# Cooperative radiation processes in two-level systems: Superfluorescence. II*

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(Received 4 November 1974; revised manuscript received 18 February 1975)

In this paper we discuss the quantum features of cooperative radiation processes in a two-level system taking into account non-Markovian effects. If the initial condition is spatially uniform, the many-mode atom-field master equation which has been derived and semiclassically discussed in a previous paper, reduces to the well-known single-mode model laser master equation. Hence, using a suitable projection operator, we derive a generalized non-Markovian master equation which turns out to be correct already in the first Born approximation. This master equation incorporates the effects both of stimulated emission and absorption and of spontaneous emission. When stimulated processes can be neglected, it reduces to a previously known Markovian superradiance master equation, whereas in general it describes non-Markovian oscillations of the radiated intensity. Finally we derive a linear system of equations for the probability of occupation of the Dicke states which generalizes a previously discussed system, taking into account the contribution of stimulated processes.

## I. INTRODUCTION

In a previous paper[1] (hereafter referred to as I), we have given a first-principles treatment of cooperative emission from a system of $N$ two-level atoms. Cooperative emission occurs when the radiation rate is proportional to $N^2$. In particular we have studied superfluorescence, which is defined as the cooperative emission from an atomic state initially excited with a zero macroscopic dipole moment, and a uniform population difference between the excited and the fundamental states. In I a general atom-field master equation (ME) was derived for the system atoms plus field internal to the active volume; in such an equation the atoms are described by means of collective dipole operators and the field internal to the active volume by damped quasimodes. The damping arises from the free propagation of the Maxwell field from the inside to the outside of the active volume. Inhomogeneous broadening appears simply via a time-dependent atom-field coupling constant. In I we discussed superfluorescence deriving from the atom-field ME in the semiclassical approximation, a pendulum equation for the Bloch angle. The initial state with no polarization corresponds in the semiclassical theory to the unstable equilibrium point of the pendulum. Therefore we have simulated a uniform initial "noise" polarization which starts the movement of the pendulum. This polarization corresponds to the initial value

$$\varphi(0) = (2N)^{-1/2} \tag{1.1}$$

of the Bloch angle measured from the unstable point.

In the present paper we want to give a fully quantum-mechanical treatment of superfluorescence. Our first aim is to justify the initial condition (1.1) for the pendulum equation. In Sec. II we study the atom-field ME for a pencil-shaped geometry. Rephrasing the semiclassical treatment of I, we show that with our initial condition all nonresonant modes can be neglected. In this way we reduce to a two-mode ME which, in the self-consistent-field approximation (SCFA) leads to the pendulum equation with the correct initial value (1.1) for the Bloch angle.

Furthermore in Sec. IV we show that such a two-mode ME can be suitably reduced by means of two independent one-mode models, in each of which the single mode interacts with $\frac{1}{2}N$ atoms with coupling constant $g_0\sqrt{2}$. Therefore in the following we restrict ourselves to consider the one-mode ME, which for $T_2^* = \infty$ coincides with the laser model, which is the starting point of the analysis of Ref. 2.

The main purpose of this paper is to derive a generalized master equation,[3] appropriate for treating both pure single-pulse superfluorescence and oscillatory non-Markovian superfluorescence. As shown in Ref. 2 pure superfluorescence occurs when the "cooperation time" $\tau_c$ is much larger than the maximum transit time of photons in the active volume $2L/c = K^{-1}$. In fact for $K\tau_c \gg 1$ the photons escape from the active volume so rapidly that they cannot react with the atomic system; in this condition the photons follow the motion of the atom adiabatically, so that non-Markovian effects are negligible. As a result one can eliminate the field adiabatically, obtaining a Markovian ME for the atomic system alone. Such a master equation has been generalized to the case of many modes in Ref. 4 and has been translated into a Fokker-Planck equation by Narducci *et al.*[5] and by Glauber and Haake.[6]

In Sec. VII, along the lines of a general theory for open systems developed by one of us[7] we introduce a suitable projection operator which yields a generalized non-Markovian ME. *The peculiarity of our projection operator lies in the fact that this equation is already exact in the first Born approximation.* For $K\tau_c \gg 1$ non-Markovian effects are negligible and one obtains the superradiance ME of Ref. 2. In contrast, for $K_0\tau_c \sim 1$, the reaction of the field on the atoms, i.e., stimulated processes, become relevant and one obtains non-Markovian oscillations in the radiated intensity as observed in Ref. 8.

Finally in Sec. VIII we deduce from the generalized non-Markovian ME the equations for the probabilities of the Dicke states. Such equations are closed only for $K\tau_c \gg 1$, in which case they coincide with the ones treated in Ref. 2d. For $K\tau_c \sim 1$ the stimulated terms make these equations nonclosed and one gets a hierarchy of equations which describes the effect of higher and higher-order atom-field correlations. However, it is shown that the hierarchy can be reasonably truncated in such a way that one is left with a closed linear system of equations, which gives back the exact equations for the mean values of the photon number and of the population difference. The truncated hierarchy can be solved exactly and its solution is equivalent to a resummation of infinite terms of the Born series one would obtain with the projection operator used in Ref. 2.

We mention finally that the non-Markovian ME can be immediately generalized to take into account pump and loss of the atoms and therefore it can give a unified treatment of cooperative emission and of laser phenomena.

## II. ATOM-FIELD MASTER EQUATION: REDUCTION TO RESONANT MODES

In I we derived from first principles an atom-field master equation (AFME) in which for a pencil-shaped active region only axial modes appear:

$$\frac{dW}{dt} = -\frac{i}{\hbar}[(H_F + H_I(t)), W(t)] + \Lambda_F W(t), \tag{2.1}$$

$$H_F = \hbar \sum_\alpha (c|\alpha| - \omega_0) A^\dagger(\alpha) A(\alpha),$$

$$H_I(t) = \frac{i\hbar}{v^{1/2}} \sum_\alpha (g_\alpha e^{-t/2T_2^*} A^\dagger(\alpha) R^-(\alpha) - \text{H.c.}),$$

$$\Lambda_F W(t) = K \sum_\alpha \{[A(\alpha), W(t)A^\dagger(\alpha)] + \text{H.c.}\},$$

$$K = \frac{c}{2L}, \quad \alpha = \frac{2\pi}{L} n, \quad n = 0, \pm 1, \ldots$$

where $W(t)$ is the density operator for the atom-field system, $L$ and $v$ are, respectively, the length and the volume of the active region, $T_2^*$ is the reciprocal of the inhomogeneous linewidth, and

$$g_\alpha = (\pi c |\alpha| \mu^2/\hbar)^{1/2}. \tag{2.2}$$

$A(\vec{\alpha})$ are modes of the internal field (i.e., the radiation field inside the active volume), obeying Bose commutation relations:

$$[A(\vec{\alpha}), A^\dagger(\vec{\alpha}')] = \delta_{\vec{\alpha},\vec{\alpha}'} \tag{2.3}$$

and the $R^-(\vec{\alpha})$ are collective dipole operators, obeying angular momentum commutation relations with the half total population inversion:

$$\begin{aligned} &[R^+(\vec{\alpha}), R^-(\vec{\alpha})] = 2R_3, \\ &[R_3, R^\pm(\vec{\alpha})] = \pm R^\pm(\vec{\alpha}). \end{aligned} \tag{2.4}$$

