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Steady-state pulses and superradiance in short-wavelength, swept-gain amplifiers*

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The steady-state behavior of amplifiers in which the excitation is swept at the speed of light is discussed in the semiclassical approximation. In the present work we examine the case where the decay time of the population is comparable to that of the polarization. Pulse propagation is shown to obey a generalized sine-Gordon equation which contains the effects of atomic relaxations. The analytical expression of the steady-state pulses (SSP) gives two threshold conditions. In the region of limited gain the SSP is a broad pulse with small area which can be obtained by small signal theory. In the second region of high gain the SSP is the superradiant π pulse. Its pulse power is not limited as in usual superradiant theory because, as we show, for a swept excitation the cooperation-length limit does not exist.

In considerations involving short-wavelength lasers, it is clear that in view of the very short spontaneous lifetimes, one would like to sweep the excitation in the direction of lasing in order that the atoms be prepared in an excited state just as the radiation from previously excited atoms reaches them. In several recent papers we have considered the problem of the gain in such a system, and we have found that the short lifetimes have profound implications for amplifier behavior.[1] In particular, we find that the small-signal regime of the amplifier is highly anomalous, and that superradiance plays an important role in the nonlinear regime. In this paper we calculate the steady-state behavior for short lifetimes using a semiclassical approach. This procedure is both analytic and simple, and points out the unusual behavior of the amplifier in a straightforward and transparent fashion.

One fundamental difference between a swept excitation and simultaneous excitation is the fact that in the first case the "cooperation length limit,"[2] which is a strong limitation to the appearance of superradiance[3,4] with simultaneous excitation, does not exist. A number of previous calculations have implicitly assumed swept gain conditions and hence have shown superradiant behavior,[5-7] although it has been only recently that this has been pointed out explicitly.[4,8] The first example of this behavior was the steady-state pulse (SSP)[5] in an amplifier with linear losses described previously, assuming that the decay time of the atomic populations, T_1, is very long with respect to the decay time T_2 of the polarization. In this case a hyperbolic-secant π pulse exists or does not exist depending on the unique threshold condition $g/\kappa > 1$, where g/κ is the gain-to-loss ratio. In this paper we give an analytical description of an amplifier with $T_1 = T_2$. That is, we assume that T_1 has the shortest possible value. Starting from the Maxwell-Schrödinger equations, we derive a generalized sine-Gordon equation which contains the effects of atomic relaxation and of field linear losses. We give the exact analytical expression of the SSP, which is formed asymptotically in the amplifier after many linear absorption lengths. A crucial feature arising from a short T_1 is that the steady-state pulse exists only if spontaneous emission is explicitly taken into account (whereas if $T_1 \gg T_2$, it is enough to have an input pulse). In our semiclassical model, spontaneous emission is analytically simulated by a small initial value of the Bloch angle ϕ_0 (small polarization). Since the stochastic nature of the noise is not included, the results refer only to the mean behavior of the pulse. The small angle ϕ_0 is determined on the basis of a simple argument and turns out to be inversely proportional to the square root of the atomic density. Unlike the case in which T_1 is much larger than T_2, two different kinds of SSP can exist if $T_1 = T_1$, depending on the values of g/κ.

i. $1 < g/\kappa < \ln(1/\phi_0)$. The SSP is a broad pulse with small area and time duration of the order of T_2 which can be described by a small signal theory.

ii. $g/\kappa > \ln(1/\phi_0)$. The SSP is the hyperbolic-secant superradiant pulse whose time duration is shorter than T_2 and inversely proportional to the density of atoms. We have checked the formation of the SSP by a computer analysis in which the spontaneous-emission source is replaced by a random-noise polarization. The SSP is affected by large fluctuations in the small-signal regime. However, these fluctuations become very small in the nonlinear regime. In both cases the mean field values agree with the analytical expressions obtained by representing spontaneous emission by a small Bloch angle.

