Six-parameter Gaussian modes in free space*

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A derivation is given of free-space Gaussian modes described by six parameters. For a beam traveling in the z direction, three parameters describe the beam properties in the planes of constant y and three parameters describe the beam properties in the planes of constant x. It is further shown that the beam waist for the properties in the planes of constant y need not coincide with that for the properties in the planes of constant x. For special values of the parameters the six-parameter modes reduce to the familiar three-parameter modes.

I. INTRODUCTION

In describing laser beams in free space, a Gaussian beam ansatz has been used as the starting point for obtaining a solution of the Helmholtz equation. Presumably such an ansatz arose in analogy to properties of the mode solutions developed for laser cavities. The ansatz assumes that the beam intensity has a Gaussian distribution laterally and that the beam is characterized by three "longitudinal" parameters: The wave-front radius of curvature R, a width W, and a phase ϕ. All of these parameters change with values of the beam-drift coordinate z. When this ansatz is carried out in rectangular Cartesian coordinates, one obtains a set of Hermite polynomial modes. These modes constitute a complete orthogonal set in any plane transverse to the beam axis. We refer to these as three-parameter modes.

We have approached the description of free-space laser beams by recognizing that a more general ansatz may be useful. We retain the Gaussian intensity assumption but use *two* radii of curvature, *two* widths, *two* phases, and *two* beam-drift coordinates. These two sets of parameters refer to beam properties in two orthogonal families of planes parallel to the beam axis. We again obtain a set of Hermite polynomial modes which we call six-parameter modes. These modes provide greater flexibility within the longitudinal parameters and yet are still a complete orthogonal set.

The six-parameter modes are derived as approximate solutions of the Helmholtz equation in Sec. II. In Sec. III a number of general properties of the modes are revealed. In the Appendix we obtain an analytical expression for beam power. The conclusion summarizes the results with some perspective on why certain results are particularly important.

II. BASIC THEORY

The free-space Gaussian modes used to describe laser beams can be written in terms of three beam parameters[1]

$$\phi = \tan^{-1}(k_0 W_0^2/2z), \tag{1a}$$

$$W^2 = W_0^2[1 + (2z/k_0 W_0^2)^2], \tag{1b}$$

$$R = z[1 + (W_0^2 k_0/2z)^2], \tag{1c}$$

where W_0 is the beam-waist radius at $z = 0$ and R is the radius of curvature of the wave front on the z axis which is the beam axis. The parameter ϕ is a phase parameter. We want to show here that there exists a more general set of modes described by six parameters.

To derive these more general modes, we start with the scalar wave equation[2]

$$\nabla^2 u + k_0^2 u = 0$$

and write

$$u(r, \omega) = u_0 \chi(x, y, z) \delta(-\omega_0 + \omega) \exp(-ik_0 z) + \text{c.c.},$$

where c.c. denotes the complex conjugate with the exception that $\delta(-\omega_0 + \omega)$ is replaced by $\delta(\omega_0 + \omega)$. Neglecting $\partial^2\chi/\partial z^2$ we get

$$\frac{\partial^2\chi}{\partial x^2} + \frac{\partial^2\chi}{\partial y^2} - 2ik_0 \frac{\partial\chi}{\partial z} = 0. \tag{2}$$

At this point the usual assumption is to write

$$\chi = \alpha\beta \exp[F_2(z) - (x^2 + y^2)/F_1(z)],$$

where $F_1(z)$ describes the change in beam width with distance and a phase change in the planes of constant z due to the changing radius of curvature caused by the changing beam width. The function $F_2(z)$ describes both a change in amplitude at the beam axis due to the changing width and an added phase change which occurs as the beam propagates. The function $\alpha\beta$ turns out to be approximately

$$\alpha\beta = H_m(\sqrt{2}x/W_0) H_n(\sqrt{2}y/W_0),$$

where H_m and H_n are Hermite polynomials of order m and n.

We instead propose to write

$$\chi = H_m(\sqrt{2}x/U_1)H_n(\sqrt{2}y/U_2) \times \exp[G_3(z) + G_4(z) - x^2/G_1(z) - y^2/G_2(z)] \quad (3)$$

with the expectation that $G_1(z)$ and $G_3(z)$ describe beam properties in the planes of constant y while $G_2(z)$ and $G_4(z)$ describe beam properties in the planes of constant x. We also anticipate that $H_m(\sqrt{2}x/U_1)$ and $H_n(\sqrt{2}y/U_2)$ describe beam properties in the planes of constant y and constant x, respectively.

We now proceed to show that we can obtain an approximate solution to Eq. (2) using the form given in Eq. (3) and obtain the beam parameters for this form. Substituting Eq. (3) into Eq. (2) and neglecting[3]

$$(\sqrt{2}/U_1^2)U_1'xH_m'H_n + (\sqrt{2}/U_2^2)U_2'yH_mH_n'$$

in $\partial\chi(x,y,z)/\partial z$, and rearranging, gives

$$x^2\left(\frac{4}{G_1^2} - \frac{2ik_0G_1'}{G_1^2}\right)H_mH_n - \left(\frac{2}{G_1} + 2ik_0G_3'\right)H_mH_n + \left(\frac{2}{U_1^2}H_m''H_n - \frac{4\sqrt{2}x}{G_1U_1}H_m'H_n\right) + y^2\left(\frac{4}{G_2^2} - \frac{2ik_0}{G_2^2}G_2'\right)H_mH_n - \left(\frac{2}{G_2} + 2ik_0G_4'\right)H_mH_n + \left(\frac{2}{U_2^2}H_mH_n'' - \frac{4\sqrt{2}y}{G_2U_2}H_mH_n'\right) = 0 .$$

Now add and subtract

$$\left(\frac{4m}{U_1^2} + \frac{4n}{U_2^2}\right)H_mH_n$$

to get

$$x^2\left(\frac{4}{G_1^2} - \frac{2ik_0G_1'}{G_1^2}\right)H_mH_n - \left(\frac{2}{G_1} + 2ik_0G_3' + \frac{4m}{U_1^2}\right)H_mH_n + \left(\frac{2}{U_1^2}H_m''H_n - \frac{4\sqrt{2}x}{G_1U_1}H_m'H_n + \frac{4m}{U_1^2}H_mH_n\right) + y^2\left(\frac{4}{G_2^2} - \frac{2ik_0G_2'}{G_2^2}\right)H_mH_n - \left(\frac{2}{G_2} + 2ik_0G_4' + \frac{4n}{U_2^2}\right)H_mH_n + \left(\frac{2}{U_2^2}H_mH_n'' - \frac{4\sqrt{2}y}{G_2U_2}H_mH_n' + \frac{4n}{U_2^2}H_mH_n\right) = 0 .$$

We find this equation holds if

$$U_1^2 = G_1 , \quad U_2^2 = G_2 ,$$

and

$$G_1 = A + 2z/ik_0 , \quad G_2 = B + 2z/ik_0 ,$$

$$G_3 \approx -\tfrac{1}{2}\ln(z + \tfrac{1}{2}iAk_0) - 2mz/ik_0U_1^2 + C ,$$

$$G_4 \approx -\tfrac{1}{2}\ln(z + \tfrac{1}{2}iBk_0) - 2nz/ik_0U_2^2 + D ,$$

where for the computation of G_3' and G_4' we neglected derivatives of U_1 and U_2, respectively.