At the initial time $t=0$ no field is present and the sample is totally inverted. The atoms will begin to radiate mainly into the resonant modes $A(\pm\vec{k}_0)$, $|\vec{k}_0| = \omega_0/c$. Furthermore, the nonresonant modes will never get energy from the resonant ones, since the mode-mode coupling arises via the operator

$$R_3(\vec{\alpha} - \vec{\alpha}') = \tfrac{1}{2}[R^+(\vec{\alpha}), R^-(\vec{\alpha}')]$$

whose mean value vanishes when the situation is completely homogeneous, as is prescribed by our initial condition; i.e., we have in the mean

$$[R^+(\vec{\alpha}), R^-(\vec{\alpha}')] = 2R_3 \delta_{\vec{\alpha},\vec{\alpha}'}. \tag{2.5}$$

Then we can eliminate the nonresonant modes and write the following equation for the two resonant modes:

$$\frac{dW}{dt} = -\frac{i}{\hbar}[H_I(t), W(t)] + \Lambda_F W(t), \tag{2.6}$$

$$H_I(t) = (i\hbar/v^{1/2}) g_0 e^{-t/2T_2^*} \{A^\dagger(\vec{k}_0) R^-(\vec{k}_0) + A^\dagger(-\vec{k}_0) R^-(-\vec{k}_0) - \text{H.c.}\},$$

$$\Lambda_F W(t) = K\{[A(\vec{k}_0), W(t)A^\dagger(\vec{k}_0)] + [A(-\vec{k}_0), W(t)A^\dagger(-\vec{k}_0)] + \text{H.c.}\},$$

$$g_0 = g_{k_0}.$$

The argument used to eliminate nonresonant modes parallels the semiclassical treatment given in Sec. XI of I, in which the condition of homogeneous envelopes reduces the semiclassical equations to equations for the resonant modes only. Since our initial state has no photons and all atoms excited, the initial condition for the master equation (2.6) is

$$W(0) = |0, +\rangle\langle 0, +|, \tag{2.7}$$

with

$$A(\vec{k}_0)|0,+\rangle = A(-\vec{k}_0)|0,+\rangle = 0\,,$$
$$R_3|0,+\rangle = \tfrac{1}{2}N|0,+\rangle\,. \tag{2.8}$$

We give the following useful identity for $\Lambda_F$:

$$\mathrm{Tr}[A^{\dagger m}(\pm\vec{k}_0)A^n(\pm\vec{k}_0)e^{\Lambda_F t}W(t)] = e^{-K(m+n)t}\mathrm{Tr}[A^{\dagger m}(\pm\vec{k}_0)A^n(\pm\vec{k}_0)W(t)]\,. \tag{2.9}$$

## III. DERIVATION OF THE PENDULUM EQUATION

The discussion of superfluorescence given in I is based on the pendulum equation for the Bloch angle. We show in this section how such an equation and the correct initial value for the Bloch angle can be deduced from Eq. (2.6). From Eq. (2.6) we can derive an equation for the mean value of any observable $O$ as follows:

$$\langle\dot{O}\rangle(t) = -(i/\hbar)\langle[O, H_I(t)]\rangle(t) + \mathrm{Tr}[O\Lambda_F W(t)]\,. \tag{3.1}$$

By (2.9) we then easily obtain the equation

$$\frac{d}{dt}[\langle A^\dagger(\vec{k}_0)A(\vec{k}_0)\rangle(t) + \langle A^\dagger(-\vec{k}_0)A(-\vec{k}_0)\rangle(t) + \langle R_3\rangle(t)] = -2K[\langle A^\dagger(\vec{k}_0)A(\vec{k}_0)\rangle(t) + \langle A^\dagger(-\vec{k}_0\quad\vec{k}_0)\rangle(t)]\,. \tag{3.2}$$

Equation (3.2) is a simple energy balance between the variation of the total internal energy

$$\sum_{\vec{\alpha}=\vec{k}_0,-\vec{k}_0}[\langle A^\dagger(\vec{\alpha})A(\vec{\alpha})\rangle(t)] + \langle R_3\rangle(t)$$

and the energy output

$$2K\sum_{\vec{\alpha}=\vec{k}_0,-\vec{k}_0}[\langle A^\dagger(\vec{\alpha})A(\vec{\alpha})\rangle(t)]$$

both measured in units of $\hbar\omega_0$. Accordingly, the total number of photons radiated per unit time $I(t)$ is given by

$$I(t) = 2K\sum_{\vec{\alpha}=\vec{k}_0,-\vec{k}_0}\langle A^\dagger(\vec{\alpha})A(\vec{\alpha})\rangle(t)\,. \tag{3.3}$$

We stress that, because of the symmetry under the exchange $\vec{k}_0 \longleftrightarrow -\vec{k}_0$ in the equation of motion and in the initial condition, one has

$$\langle A^\dagger(\vec{k}_0)A(\vec{k}_0)\rangle(t) = \langle A^\dagger(-\vec{k}_0)A(-\vec{k}_0)\rangle(t)\,.$$

Furthermore, taking (2.5) into account one gets from (2.6) the constant-of-motion relation

$$\frac{d}{dt}\left[\sum_{\vec{\alpha}=\vec{k}_0,-\vec{k}_0}\langle R^+(\vec{\alpha})R^-(\vec{\alpha})\rangle(t) + \langle R_3^2\rangle(t) - \langle R_3\rangle(t)\right] = 0\,, \tag{3.4}$$

and the following equation for $\langle R_3\rangle(t)$:

$$\ddot{\langle R_3\rangle}(t) + [K + (1/2T_2^*)]\dot{\langle R_3\rangle}(t) = -\frac{2g_0^2}{v}e^{-t/T_2^*}\sum_{\vec{\alpha}=\vec{k}_0,-\vec{k}_0}\langle R^+(\vec{\alpha})R^-(\vec{\alpha})\rangle(t) - \frac{4g_0^2}{v}e^{-t/T_2^*}\left\langle\left(\sum_{\vec{\alpha}=\vec{k}_0,-\vec{k}_0}A^\dagger(\vec{\alpha})A(\vec{\alpha})\right)R_3\right\rangle(t)\,. \tag{3.5}$$

From (3.4) and the initial condition (2.7) we have

$$\sum_{\vec{\alpha}=\vec{k}_0,-\vec{k}_0}\langle R^+(\vec{\alpha})R^-(\vec{\alpha})\rangle(t) + \langle R_3^2\rangle(t) - \langle R_3\rangle(t) = \tfrac{1}{4}N^2\,, \tag{3.6}$$

where in the right-hand side we have neglected corrections of order $N$, with respect to $\frac{1}{4}N^2$.