I. BASIS FOR CALCULATION

In the usual description of amplifiers,[5,6] one deals with a medium consisting of atoms decaying to some distant states at rates γ_a, γ_b, where $a(b)$ denotes the upper (lower) atomic level. In most cases, including the present calculation, the discussion is simplified by taking $\gamma_a = \gamma_b = 1/T_1$. The decay rate of the off-diagonal elements of the density matrix (polarization) of the two-level systems is

$$1/T_2 = \tfrac{1}{2}(\gamma_a + \gamma_b) + \gamma_{\text{phase}} ,$$

where γ_{phase} is produced by atomic collisions. The atoms are excited into their upper state by a δ-function excitation swept along the sample at the velocity of light. Let us denote by $\mathcal{P}$ the envelope of the atomic polarization and by N, the atomic inversion. We further introduce the Rabi flopping frequency

$$\mathcal{E}(t,z) = (\wp/\hbar) E_0(t,z) , \qquad (1.1)$$

where $E_0(t,z)$ is the envelope of the electromagnetic field which we take, for the present analysis, to be real (i.e., constant phase).

In the retarded frame z, $\mu = t - z/c$, the coupled Maxwell-Schrödinger equations read

$$\frac{\partial \mathcal{P}}{\partial \mu} = \mathcal{E} N - \frac{\mathcal{P}}{T_2} , \qquad (1.2)$$

$$\frac{\partial N}{\partial \mu} = - \mathcal{E}\mathcal{P} - \frac{N}{T_1} , \qquad (1.3)$$

$$\frac{\partial \mathcal{E}}{\partial z} = \alpha' \mathcal{P} - \kappa \mathcal{E} . \qquad (1.4)$$

Here κ is the linear losses of the amplifier, and

$$\alpha' = g/T_2 = 4\pi\wp^2 \omega_0 \rho_0 / c\hbar = 3(\lambda^2/4\pi)\tau_0^{-1} \rho_0 , \qquad (1.5)$$

where $\tau_0^{-1} = \gamma_0$ is the rate at which the atoms decay from the upper to the lower state, ρ_0 is the density of atoms excited at time $\mu = 0$, and $\hbar\omega_0$ is the energy difference between the two atomic levels.

The steady state in the amplifier is a pulse propagating at the velocity of light without distortion. This means that $\partial\mathcal{E}/\partial z$ vanishes, or that

$$\mathcal{P}_s = (\kappa/\alpha')\mathcal{E}_s . \qquad (1.6)$$

The steady state, which is denoted everywhere with the subscript s, is found by eliminating $\mathcal{P}$ from Eqs. (1.3) and (1.4) and solving the resulting pair of nonlinear differential equations. In many cases the dephasing due to collisions dominates the decay processes so that $T_2 \ll T_1$. In that case, the decay of the atomic inversion can be neglected during the time intervals of interest. The coupled equations can then be solved exactly to give the usual hyperbolic-secant steady-state pulse

$$\mathcal{E}_s(\mu) = (1/\tau_s)\,\text{sech}(\mu/\tau_s) , \qquad (1.7)$$

where

$$\tau_s = T_2[(g-\kappa)/\kappa]^{-1} . \qquad (1.8)$$

The radiation is emitted as a pulse of intensity $I(\mu) = (1/\tau_s^2)\,\text{sech}^2(\mu/\tau_s)$, with a temporal width τ_s. If the losses are small compared to the gain ($g \gg \kappa$), then

$$\tau_s \simeq T_2\, \kappa/g \simeq c\hbar\kappa/4\pi\wp^2\omega_0\rho_0 . \qquad (1.9)$$

The pulse width decreases as $1/\rho_0$, while the intensity grows as the square of the density of initially excited atoms. Thus we see that in the limit of very high gain, the system goes "superradiant." For intermediate cases, however, one does not expect such a pure behavior, and the intensity grows as $[(g-\kappa)/\kappa]^2$. Even so, we note that this kind of behavior does not appear in lasers, where the growth of the peak power goes as $(g-\kappa)/\kappa$ above threshold.