To evaluate the constants we consider the beam at $z=0$. We find

$$A = \hat{W}_1^2 \quad (4)$$

and

$$B = \hat{W}_2^2 , \quad (5)$$

where $\hat{W}_1$ and $\hat{W}_2$ are the distances, in planes of constant y and x, respectively, at which the Gaussian beam factor has fallen to e^{-1} times its value on the beam axis. Thus they are the beam widths at the beam waist. The real part of G_1 gives an amplitude term in χ of the form

$$\exp[-\hat{W}_1^2x^2/(\hat{W}_1^4 + 4z^2/k_0^2)] .$$

We thus identify the beam width in the planes of constant y as

$$W_1^2 = \hat{W}_1^2(1 + 4z^2/\hat{W}_1^4k_0^2) . \quad (6)$$

We similarly find the beam width in the planes of constant x to be

$$W_2^2 = \hat{W}_2^2(1 + 4z^2/\hat{W}_2^4k_0^2) . \quad (7)$$

To identify the contribution from the imaginary part of G_1, we note that if R_1 is a radius of curvature in the planes of constant y then

$$\exp(-ik_0R_1) \equiv \exp[-ik_0z(1 + x^2/y^2)^{1/2}] \approx \exp(-ik_0z - ik_0x^2/2z)$$

where $x^2/z^2 \ll 1$ has been used. On $x=0$ we see $R_1 = z$ so there

$$\exp(-ik_0R_1) \approx \exp(-ik_0z - ik_0x^2/2R_1) .$$

The contribution to χ from the imaginary part of G_1 yields

$$\exp\left(\frac{-2izx^2/k_0}{\hat{W}_1^4[1 + (2z/\hat{W}_1^2k_0)^2]}\right)$$

so we identify the radius of curvature at $x=0$ in the planes of constant y as

$$R_1 = z[1 + (\hat{W}_1^2k_0/2z)^2] . \quad (8)$$

We similarly identify the radius of curvature at $y=0$ in the planes of constant x as

$$R_2 = z[1 + (\hat{W}_2^2k_0/2z)^2] . \quad (9)$$

To identify C we write G_3 as

$$G_3 = -\tfrac{1}{2}\ln[(z^2 + \tfrac{1}{4}k_0^2\hat{W}_1^4)^{1/2}\exp(i\phi_1)] - 2mz/ik_0U_1^2 + C$$

where $\phi_1 = \tan^{-1}(k_0\hat{W}_1^2/2z)$. Now we require

$$G_3(0) = 0 .$$

This yields

$$C = \tfrac{1}{2}\ln(\tfrac{1}{2}i\hat{W}_1^2k_0) . \qquad (10)$$

Similarly we find

$$D = \tfrac{1}{2}\ln(\tfrac{1}{2}i\hat{W}_2^2k_0) . \qquad (11)$$

We then find that in χ, G_3 yields the contribution

$$i^{1/2}\left(\frac{\hat{W}_1}{W_1}\right)^{1/2}\exp\left(\frac{-i\phi_1}{2}\right)\exp\left(\frac{-2mz}{ik_0U_1^2}\right).$$

We similarly find that in χ, G_4 yields the contribution

$$i^{1/2}\left(\frac{\hat{W}_2}{W_2}\right)^{1/2}\exp\left(\frac{-i\phi_2}{2}\right)\exp\left(\frac{-2nz}{ik_0U_2^2}\right).$$

It is now an elementary task to assemble $u(r,\omega)$ which appears as

$$u(r,\omega) = u_0\delta(-\omega_0+\omega)\exp(-ik_0z)\,i\left(\frac{\hat{W}_1\hat{W}_2}{W_1W_2}\right)^{1/2}H_m\left(\frac{\sqrt{2}x}{U_1}\right)H_n\left(\frac{\sqrt{2}y}{U_2}\right) \times\exp\left[\frac{-i\phi_1}{2}-\frac{i\phi_2}{2}+\frac{2imz}{k_0U_1^2}+\frac{2inz}{k_0U_2^2}-\frac{x^2ik_0}{2}\left(\frac{1}{R_1}-\frac{2i}{k_cW_1^2}\right)-\frac{y^2ik_0}{2}\left(\frac{1}{R_2}-\frac{2i}{k_0W_2^2}\right)\right]+\text{c.c.},$$

where

$$U_1^2 = \hat{W}_1^2 + 2z/ik_0 , \quad U_2^2 = \hat{W}_2^2 + 2z/ik_0 .$$

Because of the identities

$$W_1^2 \equiv 2R_1/k_0\tan\phi_1$$

and

$$W_2^2 \equiv 2R_2/k_0\tan\phi_2$$

only four of the six parameters $\hat{W}_1$, $\hat{W}_2$, R_1, R_2, ϕ_1, and ϕ_2 are independent.

It is possible to obtain an even more general solution. For that we start by replacing $\exp(-ik_0z)$ in $u(r,\omega)$ by

$$\exp[-ik_0(z+z')/2]$$

where $z' = z + a$ with constant a. Thus

$$\frac{\partial}{\partial z'} = \frac{\partial}{\partial z} .$$

We now label z as z_1 and z' as z_2. If we replace z by z_1 in U_1, G_1, and G_3, and by z_2 in U_2, G_2, and G_4 then we find the more general solution is

$$u(r,\omega) = u_0\delta(-\omega_0+\omega)\exp\left(\frac{-ik_0}{2}(z_1+z_2)\right)i\left(\frac{\hat{W}_1\hat{W}_2}{W_1W_2}\right)^{1/2}H_m\left(\frac{\sqrt{2}x}{U_1(z_1)}\right)H_n\left(\frac{\sqrt{2}y}{U_2(z_2)}\right) \times\exp\left[\frac{-i\phi_1(z_1)}{2}-\frac{i\phi_2(z_2)}{2}+\frac{2imz_1}{k_0U_1^2(z_1)}+\frac{2inz_2}{k_0U_2^2(z_2)}-\frac{x^2ik_0}{2}\left(\frac{1}{R_1(z_1)}-\frac{2i}{k_0W_1^2(z_1)}\right) -\frac{y^2ik_0}{2}\left(\frac{1}{R_2(z_2)}-\frac{2i}{k_0W_1^2(z_2)}\right)\right]+\text{c.c.} \qquad (12)$$

To obtain the three parameter modes in terms of the parameters of Eqs. (1), we set $\hat{W}_1 = \hat{W}_2 = W_0$, $R_1 = R_2 = R$, $z_1 = z_2 = z$. The approximation $U_1 = U_2 \simeq W_0$ is also frequently used.

