Heisenberg-picture operator perturbation theory*

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We develop a systematic method of operator perturbation theory in the Heisenberg picture, which is the formal analog of time-independent stationary-state perturbation theory in the Schrödinger picture. However, we use the eigenoperators and operator eigenfrequencies to calculate the time evolution of the dynamical variables of interest. The spectrum of the unperturbed Hamiltonian is assumed to be discrete. The method is especially well-suited for treating systems where the unperturbed Hamiltonian represents noninteracting fermions and/or bosons, and therefore, we illustrate the method using the exactly solvable model of Jaynes and Cummings: a single two-level atom interacting with a single quantized field mode. The time evolution of these operator solutions is shown to be unitary order by order in powers of the interaction strength. We find that the time evolution of exact operator solutions is well approximated for significantly longer times by the time evolution of this method's operator solutions than by the time evolution of the same operator solutions calculated using the Dyson expansion. We also demonstrate that time-dependent operator solutions are very convenient for computing quantities such as correlation functions.

I. INTRODUCTION

A. Preliminary remarks

Let us consider a quantum system which is described by the Hamiltonian $H = H_0 + \lambda H_1$, where λ is a small parameter. Usually when we study systems described by a Hamiltonian of this type, we initially know the energy eigenstates and eigenvalues of the Hamiltonian H_0. What we would like to find is the time evolution of the known system H_0 under the influence of the perturbation λH_1. In most cases of interest this cannot be done exactly and, as a result, various time-dependent perturbation methods have been developed. We will only consider the case where H_1 is explicitly time independent.

One of the more general and widely used time-dependent perturbation methods was devised by Dyson.[1] The time-evolution operator is expressed as the product of the noninteracting time-evolution operator $U_0(t)$ with an unknown interacting time-evolution operator $U_1(t)$. The solution for $U_1(t)$ is found as a power series in λ and, in general, to some finite order in λ. In this expansion each power of λ corresponds to a power in t such that the exact solution is not well approximated for sufficiently long times t. If the exact solution is periodic, then the approximate nth-order solution is not substantially improved by simply calculating the $(n+1)$th-order solution.

Another method of time-dependent perturbation theory is the phase-operator formalism developed by Białynicki-Birula, Mielnik, and Plebański for Hamiltonians with continuous spectra.[2] A full perturbation expansion of the phase operator is developed with enables the time-evolution operator to preserve unitarity in every order of the perturbation and which preserves the exponential character of the time-evolution operator in every order of the perturbation. However, it has been shown by Rzążewski that if one uses only a finite number of terms in the perturbation expansion, then the time evolution is unsatisfactory for sufficiently long times.[3] As in the case of the Dyson expansion, powers of λ correspond to powers of t such that the nth-order solution is not substantially improved by calculating the $(n+1)$th-order solution.

Another method of time-dependent perturbation theory is Dirac's Heisenberg-picture constants-of-the-motion method.[4] Operator constants of the motion are calculated such that their interacting time evolution is in the form of a power series in λ. The time evolution of the dynamical variables of interest is found by using these operator constants of the motion. These solutions, however, are the same as those found using the Dyson expansion. As a result, the limitations in the validity of the time evolution of the Dyson expansion also apply to the time evolution of the operator solutions found by Dirac's method.

A general perturbation method which overcomes the long-time limitations of the other three methods is the Weisskopf-Wigner method.[5] The Schrödinger equation is solved for the time evolution of the unperturbed Hamiltonian's state vectors by re-

stricting the Hilbert space to only those state vectors which are "essential" to the particular problem being solved. However, while these solutions correctly describe the long-time evolution of the system, they are not calculated in a systematic manner. There is no order-by-order procedure for improving these solutions.

After observing the limitations of these general methods of time-dependent perturbation theory, we see that it would be desirable to develop a systematic method of time-dependent perturbation theory in which powers of λ do not correspond to powers of t such that solutions calculated to some finite order in λ are valid for relatively large values of t. For periodic solutions this would mean that the $(n+1)$th-order solution is valid in time for many more periods of oscillation than the nth-order solution.

In this paper we will develop a time-dependent perturbation method in the Heisenberg picture whose operator solutions are suitable for approximating periodic functions and are valid for large values of t. The usefulness of an operator approach is that we work directly with the dynamical variables of interest. In addition, operator solutions may be more convenient for calculating important physical quantities of interest, such as correlation functions. The operator perturbation method closely parallels and extends, on a formal level, the Schrödinger-picture method of stationary-state perturbation theory.[6] Eigenoperators and operator eigenfrequencies are computed by solving equations which formally parallel those of stationary-state perturbation theory. These solutions are then used to find the time evolution of the dynamical variables.

