3.22}$$

This can be compared to another symmetric approximation, namely,

$$\bar{F}(z) = \{[F^{-1}(z)]_{(2)}\}^{-1}, \tag{3.23}$$

the inversion of the truncation of the inverse. This is the approximation used by Forster and Martin in the classical case.[17] In this scheme the approximate kinetic equation has the kernel

$$\bar{\Sigma}(z) = [(F^{-1})_{(2)}]^{-1}[F^{-1}\Sigma(z)]_{(2)} \tag{3.24}$$

and initial condition

$$\bar{F} = [(F^{-1})_{(2)}]^{-1}. \tag{3.25}$$

In the classical case, the approximation (3.24) is equivalent to a direct truncation of $\Sigma(z)$, because of the special form of the classical F. This is not true for the quantum $\bar{\Sigma}(z)$. It should be noted, moreover, that a direct truncation of the quantum-mechanical $\Sigma(z)$ does not lead to a symmetric approximation.

The two approximation schemes defined by (3.19) and (3.23) have essentially the same physical content; they differ only in technical detail. In the classical limit, the difference can be simply described, as follows: In the first scheme, the initial condition and the static part of the kernel are given in terms of the truncated $h_c(k)$; in the second, they are given in terms of the truncated $c_c(k)$. The dynamic part of the kernel is the same in both. In the classical case, therefore, the two schemes are equally tractable. Quantum mechanically, however, the first scheme is by far the simpler procedure because the static quantities it requires can be obtained more directly.

To conclude the discussion of the approximation scheme given by Eq. (3.19), we show how it affects the sum rules of $S(k\omega pp')$. A similar analysis

can be applied to Eq. (3.23). We consider first the large-z expansion of the exact $F(kzpp')$, which is given in terms of the frequency moments of $S(k\omega pp')$ by

$$F(kzpp')=\sum_{j=0}^{\infty} z^{-j-1}\int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}\,\omega^j S(k\omega pp') \tag{3.26}$$

or

$$F(z)=z^{-1}F+iz^{-2}\dot{F}-z^{-3}\ddot{F}+\cdots. \tag{3.27}$$

Now expanding the right-hand side of Eq. (3.22), we obtain

$$\tilde{F}(z)=z^{-1}F_{(2)}+iz^{-2}\dot{F}_{(2)} -z^{-3}\left[\ddot{F}_{(2)}-(\dot{F}F^{-1}\dot{F})_{(2)}+\dot{F}_{(2)}(F_{(2)})^{-1}\dot{F}_{(2)}\right]+\cdots. \tag{3.28}$$

If each factor in (3.28) could be evaluated exactly, instead of being truncated at second order as indicated, we would recover the expansion of the exact $F(z)$, but, as given, the third and higher coefficients in the expansion of $\tilde{F}(z)$ contain terms that are not present in the expansion of $F(z)$. It is not difficult to see, however, that the net contribution of these terms always starts at third order, so that all the frequency moments of $\tilde{S}(k\omega pp')$ will, in fact, be correct to second order.

It should be apparent that the considerations of this section are not limited to the second-order potential expansion, but apply to any well-defined expansion to arbitrary order.

IV. THE SECOND-ORDER KERNEL

A. Static part

In this section we evaluate the static kernel to second order in the potential. We begin by rewriting Eq. (3.12b), Fourier transformed with respect to both r_1 and r_1', as