We now wish to write the six parameter modes in terms of another phase convention. For this we define

$$\phi_j' = \tfrac{1}{2}\pi - \phi_j = \tan^{-1}(2z_j/k_0\hat{W}_j^2) , \quad j = 1, 2$$

and assume that

$$2z_j/k_0\hat{W}_j^2 \ll 1 , \quad j = 1, 2 .$$

Then

$$\frac{2imz_1}{k_0U_1^2(z_1)} \approx im\tan^{-1}\left(\frac{2z_1}{k_0\hat{W}_1^2}\right) = im\phi_1' ,$$

and

$$\frac{2inz_2}{k_0U_2^2(z_2)} \approx in\phi_2' .$$

Within this approximation the independent variables in the Hermite polynomials become $\sqrt{2}x/\hat{W}_1$ and $\sqrt{2}y/\hat{W}_2$. If we now include the multiplicative factor i in Eq. (12) in the phase, we get

$$u(r,\omega)=u_0\delta(-\omega_0+\omega)\exp\left(\frac{-ik_0}{2}(z_1+z_2)\right)\left(\frac{\hat{W}_1\hat{W}_2}{W_1W_2}\right)^{1/2}H_m\left(\frac{\sqrt{2}x}{\hat{W}_1}\right)H_n\left(\frac{\sqrt{2}y}{\hat{W}_2}\right)$$
$$\times\exp\left[i(m+\tfrac{1}{2})\phi_1'(z_1)+i(n+\tfrac{1}{2})\phi_2'(z_2)-\frac{ik_0x^2}{2}\left(\frac{1}{R_1(z_1)}-\frac{2i}{k_0W_1^2(z_1)}\right)-\frac{ik_0y^2}{2}\left(\frac{1}{R_2(z_2)}-\frac{2i}{k_0W_2^2(z_2)}\right)\right]+\text{c.c.} \tag{13}$$

When $\hat{W}_1=\hat{W}_2$, $R_1=R_2$, and $z_1=z_2$ these become the three parameter modes in their most familiar form.[2]

III. MODE PROPERTIES

To gain some insight into the nature of these modes, we transform Eq. (13) to the time domain

$$u(r,t)=u_0\left(\frac{\hat{W}_1\hat{W}_2}{W_1(z_1)W_2(z_2)}\right)^{1/2}H_m\left(\frac{\sqrt{2}x}{\hat{W}_1}\right)H_n\left(\frac{\sqrt{2}y}{\hat{W}_2}\right)\exp\left(\frac{-x^2}{W_1^2(z_1)}-\frac{y^2}{W_2^2(z_2)}\right)$$
$$\times\cos\left\langle\omega_0t-\frac{k_0}{2}(z_1+z_2)+(m+\tfrac{1}{2})\phi_1'(z_1)+(n+\tfrac{1}{2})\phi_2'(z_2)-\frac{k_0x^2}{2R_1(z_1)}-\frac{k_0y^2}{2R_2(z_2)}\right\rangle. \tag{14}$$

We have plotted equal amplitude contours of $u(r,t)$ in Figs. 1–4 for a number of situations. Before discussing some of the analytical features of these plots we will first discuss the conventions used in them.

In each of the four figures, we fix m, n, z_1, z_2, and t. All lengths are measured in wavelengths. Thus the velocity of light c is measured in wavelengths per second. This means that as ct varies from 0 to 1, one cycle is completed. The plots are normalized so that with $\hat{W}_1$ and $\hat{W}_2$ fixed and with

$$z_1=0 \text{ and } z_2=0,$$

$u(r,t)$ has a maximum amplitude of 1 V/m. It then follows that with the same $\hat{W}_1$ and $\hat{W}_2$, but different values of z_1 and z_2, the maximum amplitude of $u(r,t)$ cannot be greater than 1. To see why this is true, let us denote by F the amplitude of $u(r,t)$;

$$F=u_0\left(\frac{\hat{W}_1\hat{W}_2}{W_1(z_1)W_2(z_2)}\right)^{1/2}H_m\left(\frac{\sqrt{2}x}{\hat{W}_1}\right)H_n\left(\frac{\sqrt{2}y}{\hat{W}_2}\right)\times\exp\left(\frac{-x^2}{W_1^2(z_1)}-\frac{y^2}{W_2^2(z_1)}\right).$$

We then search for extrema of F by requiring

$$\frac{\partial F}{\partial z_i}=0,\quad i=1,2.$$

Then we have

$$\left(-\frac{1}{2}+\frac{2x^2}{W_1^2(z_1)}\right)\frac{dW_1(z_1)}{dz_1}=0$$

with a similar expression for $W_2(z_2)$. Then either

$$\frac{dW_1(z_1)}{dz_1}=0,$$

which identifies the beam waist (which is at $z_1=0$), or

$$W_1^2(z_1)=4x^2.$$

The second extremum is due to the spreading of the beam. To see this, consider observing the beam amplitude at a point x removed some distance from the region of high-beam intensity first with $z_1=0$ as shown in Fig. 5, and then with z_1 increased enough to produce the second situation illustrated in Fig. 5, where the curves AB are constant amplitude contours of the partial amplitude factor F_x given by

$$F_x=\left(\frac{u_0\hat{W}_1}{W_1(z_1)}\right)^{1/2}H_m\left(\frac{\sqrt{2}x}{\hat{W}_1}\right)\exp\left\langle\frac{-x^2}{W_1^2(z_1)}\right\rangle.$$

Clearly the partial amplitude as viewed at x is greater at $z_1>0$ than at $z_1=0$. Since we can write a similar partial amplitude for the y related parameters and separate F in the form

$$F=F_xF_y,$$

we see that we can consider the extrema of the beam due to the x and y parameters separately. Our argument thus shows that the second extremum in the beam amplitude is due to the spreading of the beam and can therefore be ignored. We have verified this by comparing the maximum amplitude, with z_1 and z_2 set equal to 0, with the value of the amplitude as determined by our second extremum, with z_1 and z_2 both different from zero, for mode numbers between 0 and 3.

With z_1 and z_2 set equal to 0, the maximum partial amplitudes were found by respectively differentiating F_x and F_y with respect to x and y. The maximum partial amplitudes were then multiplied to give the maximum value of F. We chose values of u_0 which made the maximum value of F unity.

For each plot, we have also calculated the power the laser beam would need to produce a pattern corresponding to that plot. This was done by using the relation for the beam power which is derived in the Appendix.

Normalizing in a way that assured us that the amplitudes were between −1 and +1 made it possible for us to plot constant amplitude contours, first in increments of 0.1 between −1 and +1 and later in finer increments which we will refer to as subintervals. The subintervals plotted were chosen as follows: first, the maximum and minimum amplitudes achieved in the plot were found to an accuracy of 0.1 while generating the first set of contours. Let us suppose that they were 0.8 and −0.3 respectively. The computer then searched for subintervals of the form

$$0.8 + J(0.1)^I \text{ and } -0.3 - J(0.1)^I ,$$

where a number of values of J were specified and where I was progressively chosen as 2, 3, ... until $(0.1)^I$ was less than a specified limit which we will call Γ. We have not labeled each contour in the plot. Instead, we have labeled each zero-amplitude contour, and the beginning and end of each sequence of contours. The change from contour to contour within a sequence will be a fixed number. For example, in Fig. 2(d) we have labeled the zero amplitude contours and the contours with an amplitude of −0.1, −0.7, 0.003, −0.73, and −0.79. The contours between −0.1 and 0.7 change by 0.1 while the contour between −0.73 and −0.79 is −0.76.