Equations (3.2), (3.5), and (3.6) are the quantum analog of the semiclassical equations (11.6)–(11.8), from I, which we report here:

$$\frac{d}{dt}\{A_T^2 + R_3\} = -2KA_T^2\,, \tag{3.2′}$$

$$\ddot{R}_3 + \left(K + \frac{1}{2T_2^*}\right)\dot{R}_3 = -\frac{g_0^2}{v}e^{-t/T_2^*}[2R_T^2 + 4A_T^2R_3]\,, \tag{3.5′}$$

$$R_T^2 + R_3^2 = \tfrac{1}{4}N^2\,, \tag{3.6′}$$

where

$$A_T = [\langle A^\dagger(\vec{k}_0)A(\vec{k}_0)\rangle + \langle A^\dagger(-\vec{k}_0)A(-\vec{k}_0)\rangle]^{1/2}\,,$$
$$R_T = [\langle R^+(\vec{k}_0)R^-(\vec{k}_0)\rangle + \langle R^+(-\vec{k}_0)R^-(-\vec{k}_0)\rangle]^{1/2}\,.$$

Let us make the "neoclassical" approximation

$$\langle R_3^2\rangle = \langle R_3\rangle^2\,. \tag{3.7}$$

Let us introduce the "modified Bloch angle" as follows:

$$\langle R_3\rangle(t) - \tfrac{1}{2} = \tfrac{1}{2}N\cos\varphi(t)\,. \tag{3.8}$$

Hence substituting into Eq. (3.6) and neglecting $\frac{1}{4}$ with respect to $\frac{1}{4}N^2$ we obtain

$$\sum_{\vec{\alpha}=\vec{k}_0,-\vec{k}_0}\langle R^+(\vec{\alpha})R^-(\vec{\alpha})\rangle(t) = \tfrac{1}{4}N^2\sin^2\varphi(t)\,. \tag{3.8′}$$

Furthermore we perform the SCFA as follows:

$$\langle A^\dagger(\vec{\alpha})A(\vec{\alpha})R_3\rangle(t) \simeq \langle A^\dagger(\vec{\alpha})A(\vec{\alpha})\rangle(t)[\tfrac{1}{2}N\cos\varphi(t)]\,. \tag{3.9}$$

The "semiclassical approximations" (3.7) and (3.9) can be removed solving the equations for the atom-field distributions that we derive in Sec. VIII. Substituting Eqs. (3.8) into (3.2) and (3.5) we see that these equations are equivalent to

$$\sum_{\vec{\alpha}=\vec{k}_0,-\vec{k}_0}\langle A^\dagger(\vec{\alpha})A(\vec{\alpha})\rangle(t)=\frac{v}{4g_0^2}[\dot\varphi(t)]^2e^{t/T_2^*},\qquad(3.10)$$

$$\ddot\varphi(t)+\left(K+\frac{1}{T_2^*}\right)\dot\varphi(t)-\frac{g_0^2N}{v}e^{-t/T_2^*}\sin\varphi(t)=0\,.\qquad(3.11)$$

Equation (3.11) is the pendulum equation, whereas Eqs. (3.8) and (3.10) link the population inversion and the radiation intensity to the motion of the pendulum. In fact we have by (3.3)

$$I(t)=\frac{Kv}{2g_0^2}[\dot\varphi(t)]^2e^{t/T_2^*}\,.\qquad(3.12)$$

The initial condition of fully excited atoms and vacuum field gives

$$\varphi(0)=\arccos\frac{\frac{1}{2}N-\frac{1}{2}}{\frac{1}{2}N}\sim(2/N)^{1/2},\quad\dot\varphi(0)=0,\qquad(3.13)$$

which is the initial condition for Eq. (3.11), which we have used in I. The expression (3.12) for the radiation intensity coincides with that found in the semiclassical treatment.

## IV. REDUCTION TO THE ONE-MODE MODEL

The pendulum equation (3.11) follows also from a one-mode master equation, as it is shown in Ref. 2 in the case $K\tau_c\gg1$. Hence one would expect that in some way the two-mode problem can be reduced to a single-mode problem also at an operatorial level. At the moment we are not able to perform rigorously this reduction; however, we can show that the two-mode ME (2.6) can be reproduced by means of two independent one-mode models, in each of which the single mode interacts with $N'=\frac{1}{2}N$ atoms with coupling constant $g_0'=g_0\sqrt{2}$. Specifically, we consider two independent systems. System 1 is described by the field mode $A_1$ and the angular momentum operators $R_1^\pm R_3^{(1)}$; similarly for system 2. One has

$$[A_1,A_1^\dagger]=1,\qquad[R_1^+,R_1^-]=2R_3^{(1)},\quad[R_3^{(1)},R_1^\pm]=\pm R_1^\pm.\qquad(4.1)$$

The same commutation rules hold for $A_2$, $R_2^\pm$, $R_3^{(2)}$. Furthermore all operators of system 1 commute with all the operators of system 2.

Let $W^{(1)}(t)$ be the density operator for system 1; we assume the one-mode master equation (OMME)

$$\frac{dW^{(1)}}{dt}=-iL_{AF}^{(1)}W^{(1)}(t)+\Lambda_F^{(1)}W^{(1)}(t),\qquad(4.2)$$

$$L_{AF}^{(1)}W^{(1)}(t)=(1/\hbar)\,[H_{AF}^{(1)}(t),W^{(1)}(t)],$$
$$H_{AF}^{(1)}(t)=(i\hbar/v^{1/2})\,g_0'e^{-t/2T_2^*}(A_1^\dagger R_1^-+\text{H.c.}),\qquad(4.3)$$
$$\Lambda_F^{(1)}W^{(1)}(t)=K\{[A_1,W^{(1)}(t)A_1^\dagger]+\text{H.c.}\},\qquad(4.4)$$
$$g_0'=g_0\sqrt{2}\,.$$

The same OMME (4.2) is assumed for the density operator $W^{(2)}(t)$ of system 2. For $T_2^*=\infty$ Eq. (4.2) coincides with the one-mode-model laser ME, which is the starting point of the analysis of Ref. 2.

Now let us put

$$R^+(\vec{k}_0)=R_1^++iR_2^+,\quad R^+(-\vec{k}_0)=R_1^+-iR_2^+$$
$$A^\dagger(\vec{k}_0)=(A_1^\dagger+iA_2^\dagger)/\sqrt{2}\,,\quad A^\dagger(-\vec{k}_0)=(A_1^\dagger-iA_2^\dagger)/\sqrt{2}\,,\qquad(4.5)$$

and

$$R_3=R_3^{(1)}+R_3^{(2)}\,.\qquad(4.6)$$

From Eqs. (4.5) and (4.6) one gets the commutation rules (2.3) and (2.4); furthermore one gets

$$H_{AF}^{(1)}(t)+H_{AF}^{(2)}(t)=H_I(t),\qquad(4.7)$$

$$\Lambda_F^{(1)}+\Lambda_F^{(2)}=\Lambda_F\,.\qquad(4.8)$$

Then, if we assume that the density operator $W(t)$ for the full system 1+2 has the factorized form $W(t)=W^{(1)}\otimes W^{(2)}$ it is easy to see that $W(t)$ obeys the two-mode ME (2.6). One has furthermore from (4.5) and (4.6) that

$$A_1^\dagger A_1+A_2^\dagger A_2=A^\dagger(\vec{k}_0)A(\vec{k}_0)+A^\dagger(-\vec{k}_0)A(-\vec{k}_0)\qquad(4.9)$$

and

$$[R^+(\vec{k}_0),R^-(-\vec{k}_0)]=2[R_3^{(1)}-R_3^{(2)}]\,.\qquad(4.10)$$

Then we get relation (2.5), valid in conditions of uniformity, provided the state of the system is such that all moments of $R_3^{(1)}$ and $R_3^{(2)}$ are equal. In fact, in this sense, we assume that

$$R_3^{(1)}=R_3^{(2)}\,,\qquad R_3^{(1)}=R_3^{(2)}=\tfrac{1}{2}R_3\,.\qquad(4.11)$$