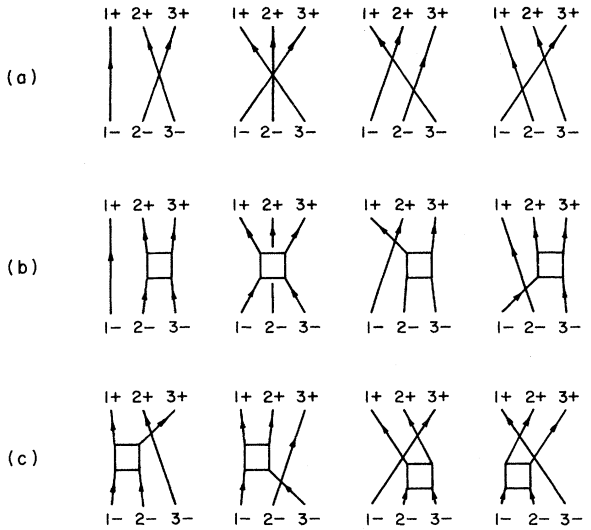

FIG. 3. Unconnected and two-connected diagrams for $F(12,3)$.

$$(2\pi)^3\delta(k+k')K^{(s)}(kpp') =-(2\pi)^{-3}\int d\bar{k}\,d\bar{p}\;V(\bar{k}p)\,F(k-\bar{k},p,\bar{k}\bar{p};k'p'), \tag{4.1}$$

where

$$V(kp)=v(k)\frac{2}{\hbar}\sinh\left(\frac{\hbar}{2}k\cdot\frac{\partial}{\partial p}\right). \tag{4.2}$$

Our task is to calculate $F(12,3)$ to first order. The diagrams for $F(12,3)$ can be classified as unconnected, two-connected, and three-connected. Since the three-connected diagrams start at second order, we need not consider them here. With each diagram taken with the two infinitesimal time orderings that correspond to the two terms of the anticommutator, the unconnected diagrams, Fig. 3(a), give

$$F_a(k_1p_1,k_2p_2;k_3p_3)=(2\pi)^6\delta(k_1)\delta(k_2+k_3)\delta(p_2-p_3)n(p_1)N(k_3p_3)+(2\pi)^6\delta(k_2)\delta(k_1+k_3)\delta(p_1-p_3)n(p_2)N(k_3p_3) +(2\pi)^3\delta(k_1+k_2+k_3)[\delta(2+1-)\delta(3+2-)\gamma n(1-)+\delta(1+2-)\delta(2+3-)\gamma n(1+)]N(k_3p_3), \tag{4.3}$$

where $\delta(2+1-)=\delta[(2+)-(1-)]$ with $2+=p_2+\frac{1}{2}\hbar k_2$, etc. Substituting (4.3) in (4.1) and writing out the effect of the V operator as a difference of displacements of p, we obtain

$$K_a^{(s)}(kpp')=\beta[v(k)+\eta\hat{v}(p-p')]M(kp)N(kp') -\eta\delta(p-p')\int d\bar{p}\;\beta\hat{v}(p-\bar{p})M(k\bar{p})N(kp'), \tag{4.4}$$

where

$$M(kp)=(\beta\hbar)^{-1}[n(p-\tfrac{1}{2}\hbar k)-n(p+\tfrac{1}{2}\hbar k)]. \tag{4.5}$$

We note that $M(kp)$ occurs in the initial value of the commutator function (3.9), and that in the classical limit it reduces to

$$\lim_{\hbar\to 0}M(kp)=k\cdot pn\phi(p). \tag{4.6}$$

With the general form (2.23) of the two-particle connected diagram, it is equally straightforward to write down the result for the two-connected diagrams of $F(12,3)$ and substitute in (4.1) to obtain their contribution to $K^{(s)}$. The diagrams of Fig. 3(b) give

$$K_b^{(s)}(kpp') = \int d\bar{p}\ \beta[v(k)+\eta\hat{v}(p-\bar{p})]M(kp)H(k\bar{p}p') - \eta\delta(p-p')\int d\bar{p}\ \beta\hat{v}(p-\bar{p})M(k\bar{p})H(kpp'), \tag{4.7}$$