To complete the specification of each plot, we

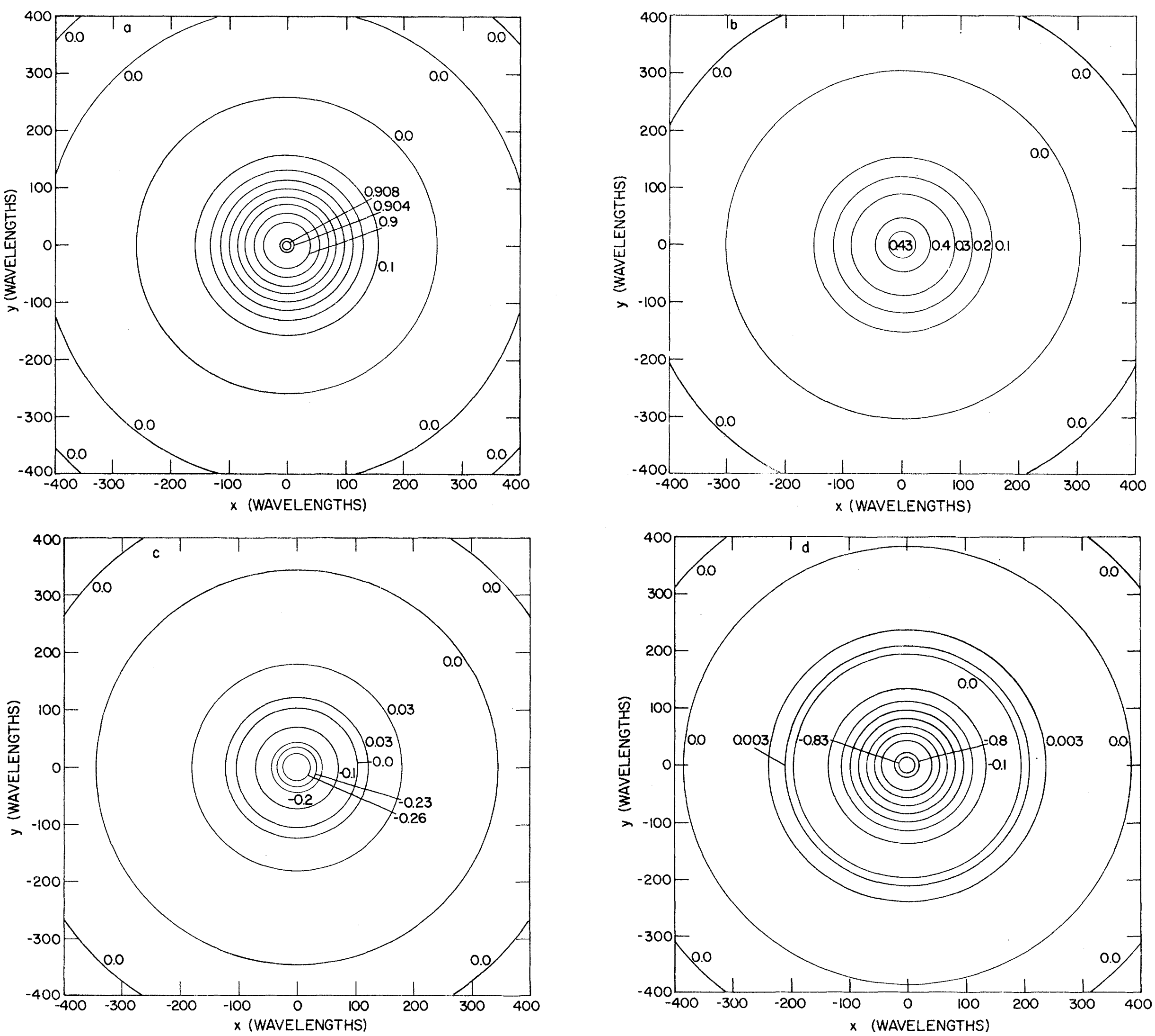

FIG. 1. $m=0$, $n=0$, $z_1=10\,000\lambda$, $z_2=10\,000\lambda$, $\hat{W}_1=100\lambda$, $\hat{W}_2=100\lambda$, (three-parameter mode). Part (a) $ct=0.000$, (b) $ct=0.125$, (c) $ct=0.250$, (d) $ct=0.375$. The power needed to produce these patterns is $20.852\lambda^2$ W.

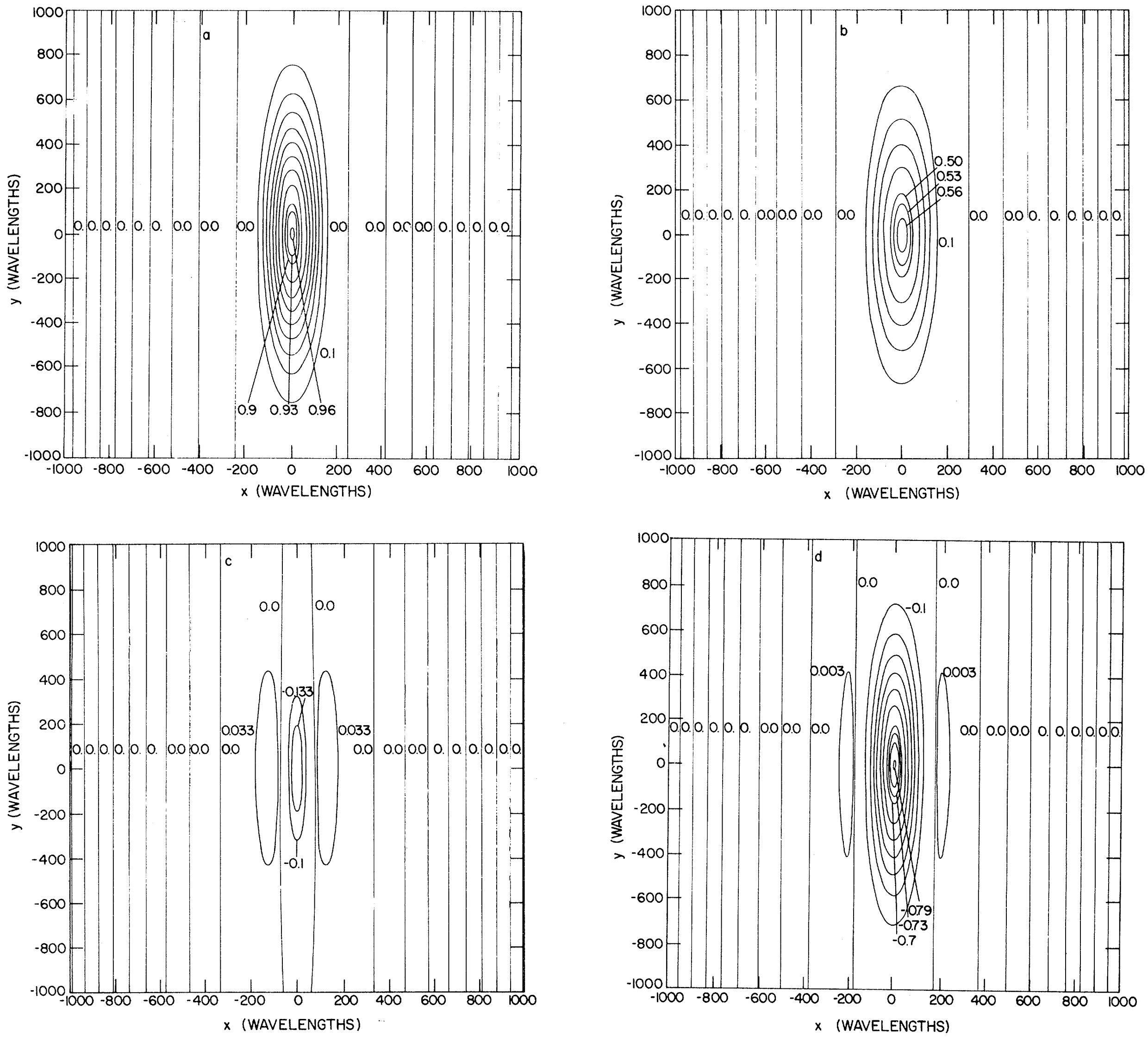

FIG. 2. $m=0$, $n=0$, $z_1=10\,000\lambda$, $z_2=10\,000\lambda$, $\hat{W}_1=100\lambda$, $\hat{W}_2=500\lambda$. Part (a) $ct=0.000$, (b) $ct=0.125$, (c) $ct=0.250$, (d) $ct=0.375$. The power needed to produce these patterns is $104.26\lambda^2$ W.

set the ranges of the x and y axes equal and chose them so that the plot would definitely include all contours of constant amplitude whose magnitude was greater than Γ. This was done using asymptotic forms for the Hermite polynomials and by specifying that the largest dimension of the plot should be an integral multiple of 100 wavelengths.