If one compares these results against the case of uniform excitation,[3] one sees that here there is no limitation on the power, whereas in the other case, the pulse width can be no shorter than the cooperation time[2] $T_c = (\alpha' c)^{-1/2}$, and thus the power is also limited. In uniform excitation there is a limitation $L \ll l_c$ ($= cT_c$) which comes from the requirement that the atom must still be excited when the pulse arrives. In a swept-gain amplifier, since atoms are excited only when the light coming from the proceeding ones reaches them, it is not necessary to impose a limit on the sample, i.e., the cooperation length is infinite.

II. METHOD OF SOLUTION

In this section we present the main points involved in finding the steady-state solution in the case $T_1 = T_2$. The details of this discussion are found in the Appendix. We consider an amplifier medium consisting of homogeneously broadened two-level atoms decaying to some distant states at the rates $\gamma_a = \gamma_b = \gamma$, and excited in their upper state by a δ-function excitation swept at the velocity of light. We assume that the effects of the collisions are negligible, such that $\gamma_{\text{phase}} = 0$, and thus

$$T_1 = T_2 = \gamma^{-1} . \qquad (2.1)$$

In this case we follow the treatment of Ref. 5 and define a Bloch angle $\phi(\mu,z)$, whose time derivative is the field amplitude $\mathcal{E}$ which obeys a generalized sine-Gordon equation

$$\frac{\partial^2\phi(\xi,z)}{\partial\xi\partial z} + \kappa\frac{\partial\phi(\xi,z)}{\partial\xi} = \alpha' \sin\phi(\xi,z) , \qquad (2.2)$$

where ξ is the reduced time,

$$\xi = (1 - e^{-\gamma\mu})/\gamma . \tag{2.3}$$

It is clear from this that all analytic solutions to the pulse propagation problem that have been found from the sine-Gordon equation can be reinvestigated in this case. The solutions will formally be the same, obtained by merely replacing μ by ξ, but the physics they describe will be very different. We choose the case of the steady-state pulse because the changes in the physics are particularly clear and important in that case. One important difference is that in this case, for an amplifier with nonzero κ, it is vital that one have a nonzero initial condition $\phi(0, z)$ that represents the effect of the noise. Otherwise, one finds that $\phi \to 0$ in the limit $z \to \infty$ (i.e., the pulse vanishes asymptotically). If we call this initial angle ϕ_0, we find the steady-state solution to be

$$\mathcal{E}_s(\mu) = (1/\tau_s)e^{-\gamma\mu}\operatorname{sech}(1/\tau_s)[\xi(\mu) - \xi_0] . \tag{2.4}$$

Here $\tau_s = \kappa T_2/g$, which differs from the previous notation, and ξ_0 is given by Eq. (A17). If we define the area of the pulse θ_s as the Bloch angle in the limit of infinite time, we find that

$$\tan\tfrac{1}{2}\theta_s = \tan\tfrac{1}{2}\phi_0 e^{g/\kappa} . \tag{2.5}$$

We discuss the properties of this steady-state solution in detail in Sec. III.

III. PROPERTIES OF THE STEADY-STATE PULSE

Before presenting the properties of the steady-state solution in detail, let us summarize its principal features and concentrate on those features which are new and interesting. The key result of this section is that, depending upon the initial atomic inversion ρ_0, the electric field can reach interesting steady-state configurations in one of two different regimes.

Introducing the explicit form of the hyperbolic secant, the steady-state electric field $\mathcal{E}(\mu)$ may be expressed as

$$\mathcal{E}(\mu) = \frac{\phi_0}{\tau_s} e^{-\gamma\mu} \frac{\exp[(1 - e^{-\gamma\mu})g/\kappa]}{1 + \frac{1}{4}\phi_0^2 \exp[2(1 - e^{-\gamma\mu})g/\kappa]} . \tag{3.1}$$