and the remaining two-connected diagrams, Fig. 3(c), give

$$K_c^{(s)}(kpp') = (2\pi\hbar)^{-3}\int dp_1\,dp_2\,dp_3\,\delta(p_1+p_2-p_3-p')[X(k)-X(-k)]$$
$$\times\{\hat{v}(p_1-p_3)[\delta(p-p_1)-\delta(p-p_3)]+\eta\hat{v}(p_1-p')[\delta(p-p_1)-\delta(p-p')]\}, \tag{4.8}$$

where

$$X(k) = [\tfrac{1}{2}+\gamma n(p'+\tfrac{1}{2}\hbar k)]\mathcal{K}(p_1-\tfrac{1}{2}\hbar k, p_2+\tfrac{1}{2}\hbar k, p_3+\tfrac{1}{2}\hbar k, p'-\tfrac{1}{2}\hbar k).$$

The combination $K_a^{(s)}+K_b^{(s)}$, taken by itself, is equivalent to a Hartree-Fock approximation for $\Sigma^{(s)}$:

$$\Sigma_{\mathrm{HF}}^{(s)}(kpp') = \beta[v(k)+\eta\hat{v}(p-p')]M(kp) - \eta\delta(p-p')\int d\bar{p}\ \beta\hat{v}(p-\bar{p})M(k\bar{p}), \tag{4.9}$$

which has been used to discuss zero sound in a Fermi liquid.[12] One can also obtain (4.9) by factoring the nonequilibrium average $\langle f(12)\rangle_{\mathrm{ne}}$ in the BBGKY equation connecting $\langle f(1)\rangle_{\mathrm{ne}}$ and $\langle f(12)\rangle_{\mathrm{ne}}$ and then linearizing the resulting collisionless kinetic equation for $\langle f(1)\rangle_{\mathrm{ne}}$.[49]

The expressions (4.4), (4.7), and (4.8) are written in terms of exact one- and two-particle distribution functions; truncated at second order, they give the full second-order expansion of the static kernel

$$K_{(2)}^{(s)} = K_1^{(s)} + K_2^{(s)}$$

in terms of the zero- and first-order static functions we have already calculated plus the first-order term in the expansion of $n(p)$, which is

$$n_1(p) = n_0(p) + n_0(p)\sigma_1(p)\tilde{n}_0(p), \tag{4.10}$$

where

$$\sigma_1(p) = -\int d\bar{p}\ \beta[\hat{v}(0)+\eta\hat{v}(p-\bar{p})]n_0(\bar{p}). \tag{4.11}$$

The first-order kernel is obtained entirely from the unconnected part $K_a^{(s)}$ and has the Hartree-Fock form:

$$K_1^{(s)}(kpp') = \left(\beta[v(k)+\eta\hat{v}(p-p')]M_0(kp) - \eta\delta(p-p')\int d\bar{p}\ \beta\hat{v}(p-\bar{p})M_0(k\bar{p})\right)N_0(kp'). \tag{4.12a}$$

The second-order term contains contributions from both $K_a^{(s)}$ and the two-connected parts $K_b^{(s)}$ and $K_c^{(s)}$, and is given by

$$K_2^{(s)}(kpp') = \int d\bar{p}\ \beta[v(k)+\eta\hat{v}(p-\bar{p})][M_0(kp)F_1(k\bar{p}p')+M_1(kp)F_0(k\bar{p}p')]$$
$$-\eta\int d\bar{p}\ \beta\hat{v}(p-\bar{p})[M_0(k\bar{p})F_1(kpp')+M_1(k\bar{p})F_0(kpp')]+K_{c2}^{(s)}(kpp'), \tag{4.12b}$$

where $K_{c2}^{(s)}$ is $K_c^{(s)}$ with $\mathcal{K}$ taken to first order from Eq. (2.24). We note that $K_{(2)}^{s}$ is odd in k, in accord with (3.14), and that its $\hbar\to 0$ limit gives the correct second-order expansion of the classical result, Eq. (3.16).[50]