An interesting property of the plots is that they can be rescaled according to the following rules, which are easily verified by examining Eq. (14) and the functional forms of $W_i(z_i)$, $R_i(z_i)$ and $\phi_i'(z_i)$, $i=1,2$. Let α and β be two real numbers determined by the following relations:

$$\alpha=(1+p/z_1), \quad p=\pm1,\pm2,\ldots,$$

$$\beta=(1+q/z_2), \quad q=\pm1,\pm2,\ldots,$$

subject to

$$p+q=0,\pm2,\pm4,\ldots.$$

Then if the parameters of a given plot are z_1, z_2, $\hat{W}_1$, $\hat{W}_2$, and t, a plot with parameters $\alpha^2 z_1$, $\beta^2 z_2$, $\alpha\hat{W}_1$, $\beta\hat{W}_2$, and t can be obtained by scaling x everywhere by α and y everywhere by β. (As in the plots, all lengths in the above relations are in units of wavelengths.)

Finally, we take up a discussion of the zeros in the beam amplitude. These occur at the zeros of the Hermite polynomials, which are straight lines parallel to the x and y axes and are independent of the time, and along contours where the phase is an odd integer of $\pi/2$, i.e., along surfaces where

$$\omega_0 t - \tfrac{1}{2}k_0(z_1+z_2) + (m+\tfrac{1}{2})\phi_1'(z_1) + (n+\tfrac{1}{2})\phi_2'(z_2) - k_0x^2/2R_1(z_1) - k_0y^2/2R_2(z_2) = \tfrac{1}{2}(2l+1)\pi\,, \quad l=0,\pm1,\pm2,\ldots.$$

This can be rewritten in the form

$$\frac{x^2}{R_1(z_1)} + \frac{y^2}{R_2(z_2)} = \mathcal{A}\,,$$

where

$$\mathcal{A} = 2ct - (2l+1)\pi/k_0 - (z_1+z_2) + [(2m+1)/k_0]\phi_1'(z_1) + [(2n+1)/k_0]\phi_2'(z_2)\,, \quad l=0,\pm1,\pm2,\ldots.$$

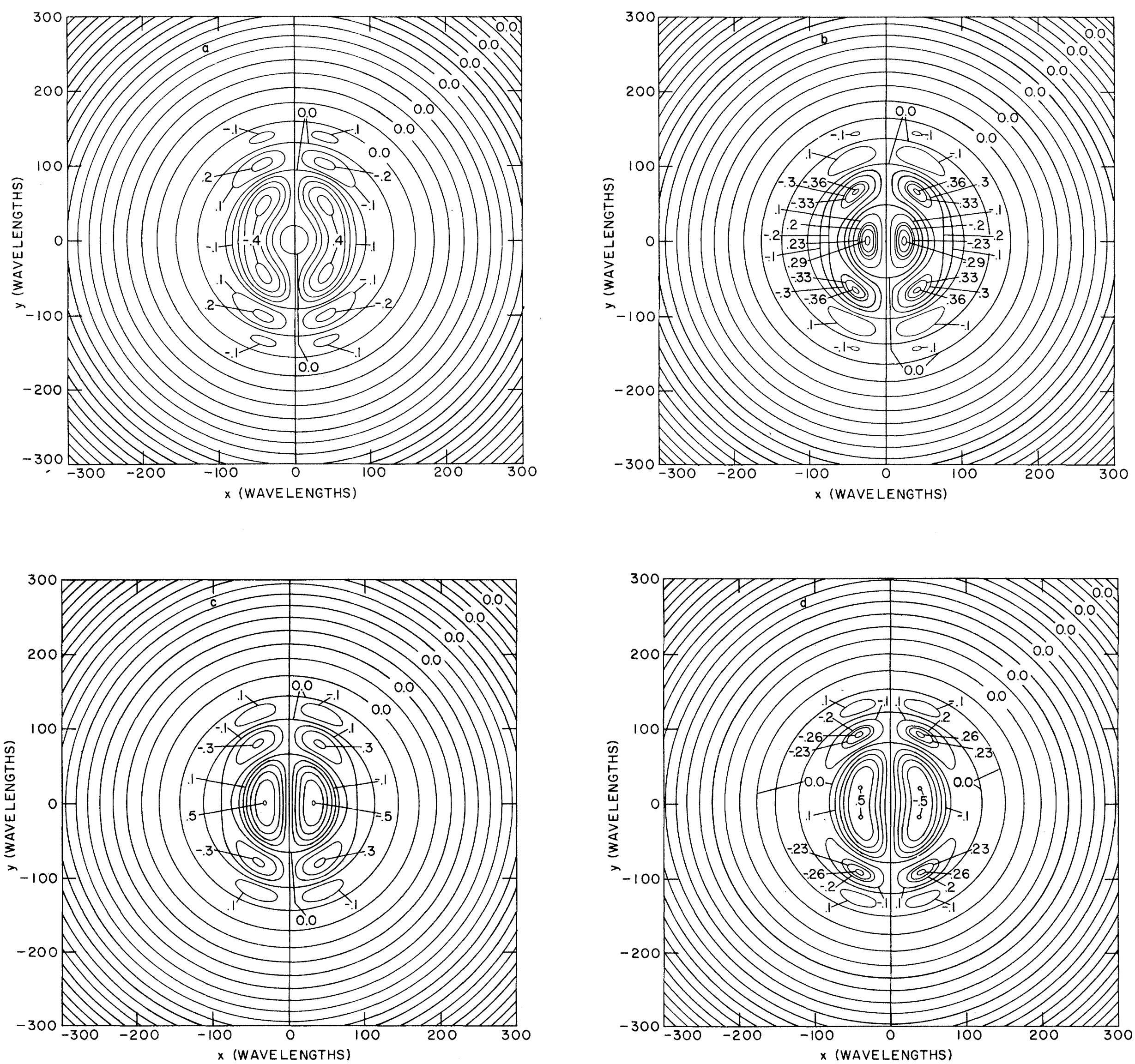

FIG. 3. $m=1$, $n=0$, $z_1=4084.07\lambda$, $z_2=7853.98\lambda$, $\hat{W}_1=36.10\lambda$, $\hat{W}_2=22.40\lambda$. Part (a) $ct=0.000$, (b) $ct=0.125$, (c) $ct=0.250$, (d) $ct=0.375$. The power needed to produce these patterns is $2.292\lambda^2$ W.