As one sees from Eqs. (4.5) and (4.6) we are decomposing $R^\pm(\pm\vec{k}_0)$ into the cos and sin polarization $R_1^\pm$ and $R_2^\pm$. The approximation we are performing consists in assuming that $R_1^\pm$ and $R_2^\pm$ obey commutation relations of two independent angular momenta. This approximation can be made plausible referring to the mean values of the commutators and using the uniformity conditions. However, we stress that the real motivation of our decoupling procedure lies, *a posteriori*, in the possibility of reconstructing the correct mean values equations starting from the two independent single-mode models. In fact, indicating the mean values by the same symbols which denote the operators, we get from Eq. (4.2) the equations

$$\frac{d}{dt}(A_1^\dagger A_1+R_3^{(1)})=-2KA_1^\dagger A_1,\qquad(4.12)$$

$$\ddot R_3^{(1)}+\left(K+\frac{1}{2T_2^*}\right)\dot R_3^{(1)}=-\frac{2g_0^2}{v}e^{-t/T_2^*}\times(2R_1^+R_1^-+4A_1^\dagger A_1R_3^{(1)}),\qquad(4.13)$$

$$R_1^+ R_1^- + (R_3^{(1)})^2 = \tfrac{1}{16} N^2 . \tag{4.14}$$

An identical set of equations holds for system 2; adding the corresponding equations and taking (4.5), (4.6), and (4.11) into account, we get the semiclassical equations (3.2′), (3.5′), and (3.6′).

Because of the symmetry in the exchange of $A_1$, $R_1$ with $A_2, R_2$ in the equations and in the initial condition, we can consider only one single-mode master equation, taking into account that by (4.9) $\langle A_1^\dagger A_1 \rangle$ is one-half of the total number of photons, so that from (3.3)

$$I(t) = 4K \langle A_1^\dagger A_1 \rangle (t) . \tag{4.15}$$

In the following we shall analyze the OMME (4.2) omitting systematically the label 1 in $W^{(1)}$, $A_1$, etc.

We shall indicate by $W_F$ and $W_A$ the density operators for the field and the atoms, respectively,

$$W_A(t) = \mathrm{Tr}_F W(t), \quad W_F(t) = \mathrm{Tr}_A W(t) \tag{4.16}$$

where $\mathrm{Tr}_F$ means partial trace on the field Hilbert space, etc. Since our initial state has no photons and all atoms excited, the initial condition for Eq. (4.2) is:

$$\begin{aligned} &W(0) = W_F(0) \otimes W_A(0) , \\ &W_F(0) = |0\rangle\langle 0|, \quad W_A(0) = |+\rangle\langle +|, \end{aligned} \tag{4.17}$$

where, by (4.11)

$$R_3 |+\rangle = \tfrac{1}{2} N' |+\rangle, \quad N' = \tfrac{1}{2} N . \tag{4.18}$$

Finally, we rephrase identity (2.9) for the one-mode model:

$$\mathrm{Tr}(A^{\dagger m} A^n e^{\Lambda_F t} W) = e^{-K(m+n)t} \mathrm{Tr}(A^{\dagger m} A^n W) . \tag{4.19}$$

## V. PROJECTION TECHNIQUE

The OMME (4.2) is still too complicated to be handled directly: in fact one gets results from it easily only in a semiclassical approximation. It is convenient to derive from (4.2), which is a coupled equation for the system atoms plus field, a closed equation for one of the two subsystems. Since the field has the smallest relaxation time $K^{-1}$, it can in some sense be considered as a "bath" for the atomic system, so that it is easier to derive from (4.2) a closed equation for the atoms.

The main tool to perform this derivation is provided by the so-called projection technique.[3] This is essentially a skillful way to rewrite the time evolution equation, which on the one hand makes the memory effects manifest, and on the other hand provides in quite a natural way a systematic approximation scheme (the Born approximation scheme). Such a technique is based on the subdivision of the density operator $W(t)$ into two parts by means of a projection operator $P$, i.e.,

$$\begin{aligned} &W(t) = \Phi(t) + \Gamma(t) , \\ &\Phi(t) = P W(t), \quad \Gamma(t) = (1-P) W(t), \quad P^2 = P . \end{aligned} \tag{5.1}$$

The meaning of the subdivision is the following: If one considers observables $O$ of the system such that

$$\mathrm{Tr}[OW(t)] = \mathrm{Tr}[O\Phi(t)], \tag{5.2}$$

one can systematically neglect the part $\Gamma(t)$ and concentrate on $\Phi(t)$.

Let us assume that

$$P\Lambda_F = \Lambda_F P, \quad P L_{AF} P = 0, \tag{5.3}$$

then one can prove that[9] $\Phi(t)$ obeys the so-called generalized master equation (GME)

$$\begin{aligned} &\frac{d\Phi(t)}{dt} = \Lambda_F \Phi(t) - \int_0^t \mathfrak{F}(t, s) \Phi(s)\, ds , \\ &\mathfrak{F}(t, s) = P L_{AF}(t)\, g(t, s) L_{AF}(s) P , \end{aligned} \tag{5.4}$$

$$g(t, s) = \zeta \exp\left[ -i \int_s^t dt'\, Q(L_{AF}(t') + i\Lambda_F) Q \right], \tag{5.5}$$

with

$$Q = 1 - P, \tag{5.6}$$

provided the initial density operator $W(0)$ satisfies the condition

$$W(0) = \Phi(0) \Longleftrightarrow \Gamma(0) = 0 . \tag{5.7}$$

In Eq. (5.5) $\zeta$ is a time-ordering operator, which orders operators of greater time arguments to the left; when $(T_2^*)^{-1} = 0$, $L_{AF}(t)$ is constant in time and $\zeta$ is unnecessary. Equation (5.4) is explicitly non-Markovian, since the second term in its right-hand side is a typical "memory" term, the memory being linked to the time behavior of the kernel $\mathfrak{F}(t, s)$. The Born approximation amounts to retain only the zero-order term in the expansion of $g(t, s)$ in power series of $L_{AF}$. In this way Eqs. (5.4) and (5.5) become

$$\frac{d\Phi(t)}{dt} = \Lambda_F \Phi(t) - \int_0^t ds\, \mathfrak{F}(t-s) \Phi(s)\, ds , \tag{5.4′}$$

$$\mathfrak{F}(t) = P L_{AF}(t)\, e^{\Lambda_F t} L_{AF}(s) P . \tag{5.5′}$$

## VI. THE SUPERRADIANCE MASTER EQUATION

In Reference 2 it is considered the case that the rate $K$, at which photons escape from the active volume $v$, is much larger than the rate $\tau_c^{-1} = g_0 (N/v)^{1/2}$ at which the photons and the atomic system exchange energy.[10]

Condition $\tau_c^{-1} \ll K$ implies that, to the effect of the motion of the atoms, the field inside the active

volume remains practically in the vacuum state always. More specifically, the photons escape too fast from the active volume to give rise to an appreciable reaction on the atoms, i.e., stimulated emission and absorption do not occur. In such conditions it is reasonable to eliminate the motion of the photons by means of the projection operator

$$P_F X = |0\rangle\langle 0|\,\mathrm{Tr}_F X\,. \tag{6.1}$$

Projector operators of this type have been first used by Argyres and Kelley.[11]