To complete the list of static quantities appearing in the initial condition $F_{(2)}(kpp')$, we give the second-order term of $n(p)$, which is

$$n_2(p) = n_0(p)[\sigma_1(p)\tfrac{1}{2}\coth^{\eta}(\tfrac{1}{4}\beta p^2)\sigma_1(p)+\sigma_2(p)]\tilde{n}_0(p), \tag{4.13}$$

where

$$\sigma_2(p) = -\int d\bar{p}\ \beta[\hat{v}(0)+\eta\hat{v}(p-\bar{p})]n_1(\bar{p}) + \frac{1}{2}\int\frac{d\bar{k}\,d\bar{p}}{(2\pi)^3}\beta^2[v(\bar{k})+\eta\hat{v}(p')]^2[\tilde{n}_0(p-\hbar\bar{k})n_0(\bar{p})\tilde{n}_0(\bar{p}+\hbar\bar{k})E_2(\beta\hbar\bar{k}\cdot p')$$
$$+\gamma n_0(p-\hbar\bar{k})\tilde{n}_0(\bar{p})n_0(\bar{p}+\hbar\bar{k})E_2(-\beta\hbar\bar{k}\cdot p')] \tag{4.14}$$

and

$$p' = p-\bar{p}-\hbar\bar{k}, \quad E_2(x) = x^{-2}(e^x-1-x).$$

The second-order term of $H(kpp')$ is given in terms of $K_2^{(s)}$ by Eq. (3.15). This is not a circular definition, because $K_2^{(s)}$ contains the first-order term H_1 but does not contain H_2.

B. Dynamic part

Up to this point we have dealt with the diagrammatic analysis of equal-time correlation functions,

for which the calculations are relatively straightforward. To obtain the dynamic part of the kinetic kernel from Eq. (3.12c), we must analyze the z-dependent function $G(12,1'2'|z)$, and the calculations will in general be more complicated. To obtain the dynamic part to second order, however, we need only the free-particle function $G_0(12,1'2'|z)$. This simplifies our work considerably. It should be noted that there is no first-order contribution to the dynamic part.

Using Eqs. (3.1) and (3.7), we find

$$G_0(12,1'2'|z)=[z-L_0(1)-L_0(2)]^{-1}G_0(12,1'2'), \tag{4.15}$$

where

$$G_0(12,1'2')=F_0(12,1'2')-\int d3\,d3'\,F_0(12,3)F_0^{-1}(3,3')\times F_0(3',1'2'), \tag{4.16}$$

so that our task is reduced to the evaluation of the zero-order static functions on the right-hand side of (4.16). The last term, which can be obtained from Eqs. (2.44) and (4.3), cancels the contribution of 20 of the 24 diagrams for $F_0(12,1'2')$. The remaining four diagrams, Fig. 4, give

$$\begin{aligned}G_0(k_1p_1,k_2p_2;k_3p_3,k_4p_4)=&\{(2\pi)^6\delta(k_1+k_3)\delta(k_2+k_4)\delta(p_1-p_3)\delta(p_2-p_4)+(2\pi)^6\delta(k_1+k_4)\delta(k_2+k_3)\delta(p_1-p_4)\delta(p_2-p_3)\\&+\gamma(2\pi)^3\delta(k_1+k_2+k_3+k_4)[\delta(3+1-)\delta(4+2-)\delta(1+4-)+\delta(1+3-)\delta(2+4-)\delta(4+1-)]\}\\&\times a(k_1p_1,k_2p_2),\end{aligned} \tag{4.17}$$

where

$$a(k_1p_1,k_2p_2)=\tfrac{1}{2}[n_0(1+)n_0(2+)\tilde{n}_0(1-)\tilde{n}_0(2-)+n_0(1-)n_0(2-)\tilde{n}_0(1+)\tilde{n}_0(2+)]. \tag{4.18}$$