Let us first consider the case

$$\tfrac{1}{2}\phi_0 e^{g/\kappa} \ll 1 , \tag{3.2}$$

which, from Eq. (2.5), is seen to be equivalent to a pulse whose area θ is much less than π. One can drop the term in the denominator in Eq. (3.1) and get

$$\mathcal{E}(\mu) \cong (\phi_0/\tau_s)e^{-\gamma\mu}\exp[(1 - e^{-\gamma\mu})g/\kappa] . \tag{3.3}$$

If $g/\kappa > 1$, it has a maximum at

$$\mu_0 = (1/\gamma)\ln(g/\kappa) . \tag{3.4}$$

If $g/\kappa < 1$, the maximum of the electric field is at $\mu = 0$, that is, the pulse is essentially a decaying exponential characteristic of spontaneous emission.

Let us now consider the case $g \gg \kappa$. From Eq. (2.5), we see that now the pulse area is of order π and the small-signal analysis is no longer valid. In this case the requirement $g \gg \kappa$ means that the term in the denominator dominates the decay of the trailing edge of the pulse, and furthermore, that the entire pulse grows and dies in a time that is short compared to γ^{-1}. In this limit the reduced time is equivalent to the retarded time over the interval of interest, and we see that

$$E(\mu) = (1/\tau_s)\operatorname{sech}[(\mu/\tau_s) - \xi_0] , \tag{3.5}$$

i.e., we recover the hyperbolic-secant solution.[5] Since this occurs in the regime $g/\kappa \gg 1$, we see that this pulse has the superradiant properties discussed in Sec. II.

In Fig. 1 we have plotted the area θ (dashed line) and width (solid line) of the pulse as a function of $(g - \kappa)/\kappa$. The area remains small for $g < \kappa\ln(1/\phi_0)$, indicating that the amplifier reaches the steady-state configuration in the small-signal regime, and the width remains large. At the point $g/\kappa \sim \ln(1/\phi_0)$, the area becomes π, and there is an abrupt decrease in the width showing the behavior of the pulse at the second threshold.

In Fig. 2 we show the various pulse waveforms that are associated with the steady state. In Fig. 2(a), we have plotted a pulse shape typical of the region $g \lesssim \kappa$. The exponential decay characteristic of the spontaneous emission is still very apparent. The pulse in Fig. 2(b) corresponds to the intermediate regime $\kappa < g < \kappa\ln(1/\phi_0)$. Here the maximum of the pulse is about four orders of magnitude higher than that of Fig. 2(a). However, the pulse area is still very small ($\theta \cong 10^{-4}$), and the width is of the order T_2. In Fig. 2(c) we show the pulse

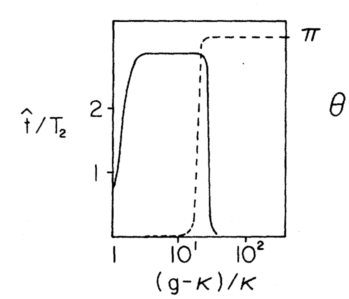

FIG. 1. Semilog graph of the width $\hat{t}$ (solid line) and the area θ (dashed line) of the pulse as a function of the parameter $(g - \kappa)/\kappa$.

for the case $g/\kappa > \ln(1/\phi_0)$ on the same scale as the other two. One sees that the width is substantially shorter, as indicated by the behavior in Fig. 1.

In Fig. 3 we show the behavior of the pulse width $\hat{t}$ as a function of $(g-\kappa)/\kappa$, on a logarithmic scale, for different values of the ratio T_1/T_2. The curves for $T_1 \neq T_2$ have been obtained by numerically solving the Bloch equations (1.2) and (1.3) with the substitution of the condition in Eq. (1.6), and the initial condition $N=1$, $\mathcal{P}=\sin\phi_0$. For large values of $(g-\kappa)/\kappa$, all cases converge on the dotted straight lines, which represent the value of the width in the limit $T_1 \to \infty$. The threshold is seen to move to progressively smaller values of $(g-\kappa)/\kappa$ as T_1 gets larger.