Conditions (5.3) hold with $P = P_F$. The initial condition (4.17) satisfies Eq. (5.7) with $\Phi(0) = P_F W(0)$. Since the field inside remains "practically" in the vacuum state, the Born approximation (5.4′) with projection $P_F$ is well justified. One gets the following closed equation for the density operator $W_A(t)$ of the atomic system alone:

$$W_A(t) = \frac{g_0'^2}{v}\int_0^t ds\, e^{-K(t-s)}\{[R^-, W_A(s)R^+] + \mathrm{H.c.}\}, \tag{6.2}$$

where, as in Ref. 2, we have put $(T_2^*)^{-1} = 0$. On the other hand in the condition $\tau_c^{-1} \ll K$ the memory involves only the initial time interval of order $K^{-1}$, so that for $t \gg K^{-1}$ (6.2) reduces to the Markovian master equation

$$W_A(t) = (g_0'^2/Kv)\{[R^-, W_A(t)R^+] + \mathrm{H.c.}\}\,. \tag{6.3}$$

Equation (6.3) is usually called the "superradiance master equation". Therefore no actually observable non-Markovian effect arises from the GME (6.2); in other words, the elimination of the motion of the photons made in Ref. 2 is an adiabatic elimination. In fact, a formula is derived in Ref. 2 that reduces the calculation of the mean values of photon operators to the calculation of mean values of atomic operators; e.g., for the mean photon number, one has

$$\langle A^\dagger A\rangle(t) = (g_0'^2/K^2 v)\langle R^+ R^-\rangle(t), \tag{6.4}$$

which shows explicitly that the field follows the atomic system adiabatically.

In the experiment of Ref. 8 the condition $\tau_c^{-1} \ll K$ is badly violated, so that Eq. (6.2) cannot account for the results of such an experiment. We want therefore to derive a new GME incorporating the stimulated effects (or, equivalently, the non-Markovian effects) which become relevant when condition $\tau_c^{-1} \ll K$ is violated. We shall call such an equation the "generalized superfluorescence master equation" (GME).

## VII. GENERALIZED MASTER EQUATION FOR SUPERFLUORESCENCE

Since when $\tau_c^{-1} \gtrsim K$ the field does not remain in the vacuum state, the Born approximation with projection (6.1) becomes very poor and one should take into account infinite terms of the Born series (5.8), which is presumably an exceedingly difficult task. This drawback is related to the fact that for $\tau_c^{-1} \gtrsim K$ the photons no longer follow the atoms adiabatically, so that the field variables become relevant and we must use a projection operator which does not eliminate the field so drastically as (6.1). Projections of this type have been introduced in a general theory of open systems recently developed by one of us.[7]

Let $|n, r, m\rangle$ $(n = 0, 1, \ldots; r \geq 0, r = \frac{1}{2}N', \frac{1}{2}N' - 1, \ldots; m = -r, -r+1, \ldots, r-1, r)$ be an orthonormal basis such that

$$\begin{aligned} A^\dagger A\,|n, r, m\rangle &= n\,|n, r, m\rangle\,, \\ R^2\,|n, r, m\rangle &= r(r+1)\,|n, r, m\rangle\,, \\ R_3\,|n, r, m\rangle &= m\,|n, r, m\rangle\,, \end{aligned} \tag{7.1}$$

where

$$R^2 = R^+R^- + R_3^2 - R_3 \tag{7.2}$$

is the square of the "total angular momentum."

For the sake of simplicity we have neglected an unessential degeneration index to label the states $|r, m\rangle$. We consider the projection $P_1$ defined as follows

$$P_1 X = \sum_{\substack{n,n';\\ (n-n')/2\,=\,\text{integer}}}{}' \sum_{r,m,r',m'} \langle n, r, m|X|n', r', m'\rangle \times |n, r, m\rangle\langle n', r', m'|, \tag{7.3}$$

where the first sum is restricted to the couples $n, n'$ such that $\frac{1}{2}(n - n')$ is an integer. Conditions (5.3) hold for $P = P_1$; the initial condition (4.17), i.e.,

$$W(0) = |0, \tfrac{1}{2}N, \tfrac{1}{2}N\rangle\langle 0, \tfrac{1}{2}N, \tfrac{1}{2}N| \tag{7.4}$$

satisfies Eq. (5.7) with $\Phi(0) = P_1 W(0)$. Furthermore $P_1$ has the property

$$\mathrm{Tr}[O(P_1 W)] = \mathrm{Tr}[(P_1 O)W]\,. \tag{7.5}$$

The reason why we consider such a projection is that on the one hand it leaves unaltered the observables $O_A$ of the atomic system, i.e.;

$$P_1 O_A = O_A\,, \tag{7.6}$$

and on the other hand it has the remarkable feature that the GME in the Born approximation (5.4′) with projection $P_1$ is *exact*. Specifically one gets:

$$\dot{\Phi}(t)=\Lambda_F\Phi(t)-\frac{g_0'^2}{v}\int_0^t ds\, e^{-(t+s)/2T_2^*}\left[(AR^+ +A^\dagger R^-), e^{\Lambda_F(t-s)}[(AR^+ +A^\dagger R^-),\Phi(s)]\right], \quad \Phi(t)=P_1W(t). \tag{7.7}$$

Equation (7.6) is the generalized master equation for superfluorescence. A simple derivation of Eq. (7.7) is given in Appendix A.

Several remarks are in order: (i). Let us consider the projection $P_2$ defined as follows:

$$P_2X=\sum_{n,n'}\sum_{\substack{r,m,r',m';\\ (m-m')/2=\text{integer}}}{}' \langle n,r,m|X|n',r',m'\rangle \times|n,r,m\rangle\langle n',r',m'|. \tag{7.8}$$

Conversely to $P_1$, $P_2$ leaves invariant the observable $O_F$ of the field

$$P_2O_F=O_F. \tag{7.9}$$

Also $P_2$ has the properties (5.3) and (7.5). One has that $P_2$ yields the same GME (7.7). Then if

$$W(0)=P_1W(0)=P_2W(0) \tag{7.10}$$

Eq. (7.7) can be used to calculate the mean values of all observables $O_A$ of the atomic system alone and of all observables $O_F$ of the field alone. The initial condition (7.4) fulfills condition (7.10). In this case it turns out that observables $O_A$ with zero diagonal part in the Dicke basis $|r,m\rangle$ have zero expectation value at all times. The same occurs to the observables $O_F$ with zero diagonal part in the photon number basis $|n\rangle$. This happens, e.g., for $R^-,A$ and it means that the phase of polarization and field is completely random at all times, as it has been anticipated in I.

(ii). In Eq. (7.7) one has a clean separation between the terms which depend on the relative phase between the polarization and the field (i.e., $[AR^+, \times e^{\Lambda_F(t-s)}[AR^+,\Phi(s)]]$ and its Hermitian conjugate) and the terms which do not (i.e., $[AR^+, e^{\Lambda_F(t-s)} \times[A^\dagger R^-,\Phi(s)]]$ and its Hermitian conjugate). We remark that the Zwanzig projection which picks out the diagonal part in the basis $|n,r,m\rangle$, i.e.,

$$P_zX=\sum_{n,r,m}\langle n,r,m|X|n,r,m\rangle|n,r,m\rangle\langle n,r,m| \tag{7.11}$$

yields in the Born approximation an equation like (7.7) but *without* the relative phase-dependent terms.