Using Eq. (2.27), we can also write $a(k_1p_1,k_2p_2)$ as

$$a(k_1p_1,k_2p_2)=N_0(k_1p_1)N_0(k_2p_2)\frac{\cosh[\frac{1}{2}\beta\hbar(k_1\cdot p_1+k_2\cdot p_2)]}{\cosh(\frac{1}{2}\beta\hbar k_1\cdot p_1)\cosh(\frac{1}{2}\beta\hbar k_2\cdot p_2)}\,. \tag{4.19}$$

We note in passing that in the classical limit, $G_0(12,1'2')$ reduces to

$$\lim_{\hbar\to 0}G_0(12,1'2')=n^2\phi(p_1)\phi(p_2)[\delta(1-1')\delta(2-2')+\delta(1-2')\delta(2-1')],$$

in agreement with a direct classical calculation.

Now substituting (4.15) and (4.17) in Eq. (3.12c) and performing several integrations, we obtain the second-order dynamic kernel in the form

$$\begin{aligned}K_2^{(d)}(kzpp')=\int\frac{d\bar{k}\,d\bar{p}}{(2\pi)^3}V(\bar{k},p)\Big[&V(\bar{k},p')\delta(p-p')-V(\bar{k}-k,p')\delta(\bar{p}-p')+\eta V\Big(\frac{p-\bar{p}}{\hbar}+\frac{k}{2},p'\Big)\delta\Big(\frac{p+\bar{p}}{2}-p'-\frac{\hbar k}{4}+\frac{\hbar\bar{k}}{2}\Big)\\&-\eta V\Big(\frac{p-\bar{p}}{\hbar}-\frac{k}{2},p'\Big)\delta\Big(\frac{p+\bar{p}}{2}-p'+\frac{\hbar k}{4}-\frac{\hbar\bar{k}}{2}\Big)\Big]\frac{a(k-\bar{k},p;\bar{k}\bar{p})}{z-(k-\bar{k})\cdot p-\bar{k}\cdot\bar{p}}\,.\end{aligned} \tag{4.20}$$

The operator $V(k,p)$, defined in Eq. (4.2), acts on everything to its right in (4.20).

It is instructive to examine the classical limit of $K_2^{(d)}$ in this form. With $\hbar=0$, the statistical factor $a(k-\bar{k},p;\bar{k}\bar{p})$ reduces to $n^2\phi(p)\phi(\bar{p})$, the terms containing η cancel, the V operator reduces to $v(k)k\cdot\partial/\partial p$, and we recover the classical expression for $[\Sigma^{(d)}(kzpp')n\phi(p')]_{(2)}$.[17,50] The statistical factor also reduces to its classical value in the $\beta\mu\to-\infty$ limit of Eq. (4.20), but the kinematic exchange and wave-diffraction effects remain.

While the classical weak-coupling kernel is naturally expressed as a generalized Fokker-Planck operator, the quantum weak-coupling kernel actually has a Boltzmann-like collisional structure, although this is not at all apparent from (4.20). We now proceed to transform the second-

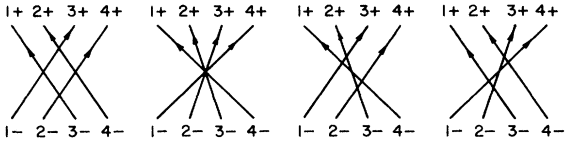

FIG. 4. Zero-order diagrams for $G(12,34)$.

order kernel to such a form, beginning by writing explicitly the displacements in p and p' indicated by the V operators. The result has two types of terms, characterized by the sign in $a(k-\overline{k}, p \pm \frac{1}{2}\overline{k}; \overline{kp})$. Corresponding to this sign, we change the variables of integration by

$$\overline{k} = \pm(p_3 - p), \quad \overline{p} = p_4 + \tfrac{1}{2}(p_3 - p \mp k)$$

and insert $\int dp_2\, \delta(p + p_2 - p_3 - p_4 \pm k)$. As we have no further need to refer to the classical limit, we have set $\hbar = 1$. Finally, we add the above to its version with p_3 and p_4 interchanged, and obtain