IV. COMPARISON WITH OTHER TECHNIQUES

In swept-gain amplifiers with finite population decays, the existence of a steady state depends in a fundamental way on the presence of spontaneous emission. As already mentioned, if spontaneous emission is not included in the description, one finds not just that nonzero SSP's do not exist, but that the pulse itself will vanish in the limit of large z. In the previous discussion, the "noise" was simulated by a nonstochastic initial value (at $\mu=0$) to the polarization described by the angle ϕ_0. In view of the vital role that the noise plays in this problem, it is worthwhile checking to see whether the nonstochastic nature of ϕ_0 affects the solution.

In the case of the small-signal pulse this checking is straightforward, since it is possible to solve the fully quantum-mechanical problem in this regime.[9] In that case, one finds that the steady-state value for the mean intensity is given by

$$\langle I_s(\mu)\rangle \propto \int_0^\infty dz\, e^{-2\gamma\mu}\{I_0([2gz(1-e^{-\gamma\mu})]^{1/2})\}^2 \times e^{-2\kappa z}\,, \tag{4.1}$$

where I_0 is the modified Bessel function. To avoid complicating the discussion, we omit the constants that appear in the equation. This formula can be put in a more transparent form if one uses the asymptotic form for I_0. This is appropriate, provided one does not evaluate the result at $\mu=0$, since the argument becomes large as $z \to \infty$. In that case, one gets

$$\langle I_s(\mu)\rangle \propto \frac{e^{-2\gamma\mu}\exp[2g/\kappa(1-e^{-\gamma\mu})]}{(1-e^{-\gamma\mu})^{1/2}}\;. \tag{4.2}$$

When this result is compared with the square of the field given by Eq. (3.1), one sees that they are the same except for a term in the denominator that alters the leading edge of the pulse. Thus the semiclassical description gives a reasonable approximation of the small-signal steady state.

To test whether an abrupt threshold exists, and whether the pulse goes over to the hyperbolic-secant form, we have taken Eqs. (1.2)–(1.4), modified them to allow for nonconstant phases, added a stochastic noise term, and solved the problem numerically. The steady state is found by carrying out the calculation for sufficiently large distances. We find that the predictions of the semiclassical theory are fully confirmed. The threshold is spanned by a factor of 2 change in g/κ, and the pulse is a hyperbolic secant above the threshold. The numerical solution also shows what happens to the fluctuations, which are absent in the semi-

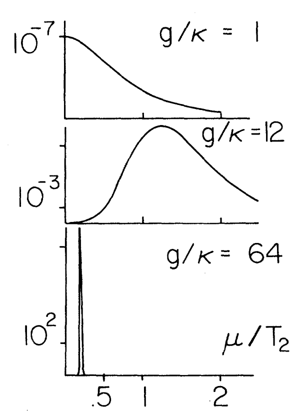

FIG. 2. Pulse intensity as a function of time, for different values of the gain-to-loss ratio. The time is in units of γ^{-1}, and the intensity is given in arbitrary units which are the same for the three cases.

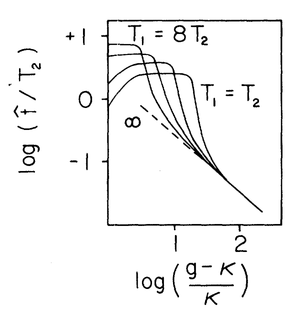

FIG. 3. Pulse width as a function of the parameter $(g-\kappa)/\kappa$, on a logarithmic scale, for different values of the ratio T_1/T_2. The curves are monotonic in T_1/T_2, so only the outer two are labeled. The dashed curve represents the limit $T_1=\infty$.