(iii). Equation (6.2) is immediately derived from Eq. (7.7) for $(T_2^*)^{-1}=0$ taking the partial trace $\mathrm{Tr}_F$ of Eq. (7.7) and putting $\Phi(t)=|0\rangle\langle 0|\otimes W_A(t)$. This confirms that the condition considered in Ref. 2 is that of atoms emitting in the vacuum of photons.

(iv). For $\tau_c^{-1}\sim K$ Eq. (7.7) cannot be reduced to a Markovian master equation. In fact, in this condition one has typical non-Markovian oscillations in the atomic decay and in the radiation intensity, as we have seen in I.

(v). Equation (7.7) allows the elimination of the atomic variables as well as of field variables. Then if one adds to the Liouvillian of Eq. (4.2) a term $\Lambda_A$ describing pumping and loss of the atoms, Eq. (7.7) can be generalized in a straightforward way yielding an equation that can treat in a unified way both cooperative emission and laser phenomena. Some results for the laser will be given in Ref. 12.

(vi). Equation (7.6) is still a coupled equation for the atom-field system. In order to obtain a closed evolution equation for the atomic system alone, one must do further work, which will be performed in Sec. VIII. On the other hand, this work is much easier than summing up infinite terms of the Born series of the GME with projection (6.1).

(vii). To give a quantitative comparison between Eqs. (6.3) and (7.7), let us consider the equations for the mean value of $R_3$. We have from Eq. (7.7) with (4.19):

$$\langle\dot{R}_3\rangle(t)=-\frac{g_0'^2}{v}\int_0^t ds\, e^{-(t+s)/2T_2^*}e^{-K(t-s)} \times[2\langle R^+R^-\rangle(s)+4\langle A^\dagger AR_3\rangle(s)], \tag{7.12}$$

or equivalently

$$\langle\ddot{R}_3\rangle(t)=-\left(K+\frac{1}{2T_2^*}\right)\langle\dot{R}_3\rangle(t) -(g_0'^2/v)e^{-t/T_2^*}[2\langle R^+R^-\rangle(t)+4\langle A^\dagger AR_3\rangle(t)]. \tag{7.12'}$$

From Eq. (6.3) we have, on the other hand,

$$\langle\dot{R}_3\rangle(t)=-(2g_0'^2/Kv)\langle R^+R^-\rangle(t). \tag{7.13}$$

Clearly Eq. (7.13) exhibits only the cooperative spontaneous emission term $\langle R^+R^-\rangle(t)$, whereas it lacks the stimulated emission and absorption term $\langle A^\dagger AR_3\rangle(t)$. Such a term, together with $\langle\ddot{R}_3\rangle(t)$, is responsible for the non-Markovian ringing in the decay which occur for $\tau_c^{-1}\sim K$. In fact, from Eq. (7.12') one derives the pendulum equation in the same way as it has been derived from Eq. (3.5). On the other hand, replacing Eq. (7.12') by Eq. (7.13) one gets the equation of the overdamped pendulum

$$\dot{\varphi}(t) = (1/\tau_R)\sin\varphi(t),$$
$$\tau_R = K\tau_c^2 = Kv/g_0'^2 N', \tag{7.14}$$

where $\tau_R$ is the time duration of the hyperbolic secant pulses which arise from (7.14).

The relative order of magnitude of the two terms on the right-hand side of Eq. (7.12′) can be evaluated using the SCFA for $\langle A^\dagger A R_3\rangle(t)$ and (6.4) for $\langle A^\dagger A\rangle(t)$. In this way one obtains

$$\frac{\langle A^\dagger A R_3\rangle}{\langle R^+ R^-\rangle} = \frac{1}{K^2\tau_c^2},$$

so that stimulated effects are negligible only if $K\tau_c \gg 1$.

## VIII. EQUATIONS FOR THE PROBABILITIES OF THE DICKE STATES

In order to arrive to a closed equation for the atomic system, let us consider the equations for the probabilities of occupation of the Dicke states $|r, m\rangle$:

$$p(r, m, t) = \sum_n \langle n, r, m|\,\Phi(t)\,|n, r, m\rangle\ , \tag{8.1}$$

following from Eq. (7.7). Since the constant of motion $R^2$ has the definite value $\frac{1}{2}N'(\frac{1}{2}N'+1)$, the motion involves only the subspace spanned by the elements $|n, \frac{1}{2}N', m\rangle$, so that we can drop the index $r$ in the following.

By (4.19) and

$$R^\pm|n, m\rangle = [(\tfrac{1}{2}N' \mp m)(\tfrac{1}{2}N' \pm m + 1)]^{1/2}|n, m\pm 1\rangle\ , \tag{8.2}$$

the equation for $p(m, t)$ arising from (7.7) is

$$\dot{p}(m,t) = -2\frac{g_0'^2}{v}\int_0^t ds\, e^{-(t+s)/2T_2^*}\, e^{-K(t-s)}\{g(m)p(m,s) - g(m+1)p(m+1,s) + [g(m)+g(m+1)]\mathfrak{N}(m,s)$$
$$- g(m+1)\mathfrak{N}(m+1,s) - g(m)\mathfrak{N}(m-1,s) + g^{1/2}(m)g^{1/2}(m-1)\mathfrak{L}(m,s)$$
$$- 2g^{1/2}(m)g^{1/2}(m+1)\mathfrak{L}(m+1,s) + g^{1/2}(m+1)g^{1/2}(m+2)\mathfrak{L}(m+2,s)\}\ , \tag{8.3}$$

where

$$g(m) = \begin{cases} (\frac{1}{2}N'+m)(\frac{1}{2}N'-m+1) & \text{for } -\frac{1}{2}N' \leq m \leq \frac{1}{2}N', \\ 0 \quad \text{otherwise}\ , \end{cases} \tag{8.4}$$

and

$$\mathfrak{N}(m,t) = \sum_n n\langle n, m|\,\Phi(t)\,|n, m\rangle = \langle m|\,\mathrm{Tr}_F a^\dagger a\Phi(t)|m\rangle\ , \tag{8.5}$$

$$\mathfrak{L}(m,t) = \mathrm{Re}\bar{\mathfrak{L}}(m,t)\ ,$$
$$\bar{\mathfrak{L}}(m,t) = \sum_n [(n+1)(n+2)]^{1/2}\langle n+2, m-2|\,\Phi(t)\,|n, m\rangle\ , \tag{8.6}$$

$\mathfrak{N}(m,t)$ and $\bar{\mathfrak{L}}(m,t)$ are atom-field distribution functions, meaning that (i) For any diagonal atomic observable $O_d$, with

$$\langle m|O_d|\,m'\rangle = (O_d)_m\delta_{m,m'} \tag{8.7}$$

one has

$$\langle A^\dagger A O_d\rangle(t) = \sum_m (O_d)_m\mathfrak{N}(m,t)\ , \tag{8.8}$$

e.g., the first moment of $\mathfrak{N}(m,t)$ gives the correlation between photon number and population inversion; (ii) For any atomic operator $\bar{O}$ (as, e.g., $R^+R^+$) such that

$$\langle m|\,\bar{O}\,|\,m'\rangle = \bar{O}_m\delta_{m',m-2}\ , \tag{8.9}$$

one has

$$\langle AA\bar{O}\rangle(t) = \sum_m \bar{O}_m\bar{\mathfrak{L}}(m,t)\ . \tag{8.10}$$

In particular

$$\langle AAR^+R^+\rangle(t) = \sum_m g^{1/2}(m)g^{1/2}(m-1)\bar{\mathfrak{L}}(m,t).$$

This quantity depends on the relative phase between field and atoms, and would be zero in the SCFA. In fact, when $W(0)$ is diagonal, $W_A(t)=\mathrm{Tr}_F W(t)$ and $W_F(t)=\mathrm{Tr}_A W(t)$ are diagonal even if $W(t)$ is not diagonal.