$$K_2^{(d)}(kzpp') = (2\pi)^{-3} \int dp_2\, dp_3\, dp_4\, [v(p-p_3) + \eta v(p-p_4)]\, W(pp_2p_3p_4, p')[\tfrac{1}{2}\hat{A}(pp_2p_3p_4|k,z) - \tfrac{1}{2}\hat{A}(pp_2p_3p_4|-k,-z)], \tag{4.21}$$

where

$$W(p_1p_2p_3p_4, p') = [v(p_1-p_3) + \eta v(p_1-p_4)]\,\delta(p_1-p') + [v(p_2-p_4) + \eta v(p_2-p_3)]\,\delta(p_2-p') - [v(p_3-p_1) + \eta v(p_3-p_2)]\,\delta(p_3-p') - [v(p_4-p_2) + \eta v(p_4-p_1)]\,\delta(p_4-p'), \tag{4.22}$$

$$\hat{A}(p_1p_2p_3p_4|k,z) = \delta(p_1 + p_2 - p_3 - p_4 + k) A_0(p_1 + \tfrac{1}{2}k, p_2 + \tfrac{1}{2}k, p_3 - \tfrac{1}{2}k, p_4 - \tfrac{1}{2}k|z), \tag{4.23}$$

$$A_0(p_1p_2p_3p_4|z) = \frac{\frac{1}{2}n_0(p_1)n_0(p_2)\tilde{n}_0(p_3)\tilde{n}_0(p_4) + \frac{1}{2}\tilde{n}_0(p_1)\tilde{n}_0(p_2)n_0(p_3)n(p_4)}{z - \frac{1}{2}(p_1^2 + p_2^2 - p_3^2 - p_4^2)}. \tag{4.24}$$

To clarify the structure of the dynamic kernel as given by (4.21), it is useful to examine the following limits. For $k = 0$, the δ function in $\hat{A}$ may be used to reduce the W factor to

$$[v(p-p_3) + \eta v(p-p_4)][\delta(p-p') + \delta(p_2-p') - \delta(p_3-p') - \delta(p_4-p')];$$

for $k = 0$, $z \to \omega + i0^+$ the denominators of the $\hat{A}$ factors produce

$$\delta(\omega \pm \tfrac{1}{2}[p_1^2 + p_2^2 - p_3^2 - p_4^2])$$

and the numerators then differ only by a factor of $e^{\beta\omega}$. For k and ω equal to zero, therefore, the second-order dynamic kernel reduces to the linear Uehling-Uhlenbeck collision kernel with the Born approximation cross section, in the form

$$\lim_{\epsilon \to 0^+} K_2^{(d)}(0, i\epsilon; pp') = -i\pi(2\pi)^{-3} \int dp_2\, dp_3\, dp_4\, [v(p-p_3) + \eta v(p-p_4)]^2 [\delta(p-p') + \delta(p_2-p') - \delta(p_3-p') - \delta(p_4-p')] \times \delta(p + p_2 - p_3 - p_4)\,\delta(\tfrac{1}{2}[p^2 + p_2^2 - p_3^2 - p_4^2])\, n_0(p) n_0(p_2) \tilde{n}_0(p_3) \tilde{n}_0(p_4). \tag{4.25}$$

The Uehling-Uhlenbeck kernel can be interpreted in terms of energy- and momentum-conserving collisions between free particles with incoming momenta p and p_2 and outgoing momenta p_3 and p_4. Correspondingly, the k, z-dependent kernel involves collisions in the presence of a medium of other particles, whose collective effect is represented by a momentum k and an energy $\omega = \mathrm{Re}z$. Except at $k = 0$, however, the collisions are not simply described by a cross section.