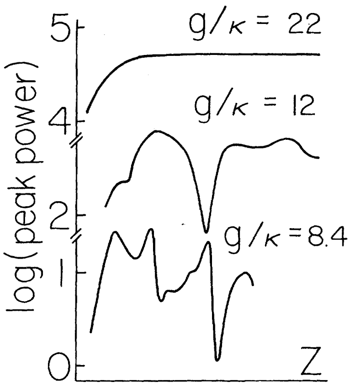

FIG. 4. Behavior of the peak power of the pulse on a logarithmic scale as a function of distance z (arbitrary units). The curves correspond to gain-to-loss ratios of 8.4, 12, and 22, which correspond to the three different regimes of the amplifier.

classical result. In Fig. 4, we show the behavior of the peak power of the pulse on a logarithmic scale as a function of distance z in the asymptotic (large-z) limit. Here we take $T_1 = T_2$ and $\kappa = 1$, and show three different values of the gain. For $g = 8.5$, the pulse is so small that the nonlinearities are completely negligible. The large intensity fluctuations are characteristic of the Gaussian fluctuations in the spontaneous emission. The case $g = 12$ represents a case where one is near but not yet at the threshold. Here the nonlinearities play a small but non-negligible role, and the fluctuations are slightly suppressed. Once the threshold is passed (i.e., $g = 22$), the fluctuations are substantially suppressed and the intensity is practically constant.[10]

We see, then, that the simple semiclassical results are well verified by the numerical and analytical calculations that take into explicit account the fluctuating feature of the noise. We see that the small-signal steady state is characterized by large fluctuations, whereas the fluctuations on the π pulse are comparatively small.

APPENDIX

In this appendix we discuss in detail the method of solution for this problem. Following the notation introduced in Sec. II, we take $1/T_2 = 1/T_1 = \gamma$. We then introduce the new variables,

$$\tilde{\mathcal{P}} = \mathcal{P} e^{\gamma\mu} , \tag{A1}$$

$$\tilde{N} = N e^{\gamma\mu} . \tag{A2}$$

Equations (1.2)–(1.4) become

$$\frac{\partial \tilde{\mathcal{P}}}{\partial \mu} = \mathcal{E}\tilde{N} , \tag{A3}$$

$$\frac{\partial \tilde{N}}{\partial \mu} = -\mathcal{E}\tilde{\mathcal{P}} , \tag{A4}$$

$$\frac{\partial \mathcal{E}}{\partial z} = \alpha' \tilde{\mathcal{P}} e^{-\gamma\mu} - \kappa\mathcal{E} . \tag{A5}$$

It is straightforward to show that

$$\tilde{\mathcal{P}}^2 + \tilde{N}^2 = 1 \tag{A6}$$

is a constant of motion of these equations. (This constant is equal to one with our particular normalization.)

From now on, the method of solution parallels that of Arecchi and Bonifacio.[5] Consistently with (A6), we define an angle ϕ such that

$$\tilde{\mathcal{P}} = \sin\phi(\mu, z) , \tag{A7}$$

$$\tilde{N} = \cos\phi(\mu, z) . \tag{A8}$$

From (A3) and (A5), we infer that

$$\mathcal{E}(\mu, z) = \frac{\partial}{\partial \mu} \phi(\mu, z) . \tag{A9}$$

Introducing (A7) and (A9) in (A5), we obtain the generalized sine-Gordon equation

$$\frac{\partial^2 \phi}{\partial \mu \partial z} = \alpha' e^{-\gamma\mu} \sin\phi - \kappa \frac{\partial \phi}{\partial \mu} . \tag{A10}$$

In the steady state, the z derivative vanishes, and the equation reduces to

$$\frac{d\phi_s}{d\mu} = \frac{1}{\tau_s} e^{-\gamma\mu} \sin\phi_s , \tag{A11}$$

where we denote

$$\tau_s = (\kappa/g) T_2 = \tau_0 / N_c , \tag{A12}$$

which differs from the previous notation [Eq. (1.8)] in that the dependence never goes as $g - \kappa$. We introduce N_c to show that this expression relates to the usual superradiant decay times,[2-4] by the replacement of the sample length with the absorption length $l_a = 1/\kappa$. The "cooperation number" N_c is defined as

$$N_c = (3/4\pi)\rho\lambda^2 l_a . \tag{A13}$$

The "coherence brightening" criterion for having a superradiant decay time much shorter than τ_0 leads to the condition $N_c \gg 1$.