Equation (8.3) is not closed in $p(m,t)$ and must therefore be coupled to the equations for $\mathfrak{N}(m,t)$ and $\mathfrak{L}(m,t)$. We have from (7.7), (4.19), and (8.2)

$$\dot{\mathfrak{N}}(m,t)=-2K\mathfrak{N}(m,t)-\frac{2g_0'^2}{v}\int_0^t ds\, e^{-(t+s)/2T_2^*}$$
$$\times\{e^{-K(t-s)}\{-g(m+1)[p(m+1,s)+\mathfrak{N}(m+1,s)-\mathfrak{N}(m,s)]$$
$$-g^{1/2}(m)g^{1/2}(m+1)\mathfrak{L}(m+1,s)+g^{1/2}(m+1)g^{1/2}(m+2)\mathfrak{L}(m+2,s)\}$$
$$+e^{-3K(t-s)}\mathfrak{F}_1(\mathfrak{L},\mathfrak{N},\mathfrak{N}_2,\ldots,s)\},\tag{8.11}$$

where $\mathfrak{F}_1$ is a suitable functional of $\mathfrak{L}(m,t)$, $\mathfrak{N}(m,t)$, and other higher-order distribution functions as, for example,

$$\mathfrak{N}_2(m,t)=\sum_n n^2\langle n,m|\,\Phi(t)\,|n,m\rangle\equiv\langle m|\,\mathrm{Tr}(a^+)^2a^2\Phi\,|m\rangle\ .$$

Clearly proceeding in this way we get a hierarchy of equations, involving atom-field distribution functions of higher and higher order. To disentangle this situation one can introduce different approximations. The simplest one is to retain only the terms with $p$ in Eqs. (8.3) and (8.11). This is possible only for $\tau_c^{-1}\ll K$, in which case the Markov approximation also holds. In such a way (8.3) becomes a closed equation for the $p$'s, which for $T_2^*=\infty$ has been both analytically and numerically solved in Ref. 2. Furthermore, as is shown in Appendix B, Eq. (8.11) leads to the adiabatic formula

$$\langle A^\dagger O_d A\rangle(t)=e^{-t/T_2^*}(g_0'^2/K^2v)\langle R^+O_dR^-\rangle(t)\ ,\tag{8.12}$$

where $O_d$ is an atomic observable of type (8.7). Hence $A$ and $A^\dagger$ can be replaced, respectively, by $R^-$ and $R^+$ provided the $A$'s are on the right-hand side of the atomic operators and the $A^\dagger$'s on the left-hand side. This is a generalized normal ordering prescription for atom-field operators.

A more general approximation is achieved by neglecting in Eqs. (8.3) and (8.11) the effect of off-diagonal atom-field correlations described by $\mathfrak{L}$ and the rapidly decaying terms $e^{-3K(t-s)}$. In this way one obtains

$$\dot{p}(m,t)=-\frac{2g_0'^2}{v}\int_0^t e^{-K(t-s)}e^{-(t+s)/2T_2^*}\{g(m)p(m,s)-g(m+1)p(m+1,s)+[g(m)+g(m+1)]\mathfrak{N}(m,s)$$
$$-g(m+1)\mathfrak{N}(m+1,s)-g(m)\mathfrak{N}(m-1,s)\}\tag{8.13}$$

$$\dot{\mathfrak{N}}(m,t)=-2K\mathfrak{N}(m,t)+\frac{2g_0'^2}{v}\int_0^t ds\, e^{-K(t-s)}e^{-(t+s)/2T_2^*}\{g(m+1)[p(m+1,s)+\mathfrak{N}(m+1,s)-\mathfrak{N}(m,s)]\}\ .\tag{8.14}$$

Equations (8.13) and (8.14) are a closed system of linear equations with a very remarkable feature: They lead without approximation to the nonlinear nonclosed system of equations (7.12) and (4.12) for the mean values of $A^\dagger A$ and $R_3$. This can be easily verified observing that from the above definitions:

$$\langle R_3'\rangle(t)=\sum_m m'p(m,t)\ ,$$
$$\langle A^\dagger A\rangle(t)=\sum_m \mathfrak{N}(m,t)\ ,\tag{8.15}$$
$$\langle A^\dagger AR_3\rangle(t)=\sum_m m\mathfrak{N}(m,t)\ ,$$

and using the following algebraic identities:

$$g(m)-g(m+1)=2m,$$

$$\sum_m g(m+1)F(m+1)=\sum_m g(m)F(m)\ ,$$

which hold whatever is $F(m)$.

Hence our truncation of the hierarchy of equations for the distribution function gives back the *exact equations* for the mean values. On the other hand, we have shown that Eqs. (7.12) and (4.12), in the semiclassical limit, reduce to the pendulum equations for the Bloch angle describing both Markovian and non-Markovian superfluorescence. Hence Eqs. (8.13) and (8.14) generalize the semiclassical description, taking fully into account the effect of quantum fluctuations $\langle R_3^2\rangle-\langle R_3\rangle^2$ and of atom-field correlations $\langle A^\dagger AR_3\rangle-\langle A^\dagger A\rangle\langle R_3\rangle$ whose explicit expression in terms of $p$ and $\mathfrak{N}$ is immediately obtained by (8.15). The analytic solution by Laplace transform of Eqs. (8.13) and

(8.14) for $p$ and $\mathfrak{N}$ is described in Appendix C for $T_2^* = \infty$. The explicit form of this solution has, however, a very complicated structure so that a numerical evaluation is needed. Here we only stress that the coupled equations (8.13) and (8.14) can be rephrased into a closed equation for $p(m,t)$ with a kernel containing the coupling constant $g_0'^2$ to all orders. Hence this equation is equivalent to a resummation of infinite contributions to the Born series with the Argyres and Kelley's projection (6.1) used in Ref. 2.

We would like to conclude by showing how our theory allows a calculation of all moments of the photon distribution. In fact let us define the normally ordered conditional moments which suitably generalize the function $\mathfrak{N}(m,t)$:

$$\mathfrak{N}_l(m,t) = \langle m | \operatorname{Tr}_F (A^\dagger)^l A^l \Phi(t) | m \rangle \ . \tag{8.16}$$

This is the mean value of $A^{\dagger l} A^l$ assuming that the atoms are in a Dicke state $|m\rangle$. In particular one has

$$\langle A^{\dagger l} A^l \rangle_t = \sum_m \mathfrak{N}_l(m,t) \ . \tag{8.17}$$

The function $\mathfrak{N}(m,t)$ is obtained specializing to $l=1$. It is easy to show that the following equation holds:

$$\dot{\mathfrak{N}}_l(m,t) = -2lK\mathfrak{N}_l(m,t) + \frac{2lg_0'^2}{v} \int_0^t ds\, e^{-(t+s)/2T_2^*} e^{-(2l-1)k(t-s)} g(m+1) \times \{ l\mathfrak{N}_{l-1}(m+1,s) + \mathfrak{N}_l(m+1,s) - \mathfrak{N}_l(m,s) \} \ , \tag{8.18}$$

where $\mathfrak{N}_0 \equiv p(m,t)$.[13]

This equation reduces to (8.14) for $l=1$ and allows to calculate $\mathfrak{N}_l(m,t)$ from $\mathfrak{N}_{l-1}(m,t)$. In this way we can calculate successively higher and higher moments from the knowledge of the first moment.