To conclude our derivation, we write the dynamic kernel in a form which clearly displays its full symmetry. First we insert $\int dp_1\, \delta(p - p_1)$ in Eq. (4.21) and perform the changes of variable $(1 \rightleftharpoons 2, 3 \rightleftharpoons 4)$, $(1 \rightleftharpoons 3, 2 \rightleftharpoons 4)$, and $(1 \rightleftharpoons 4, 2 \rightleftharpoons 3)$, where 1 stands for p_1, etc. Using the symmetries

$$W(1234, p') = -W(3412, p') = \eta W(2134, p'), \tag{4.26}$$

$$\hat{A}(1234|k,z) = -\hat{A}(3412|-k,-z) = \hat{A}(2134|k,z), \tag{4.27}$$

we can then write the sum of these four expressions as

$$K_2^{(d)}(kzpp') = \frac{1}{4} \int \frac{d1\, d2\, d3\, d4}{(2\pi)^3} W(1234, p) W(1234, p') \times [\tfrac{1}{2}\hat{A}(1234|k,z) - \tfrac{1}{2}\hat{A}(1234|-k,-z)]. \tag{4.28}$$

From this expression, it follows immediately that the second-order kernel satisfies the positivity and symmetry conditions (3.14).

V. CONCLUSION

We have presented what we believe is the first explicit kinetic equation for a quantum-mechanical fluid that is meaningful on all scales of length and time. Although the second-order kernel is not directly applicable to a real fluid with a strong repulsive interaction, it provides a model kinetic equation containing features which should also appear in any improved theory. The symmetries of the kernel, which are related to the conservation laws, and the positivity or stability condition, which ensures that $S(k\omega)$ and the transport coeffi-

cients are positive, are maintained exactly. The short-time limit reflected in the sum rules, and the long-time large-distance limit reflected in the Uehling-Uhlenbeck kernel are correctly reproduced to the order of the approximation. In a future publication we will give a detailed discussion of the conservation laws and the transport coefficients determined by this second-order theory.

We have presented our theory in terms of the anticommutator function $F(1,1'|t)$. If we had worked instead with the linear response function $L(1,1'|t)$, we would have obtained an approximate kinetic equation similar in form to the one for $F(1,1'|t)$ but differing by detailed-balancing factors corresponding to the factor $\tau(x)$ in Eq. (2.32). The second-order dynamic kernel, for example, would have the same form as (4.21) but the function $A_0(p_1p_2p_3p_4|z)$ in (4.24) would be multiplied by $\tau[\frac{1}{2}\beta(p_1^2+p_2^2-p_3^2-p_4^2)]$. Similarly, the first-order term (4.12a) of the static kernel would be multiplied by $\tau(\beta\hbar k\cdot p')$. In the long-time large-distance limit the effect of these factors would disappear and we would again recover the Uehling-Uhlenbeck kernel (4.25).

While the approach to quantum kinetic theory we have formulated here is limited to the linear-response domain, it is independent of any appeal to coarse graining in space or time and of assumptions such as the Bogoliubov functional ansatz.[4,51] It is clear *a priori* that assumptions of the former kind preclude an accurate description of the short-time behavior. Likewise, the Bogoliubov theory does not attempt to describe the short-time behavior, and in fact it has been shown in the classical case that the ansatz is correct only for vanishing wave vector and frequency.[52]

In this paper we have emphasized the consistent application of a second-order perturbation calculation of the kernel and the initial condition. We do not mean to suggest, however, that calculating all terms of an nth-order expansion would be of great value; selective summation of diagrams or other techniques appropriate to a particular physical system are more likely to be fruitful. The formulation of Secs. II and III is suitable for a density or fugacity expansion, and in the future we intend to discuss an approximate kernel containing all effects of binary collisions.[53] For a dense fluid or a system with long-range forces, it would probably be desirable to modify the theory to eliminate any direct reference to the potential, as has been suggested in the classical case.[23,26-28] We note again that in its present form the theory is not applicable to a superfluid, as we have not allowed for Bose-Einstein condensation in the noninteracting system.

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