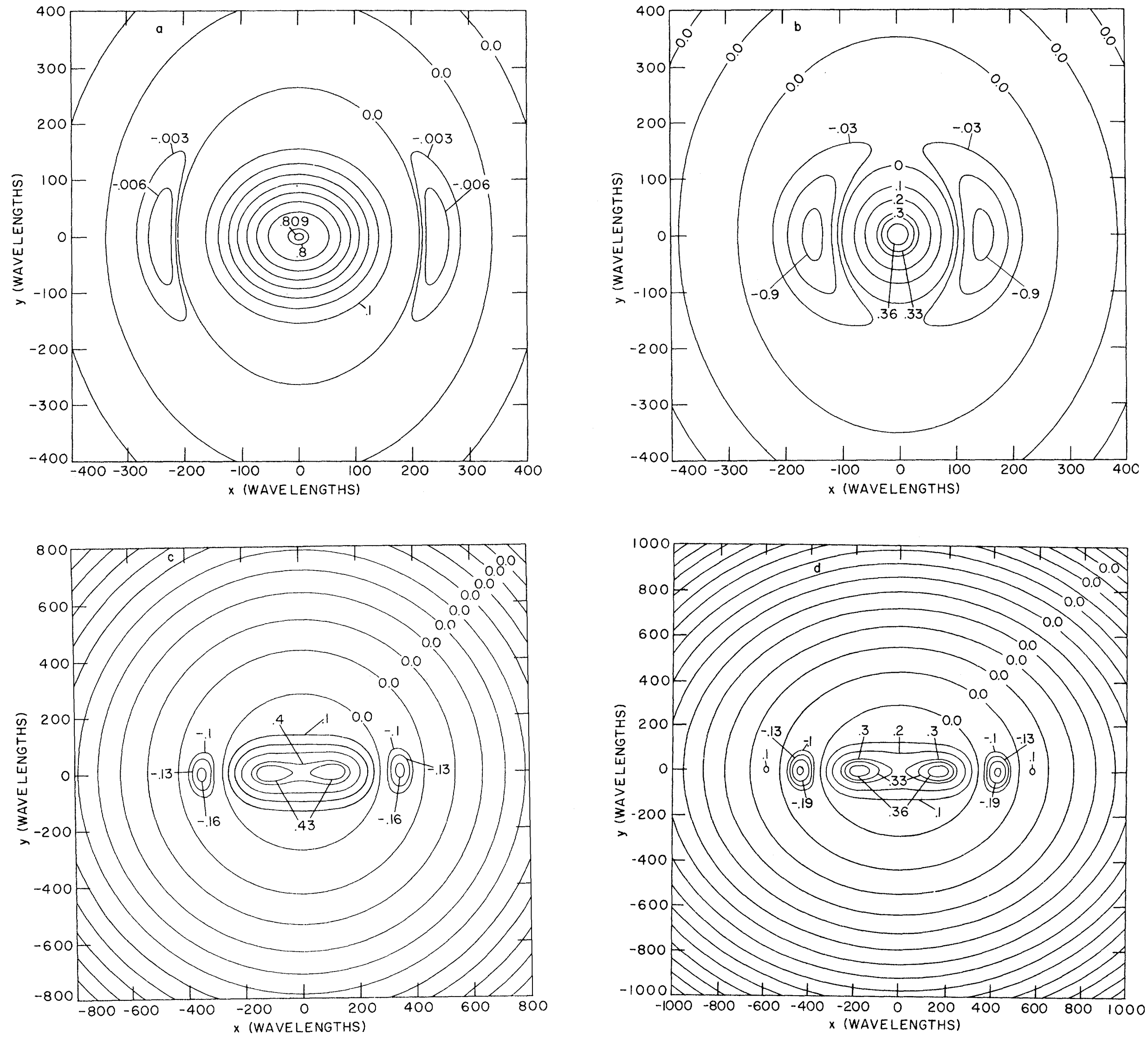

FIG. 4. These plots show the effect of increasing z_1, while fixing z_2, $\hat{W}_1$, $\hat{W}_2$, and ct (see text). For all parts $m=0$, $n=0$, $z_2=10\,000\lambda$, $\hat{W}_1=100\lambda$, $\hat{W}_2=100\lambda$, $ct=0.000$. Part (a) $z_1=20\,000.00\lambda$, (b) $z_1=20\,000.50\lambda$, (c) $z_1=98\,696.04\lambda$, (d) $z_1=150\,000.00\lambda$. The power required to produce this pattern is $20.852\lambda^2$ W.

Since we can always find values of l that allow $\mathfrak{a}$ to be positive, we see that at a fixed z_1, z_2, and t, the zeros of $u(r,t)$ fall along ellipses whose semiaxes are given by

$$[\mathfrak{a}R_1(z_1)]^{1/2} \text{ and } [\mathfrak{a}R_2(z_2)]^{1/2}.$$

Since $\mathfrak{a}$ increases linearly with time, the semiaxes of these ellipses increase as $t^{1/2}$.

It is interesting to note that when

$$\hat{W}_1=\hat{W}_2,$$

and with the beam waist of one parameter (x or y) always in the same relation with that of the other parameter (i.e., always behind or always in front of it), the semimajor axis of an ellipse of zero amplitude can be along either the x or the y axis, or the two semiaxes can be equal. This can be seen without loss of generality by assuming

$$z_1 > z_2;$$

then the semiaxes of an ellipse of zero amplitude will fall into one of the above three categories depending on which of the following conditions is fulfilled: (a) the semimajor axis will be along the y axis when

$$z_1 z_2 < \hat{W}_1^4 k_0^2/4;$$

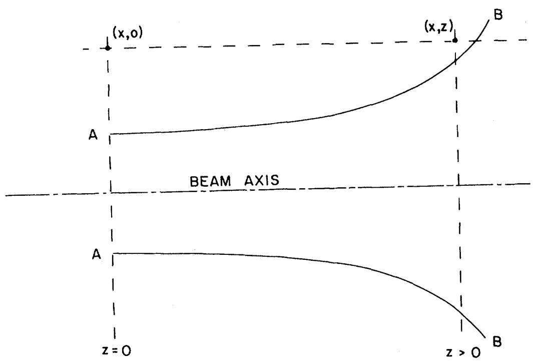

FIG. 5. Illustration of the origin of the second extremum in beam amplitude. The curves AB are constant amplitude contours of the partial amplitude F_x as described in the text, and the point x is far from the intense portion of the beam. At $z = 0$, x is located at the beam waist, while at $z > 0$, x is near the curve AB.

(b) the lines of zero amplitude will be circles when

$$z_1 z_2 = \hat{W}_1^4 k_0^2/4\,;$$

(c) the semimajor axis will be along the x axis when

$$z_1 z_2 > \hat{W}_1^4 k_0^2/4\,.$$

For example, suppose condition (a) is met; then if we first look at the beam with the waists at the same location and of the same width and proceed to slip the x-z plane waist to greater and greater distances behind the y-z plane waist while keeping the location of the y-z plane waist fixed at a value such that

$$z_2 < \hat{W}_1^2 k_0/2$$

the lines of zero amplitude determined by the phase first appear as ellipses with their semimajor axes along the y axis, then as circles, and finally as ellipses with their semimajor axes along the x axis. Intuitively one might expect the ellipses to always have their semimajor axes along the x axis since the x beam width will be larger than the y beam width at the plane of observation. This shows the importance of the phase terms in the expression for u and is illustrated in Fig. 4. In Fig. 4(a), the semimajor axes of the zero amplitude ellipses are along the y axis. Note that the contours interior to the first zero-amplitude contour have their long axes along the x axis. In Fig. 4(b), the beam waists have been slipped further by a half wavelength. Note that now all contours have their long axes along the y axis. Figures 4(c) and 4(d) illustrate the transformation of the zero-amplitude contours into circles and then into ellipses with their semimajor axes along the x axis.