We now introduce the "reduced time" $\xi(\mu)$ from Eq. (2.3), and we see that we recover Eq. (2.2). In the present case Eq. (2.2) becomes the equation for an overdamped pendulum, whose solution is

$$\tan\tfrac{1}{2}\phi_s = (\tan\tfrac{1}{2}\phi_0) e^{\xi(\mu)/\tau_s} . \tag{A14}$$

ϕ_0 is the value of ϕ at time $\mu = 0$, and its value will be discussed later on. From this formula, one obtains immediately that the area under the steady-state pulse, defined as

$$\theta_s = \phi_s(\mu)|_{\mu\to\infty}, \quad \text{(A15)}$$

is given by

$$\tan\tfrac{1}{2}\theta_s = (\tan\tfrac{1}{2}\phi_0)e^{\mathcal{E}/\kappa}. \quad \text{(A16)}$$

Introducing the value ξ_0, where

$$\xi_0 = \tau_s \ln\cot\tfrac{1}{2}\phi_0, \quad \text{(A17)}$$

we use (A7) and (1.6) to write the electric field of the steady-state pulse in the usual hyperbolic-secant form:

$$\mathcal{E}_s(\mu) = (1/\tau_s)e^{-\gamma\mu}\,\text{sech}(1/\tau_s)[\xi(\mu) - \xi_0]. \quad \text{(A18)}$$

The basic difference between the electric field in that case and that in the usual case ($T_1 >> T_2$) is that here it is symmetric in the reduced time ξ instead of the retarded time μ. Transforming back, the solution shows that the steady-state pulse is *asymmetric* in this kind of an amplifier.

The solution (A18) is not complete until we have specified the value of ϕ_0. This cannot be done rigorously in a semiclassical theory such as that presented here; we have to introduce an *ad hoc* noise term taking into account at least the most important features of a complete QED theory, namely that even with no initial field and no initial polarization, the vacuum fluctuations induce atomic decay and thus contribute to the growth of the electric field. This process is intrinsically random, and should be simulated semiclassically by a random Langevin-type force (as it is the case on computer simulations). However, we shall not do this in the present work, but rather we simulate the effects of spontaneous emission by a small, nonstochastic angle ϕ_0. This procedure turns out to give satisfactory results in the analysis of the mean properties of the steady-state pulses.

We evaluate this small angle ϕ_0 by imposing that at time $\mu = 0$, the atomic inversion decays as

$$\dot{N}(\mu=0) = -\gamma_0 N(\mu=0) = -\gamma_0, \quad \text{(A19)}$$

where $\gamma_0 = \tau_0^{-1} << \gamma$ is the rate at which the atom decays from the upper to the lower level by spontaneous emission. (The effects of the decay to the distant ground states have already been accounted for in γ and are not relevant for the evaluation of ϕ_0.) For the purpose of evaluating the effect of γ_0, we drop the γ from Eqs. (1.2)–(1.4) to find

$$\dot{N} \cong (\sin\phi_0)\dot{\phi}_0. \quad \text{(A20)}$$

But, from Eq. (A11),

$$\frac{d\phi}{d\mu} = (1/\tau_s)\sin\phi \quad \text{(A21)}$$

for $\xi \cong \mu \cong 0$. Therefore, combining Eqs. (A19)–(A21) gives an approximate value of ϕ_0:

$$\phi_0 \cong \sin\phi_0 = N_c^{-1/2} \quad (<<1). \quad \text{(A22)}$$

Hence, we finally obtain for the steady-state solution

$$\mathcal{E}(\mu) = \frac{1}{\tau_s}e^{-\gamma\mu}\,\text{sech}\left[\frac{1}{\tau_s}\left(\frac{1-e^{-\gamma\mu}}{\gamma} - \ln(N_c^{1/2})\right)\right]. \quad \text{(A23)}$$

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