In conclusion we have explicitly shown how to obtain analytical closed solutions for atomic fluctuations, atom-field correlations, and photon statistics during superfluorescent non-Markovian decay. In particular, the problem of the validity of the semiclassical assumptions is now reduced to the numerical evaluation of these solutions, which will be presented elsewhere.

ACKNOWLEDGMENTS

One of us (R.B.) is grateful to Professor H. Haken and Professor W. Weidlich for their hospitality at Stuttgart University, where some results of this paper were worked out under the stimulation of several discussions with Professor H. Haken, Professor W. Weidlich, Professor F. Haake, and Dr. P. Schwendimann. We also acknowledge helpful discussions with Professor F. T. Arecchi, Professor P. Caldirola, and Dr. M. Gronchi.

## APPENDIX A

The formal solution of Eq. (4.2) can be written as follows:

$$W(t) = U(t) W(0) \ ,$$

where

$$U(t) = e^{\Lambda_F t} - i \int_0^t ds\, e^{\Lambda_F (t-s)} L_{AF}(s) U(s) \ . \tag{A1}$$

Iterating this equation once one gets

$$U(t) = e^{\Lambda_F t} - i \int_0^t ds\, e^{\Lambda_F (t-s)} L_{AF}(s)\, e^{\Lambda_F s} - \int_0^t ds_1 \int_0^{s_1} ds_2\, e^{\Lambda_F (t-s_1)} L_{AF}(s_1) \times e^{\Lambda_F (t-s_2)} L_{AF}(s_2) U(s_2) \ . \tag{A2}$$

Substituting Eq. (A2) into (A1) and differentiating one gets

$$\frac{dW(t)}{dt} = \Lambda_F W(t) - iL_{AF}(t) e^{\Lambda_F t} W(0) - \int_0^t ds L_{AF}(t)\, e^{\Lambda_F (t-s)} L_{AF}(s) W(s) \ . \tag{A3}$$

Then, taking into account that $P_1$ satisfies conditions (5.3) and (5.7) and that it commutes with the kernel of Eq. (A3), one has that $\Phi(t) - P_1 W(t)$ obeys Eq. (7.7).

## APPENDIX B

Retaining only the $p$ terms in Eq. (8.11) and defining $\tau = t - s$ we obtain

$$\dot{\mathfrak{N}}(m,t) = -2K\mathfrak{N}(m,t) + \frac{2g_0'^2}{v} \int_0^t d\tau\, e^{-(K+1/2T_2^*)\tau} e^{-(t-\tau)/T_2^*} g(m+1) p(m+1, t-\tau) \ . \tag{B1}$$

Assuming $K\tau_c >> 1$ and $K >> 1/2T_2^*$ we can perform the Markov approximation on the integral and the adiabatic approximation, which amounts to neglecting $\dot{\mathfrak{R}}$ with respect to $2K\mathfrak{R}$. In this way we obtain

$$\mathfrak{R}(m,t) \simeq e^{-t/T_2^*}(g_0'^2/K^2)\, g(m+1)\, p(m+1,t)\ . \tag{B2}$$

Let us now calculate $\langle R^+ O_d R^- \rangle$

$$\begin{aligned}\langle R^+ O_d R^- \rangle &= \sum_m \langle m|\, R^+ O_d R^- \Phi_A |\, m\rangle \\ &= \sum_{m,m'} \langle m|\, R^+ O_d R^- |\, m'\rangle \langle m'|\, \Phi_A |\, m\rangle \\ &= \sum_m g(m)(O_d)_{m-1}\, p(m,t) \\ &= \sum_m g(m+1)(O_d)_m\, p(m+1,t) \\ &= e^{t/T_2^*}\frac{K^2}{g_0'^2}\sum_m \mathfrak{R}(m,t)(O_d)_m\end{aligned}$$

with $\Phi_A = \mathrm{Tr}_F \Phi$. Using Eq. (8.8) one obtains immediately (8.12).

## APPENDIX C

Let $\tilde{p}(m,z)$, $\tilde{\mathfrak{R}}_l(m,z)$ be the Laplace transforms of $p(m,t)$, $\mathfrak{R}_l(m,z)$, respectively. Then for $T_2^* = \infty$, taking into account the initial condition (4.17),

$$\mathfrak{R}_l(m,0) = 0\ , \tag{C1}$$

one has by a Laplace transform of Eq. (8.17) that

$$\tilde{\mathfrak{R}}_l(m,z) = \sum_{s=m+1}^{N/2} c_l(m,s,z)\tilde{\mathfrak{R}}_{l-1}(s,z)\ , \tag{C2}$$

with

$$c_l(m,s,z) = l \prod_{i=m+1}^{s} \frac{KI\, g(i)}{(z+2lK)[z+(2l-1)K]+lKIg(i)}\ , \tag{C3}$$

where we have put as in Ref. 2

$$I = 2g_0'^2/Kv\ . \tag{C4}$$

In particular, for $l=1$ we have

$$\tilde{\mathfrak{R}}(m,z) = \sum_{s=m+1}^{N/2} c_1(m,s,z)\, p(s,z)\ . \tag{C5}$$

Then substituting Eq. (C5) into the Laplace tranform of Eq. (8.13) we get the equations for $\tilde{p}(m,z)$:

$$A(m,z)\tilde{p}(m,z) = (z+K)p(m,0) + \sum_{s=m+1}^{N/2} d(m,s,z)\tilde{p}(s,z)\ , \tag{C6}$$

where

$$A(m,z) = z(z+K) + KI\, g(m)[1 - c_1(m-1,m,z)] \tag{C7}$$

and

$$d(m,s,z) = -KI\{-g(m+1)\delta_{s,m+1} + [g(m)+g(m+1)]\, c_1(m,s,z) - g(m+1)c_1(m+1,s,z) - g(m)\, c_1(m-1,s,z)\}\ . \tag{C8}$$

Equation (C6) is equivalent to an integrodifferential equation for $p(m,t)$ with a kernel containing the coupling constant $g_0$ at all orders. One verifies that Eq. (C6) is solved by

$$\tilde{p}(m,z) = (z+K)A^{-1}(m,z)\sum_{i=m}^{N/2} p(i,0)\left[\delta_{i,m} + \sum_{K_1=m+1}^{i} d(m,k,z)A^{-1}(k_1,z) \sum_{K_2=K_1+1} d(K_1,K_2,z)A^{-1}(K_2,z)\cdots\right]\ , \tag{C9}$$

where the product stops as soon as an index $K_s$ takes the value $i$. We recall the initial condition, following from (4.17)

$$p(i,0) = \delta_{i,N/2}\ . \tag{C10}$$

*Work supported in part by the Deutsche Forshungsgemeinshaft (Richard Merton Fonds).

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