Another surprising feature is the fact that phase effects dominate over the spreading of the beam in regard to the dimensions of the contours of zero amplitude. For example, suppose z_1, z_2, and t are positive numbers, that z_2 and t are fixed, and that z_1 is increased to a new value z_1' where

$$z_1' = z_1 + s\,, \quad s > 0\,.$$

If s is sufficiently large, the result is to decrease the value of $\mathfrak{A}$, and so to decrease the semiaxes of the zero-amplitude contours. Pursuing this further, let us consider what happens to the smallest contour. Suppose that l is such that

$$k_0 \mathfrak{A}/2 < \pi$$

when $\mathfrak{A}$ is evaluated at z_1, z_2, and t. Or, stated more explicitly, suppose l is such that

$$\omega_0 t - \frac{(2l+1)\pi}{2} - \frac{(z_1+z_2)}{2} + \frac{(2m+1)}{2}\phi_1'(z_1) + \frac{(2n+1)}{2}\phi_2'(z_2) < \pi\,. \tag{15}$$

If l is increased by 1, $\mathfrak{A}$ becomes negative, so this value of l corresponds to the smallest zero amplitude ellipse. We know that

$$\frac{x^2 k_0}{2R_1(z_1+s)} + \frac{y^2 k_0}{2R_2(z_2)} = \omega_0 t - \frac{(2l+1)\pi}{2} - \frac{z_1+s+z_2}{2}k_0 + \frac{2m+1}{2}\phi_1'(z_1+s) + \frac{2n+1}{2}\phi_2'(z_2)\,. \tag{16}$$

Then since

$$0 \leqslant \phi_1'(z_1+s) \leqslant \pi/2$$

and

$$\phi_1'(z_1) \geqslant 0\,,$$

we can certainly write

$$\phi_1'(z_1+\Delta) \leqslant \phi_1'(z_1) + \pi/2\,.$$

Inserting this into Eq. (16), we have

$$\frac{x^2}{2R_1(z_1+\Delta)} + \frac{y^2}{2R_2(z_2)} \leqslant \frac{1}{k_0}\left(\omega_0 t - \frac{(2l+1)\pi}{2} - \frac{z_1+z_2}{2} + \frac{2m+1}{2}\phi_2'(z_2)\right) - \frac{s}{2} + \frac{\pi}{2k_0}\left(\frac{2m+1}{2}\right).$$

Making use of inequality (15), we thus have

$$\frac{x^2}{2R_1(z_1+\Delta)} + \frac{y^2}{2R_2(z_2)} < \frac{1}{k_0}\left[\pi\left(\frac{m}{2}+\frac{5}{4}\right) - \frac{sk_0}{2}\right].$$

Thus, when one increases z_1 to the point that

$$s \geq \tfrac{1}{2}\lambda(m + \tfrac{5}{2})$$

the innermost ellipse of zero amplitude ceases to exist. Since $u(r,t)$ is a continuous function, this means that as s is increased and the innermost zero-amplitude ellipse shrinks, the absolute value of the amplitude at points within this ellipse must decrease. By conservation of energy, this means that the net power outside of this ellipse increases as s increases. This explains why lobes frequently appear in the plots when z_1 and z_2 are not equal.

Finally, it is worth noting that the amplitudes on opposite sides of lines of zero amplitude have opposite signs as we now demonstrate. For the lines of zero amplitude generated by the Hermite polynomials, this is easily seen since

$$\frac{d}{dx}H_m\left(\frac{\sqrt{2}x}{\hat{W}_1}\right) = \frac{2\sqrt{2}}{\hat{W}_1}mH_{m-1}\left(\frac{\sqrt{2}x}{\hat{W}_1}\right)$$

and since the zeros of $H_m(\sqrt{2}x/\hat{W}_1)$ and $H_{m-1}(\sqrt{2}x/\hat{W}_1)$ never coincide. Thus $H_m(\sqrt{2}x/\hat{W}_1)$ is never extremal at its zeros, and therefore changes sign as it crosses a zero. For phase induced zeros, we note first that any ellipse having semiaxes along the x and y axes is a curve of constant phase. Let us call the total phase along an ellipse e_n, $\Phi(e_n)$. If we consider an ellipse e_1 just inside of an ellipse of zero amplitude e_0, it will correspond to a phase slightly larger than the phase along e_0, while if we consider an ellipse e_2 just outside of e_0 it will correspond to a phase slightly smaller than that along e_0. Thus, since

$$\Phi(e_0) = [(2l+1)/2]\pi, \quad l = 0, \pm 1, \pm 2, \ldots,$$

then

$$\cos\langle\Phi(e_1)\rangle / \cos\langle\Phi(e_2)\rangle < 0 .$$

Thus, $u(r,t)$ also differs in sign across phase induced zero-amplitude contours. We see also that, across a vertex where phase-induced zero-amplitude contours cross zero-amplitude contours generated by Hermite polynomials, there is no sign reversal since the sign reversal produced by the Hermite polynomials and by the phase will then cancel.

IV. DISCUSSION

A more flexible set of free-space Gaussian modes has been described. These modes are characterized by the independence of the symmetry properties in the x-z and y-z planes. The ordinary modes occur when the symmetry properties of the x-z and y-z planes coincide.

We have presented identities which relate the parameters W_1, R_1, and ϕ_1 as well as W_2, R_2, and ϕ_2. Thus actually only four of these six parameters are independent so it might seem appropriate to describe our modes as four- rather than six-parameter modes. However, we have seen that we have two additional independent parameters z_1 and z_2 which have essential roles. Because of this we have chosen to designate the new modes as six-parameter modes and the usual modes as three-parameter modes.

We have given detailed illustrations of field amplitude contours. We have shown how to scale the modes. This was achieved by using the periodicity of the sine and cosine functions. As a result the scaling is restricted to an infinite set of discrete scaling factors. These factors allow flexible scaling except near the waist. It is the ability to scale the modes that makes detailed mode illustrations useful.

The six-parameter modes do not yield approximate solutions of the Helmholtz equation which cannot also be obtained by an infinite expansion using three-parameter modes. This is because of the completeness of the Hermite polynomials. The usefulness of the six-parameter modes should, however, be readily apparent.

APPENDIX: CALCULATION OF BEAM POWER FOR SIX-PARAMETER MODES

We will use as an approximation, the assumption that the $\vec{E}$ and $\vec{H}$ fields are normal to the z axis. We can then write for the magnitude of the Poynting vector

$$S = (\epsilon_0/\mu_0)^{1/2}E^2 .$$

This means that the instantaneous power of the beam is given by

$$P = \left(\frac{\epsilon_0}{\mu_0}\right)^{1/2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E^2\, dx\, dy .$$

We deal only with the simple case where $\vec{E}$ has a single six-parameter Gaussian mode component polarized in a direction normal to the y axis. Then

$$P = \frac{1}{2}\left(\frac{\epsilon_0}{\mu_0}\right)^{1/2} u_0^2\left(\frac{\hat{W}_1\hat{W}_2}{W_1(z_1)W_2(z_2)}\right)[\mathrm{Re}(I_1) + I_2],$$

where

$$I_1 = \exp[2i\omega t - ik_0(z_1+z_2) + i(2m+1)\phi_1'(z_1) + i(2n+1)\phi_2'(z_2)]$$
$$\times \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} H_m^2\left(\frac{\sqrt{2}x}{\hat{W}_1}\right)H_n^2\left(\frac{\sqrt{2}y}{\hat{W}_2}\right)\exp\left(\frac{-2x^2}{W_1^2(z_1)} - \frac{ik_0x^2}{R_1(z_1)}\right)\exp\left(\frac{-2y^2}{W_2^2(z_2)} - \frac{ik_0y^2}{R_2(z_2)}\right)dx\,dy ,$$

and

$$I_2=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}H_m^2\left(\frac{\sqrt{2}x}{\hat{W}_1}\right)H_n^2\left(\frac{\sqrt{2}y}{\hat{W}_2}\right)\times\exp\left(\frac{-2x^2}{W_1^2(z_1)}-\frac{2y^2}{W_2^2(z_2)}\right)dx\,dy\,.$$

Since the total power of the beam should be the same at any point along the beam, we are at liberty to arbitrarily specify z_1. It will be convenient to choose.

$$z_1=0\,.$$

Then, since z_1 and z_2 are always separated by a fixed amount, say Δ, we must set

$$z_2=\Delta\,.$$

Then, since

$$W_1(0)=\hat{W}_1$$

and

$$R_1(0)\to\infty\,,$$

$$I_1=2^m m!\,(\pi/2)^{1/2}\hat{W}_1\exp[2i\omega_0 t-ik_0\Delta+i(2m+1)\phi_1'(0)+i(2n+1)\phi_2'(\Delta)]\times\int_{-\infty}^{\infty}H_n^2\left(\frac{\sqrt{2}y}{\hat{W}_2}\right)\exp\left(\frac{-2y^2}{W_2^2(\Delta)}-\frac{ik_0y}{R_2(\Delta)}\right)dx\,dy\,,$$

$$I_2=2^m m!\,(\pi/2)^{1/2}\hat{W}_1\int_{-\infty}^{\infty}H_n^2\left(\frac{\sqrt{2}y}{\hat{W}_2}\right)\exp\left(\frac{-2y^2}{\hat{W}_2^2(\Delta)}\right)dy\,.$$

The remaining integrals are difficult to evaluate in their present form. The beam waists can be shifted by means of a thin cylindrical lens in the following manner. Let z_{10} and z_{20} be the untransformed drift coordinates at the lens and let $\hat{W}_1$ and $\hat{W}_2$ be the beam widths at the waists for the untransformed beam. For the transformed mode we assume the lens is at z_{10}' and z_{20}'. Then the beam widths at the waists are $\hat{W}_1'$ and $\hat{W}_2'$. At the lens the boundary conditions are

$$W_j'^2(z_{j0}',\hat{W}_j')=W_j^2(z_{j0},\hat{W}_j)\,,\quad j=1,2\,,$$

$$\frac{1}{R_j'(z_{j0},\hat{W}_j')}=\frac{1}{R_j(z_{j0},\hat{W}_j)}-\frac{1}{f_j}\,,\quad j=1,2\,,$$

where f_1 and f_2 are the focal lengths of the lens in a medium where the propagation vector is $\vec{k}$. We then find that

$$\hat{W}_j'=\Omega_j\hat{W}_j\,,\quad j=1,2$$

and

$$z_{j0}'=\frac{k_0^2\hat{W}_j^4\Omega_j}{4z_{j0}f_j}\left[(f_j-z_{j0})\left(\frac{2z_{j0}}{\hat{W}_j^2k_0}\right)^2-z_{j0}\right]\,,\quad j=1,2\,,$$

where

$$\Omega_j=\left(\frac{1+(2z_{j0}/k_0W_j^2)^2}{1+(k_0\hat{W}_j^2/2z_{j0}f_j)^2\langle(f_j-z_{j0})(2z_{j0}/\hat{W}_j^2k_0)^2-2z_{j0}\rangle^2}\right)^{1/2}.$$

We choose our cylindrical lens to have $f_1=\infty$, and f_2 such that when the lens is placed at $z_{20}=\Delta$ then $z_{20}'=0$. Then we have moved the transformed beam waist $\hat{W}_2'$ to the lens which is also the location of the waist $\hat{W}_1'$. This simplifies evaluation of the remaining integrals since the insertion of such a lens should not affect the power of the beam. The appropriate choice of f and Ω is given by

$$f=\Delta\left[1+\left(\frac{k_0\hat{W}_2^2}{2\Delta}\right)^2\right],$$

$$\Omega=\left[1+\left(\frac{2\Delta}{k_0\hat{W}_2^2}\right)^2\right].$$

The beam waist $\hat{W}_2'$ of the transformed beam is given by

$$\hat{W}_2'=\Omega\hat{W}_2\,.$$

Then

$$I_1=\Omega 2^{m+n-1}m!\,n!\,\pi\hat{W}_1\hat{W}_2\times\exp[2i\omega_0t-ik_0\Delta+i(2m+1)\phi_1'(0)+i(2n+1)\phi_2''(0)]$$

and

$$I_2=\Omega 2^{m+n-1}m!\,n!\,\pi\hat{W}_1\hat{W}_2$$

where $\phi_2''(0)$ is the transformed y-parameter phase. Since the amplitude of the beam is unchanged by the lens, we then have

$$P=\tfrac{1}{2}(\epsilon_0/\mu_0)^{1/2}u_0^2\left(\frac{\hat{W}_1}{W_1(0)}\,\frac{\hat{W}_2}{W_2(\Delta)}\right)\Omega 2^{m+n-1}m!\,n!\times\pi\hat{W}_1\hat{W}_2[\cos\langle 2\omega_0t-k_0\Delta+(2m+1)\phi_1'(0)+(2n+1)\phi_2''(0)\rangle+1]\,.$$

The lens cannot produce any discontinuities in the beam, so

$$W_2(\Delta)=W_2'(0)$$

or

$$W_2(\Delta)=\Omega\hat{W}_2\,.$$

Therefore

$$\langle P\rangle=(\epsilon_0/\mu_0)^{1/2}u_0^2 2^{m+n-2}m!\,n!\,\pi\hat{W}_1\hat{W}_2\,.$$

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[1]A. Maitland and M. H. Dunn, *Laser Physics* (North-Holland, Amsterdam, 1969). These authors derive the lowest-order three-parameter mode using the beam parameters of Eqs. (1). They briefly outline the form of the higher three-parameter modes but do not mention that the form they give is based on a different phase convention, namely the phase convention used in Ref. 2.

[2]H. Kogelnik and T. Li, Appl. Opt. 5, 1550 (1966).

[3]In neglecting these terms we are assuming that U_1' and U_2' are small. One finds that this is equivalent to assuming that the beam width is not a rapidly varying function of z.