This method is very general and can be used for solving problems in many areas of physics, i.e., in quantum optics, solid-state physics, and many-body physics. In particular, in this paper we will place special emphasis on problems where H_0 is expressed directly in terms of creation and destruction operators for fermion and/or boson systems. Any other operators of interest can be calculated from a power series of these operators. The method in this paper will be restricted to the case where the spectrum of H_0 is discrete, i.e., H_0 consists only of a finite number of different creation and destruction operators. In Sec. VI we will comment on the generalization of the method for treating the case where H_0 has a continuous spectrum.

B. Organization of paper

In Sec. II we develop a nondegenerate operator perturbation method in the Heisenberg picture whose solutions enable the time evolution of the dynamical variables to be calculated. The close parallel between this formalism and Schrödinger-picture stationary-state perturbation theory is discussed in Sec. III. In addition, an underlying Hilbert-space structure for the operator perturbation method is established.

In Sec. IV we illustrate the operator perturbation method using the Jaynes-Cummings exactly solvable model of a two-level atom interacting with a single mode of the quantized radiation field.[7] A comparison of the exact operator solutions with the perturbation method's solutions to second order in λ is made. The advantage of using time-dependent operators for calculating correlation functions is also shown. In the Appendix, we explicitly demonstrate that the operator algebra is preserved for all time by these approximate solutions to second order in λ.

In Sec. V a method of degenerate operator perturbation theory is developed which is based on the nondegenerate perturbation method of Sec. II. This method is again illustrated using the Jaynes-Cummings model where the energy-level separation of the two-level atom is chosen to be degenerate with the energy-level separation of the single field mode.

II. FORMALISM

A. Preliminary remarks

In this section we will develop a method of operator perturbation theory in the Heisenberg picture suitable for attacking problems which can be described by the Hamiltonian

$$H = H_0 + \lambda H_1 , \tag{2.1}$$

where H, H_0, and H_1 are Hermitian and explicitly time independent. H_0 represents one or more noninteracting systems, and λH_1 represents a coupling between, or a perturbation of, the noninteracting systems. The parameter λ represents the strength of the coupling or perturbation, and it is assumed to be small.

For convenience, we will define the Liouville operator $\hat{L}$ as

$$\hat{L}R = [H, R] , \tag{2.2}$$

where $\hat{L}$ is a "super-operator" in that it maps ordinary operators R into ordinary operators $\hat{L}R$. We will, in addition, define the unperturbed and interaction Liouville operators $\hat{L}_0$ and $\hat{L}_1$ as

$$\hat{L}_0 R = [H_0, R] \tag{2.3}$$

and

$$\hat{L}_1 R = [H_1, R] . \tag{2.4}$$

In order to use the method of operator perturba-

tion theory, we require an *a priori* knowledge of the unperturbed system H_0. We assume that we have solved the eigenvalue equation

$$H_0|\psi_n^{(0)}\rangle = E_n^{(0)}|\psi_n^{(0)}\rangle \tag{2.5}$$

for the unperturbed Hamiltonian's energy eigenstates $|\psi_n^{(0)}\rangle$ and eigenvalues $E_n^{(0)}$. The unperturbed eigenstates are used to specify the initial state of the system $|\psi(0)\rangle = \sum_n c_n|\psi_n^{(0)}\rangle$. In addition, we assume that we have found a basis set of unperturbed operators $R_m^{(0)}$ which satisfy the operator eigenvalue equation

$$\hat{L}_0 R_m^{(0)} = \omega_m^{(0)} R_m^{(0)}, \tag{2.6}$$

where $\hbar = 1$, and that H_1 can be written in terms of this basis set of operators $R_m^{(0)}$. The frequencies $\omega_m^{(0)}$, which may be operators, are chosen such that

$$\hat{L}_0 \omega_m^{(0)} = 0, \tag{2.7}$$

allowing the unperturbed time evolution of the operators to be exponential:

$$R_m^{(0)}(t) = e^{i\omega_m^{(0)} t} R_m^{(0)}(0). \tag{2.8}$$

For systems H_0 which are boson or fermion, the operators $R_m^{(0)}$ can be conveniently chosen simply to be the usual harmonic-oscillator or spin-$\frac{1}{2}$ creation and destruction operators, where the frequencies $\omega_m^{(0)}$ are c numbers which trivially satisfy condition (2.7). Another possible choice for $R_m^{(0)}$, which will be discussed further in Sec. III, is

$$R_{ij}^{(0)} = |\psi_i^{(0)}\rangle\langle\psi_j^{(0)}|, \tag{2.9}$$

where the single operator index m in the definition of (2.6) implies both indices i and j in (2.9). The frequencies $\omega_{ij}^{(0)}$ are found from (2.6) to be

$$\omega_{ij}^{(0)} = E_i^{(0)} - E_j^{(0)}. \tag{2.10}$$

B. Perturbation method

We would like to find the time evolution of the system, which is described by the total Hamiltonian $H = H_0 + \lambda H_1$, by solving the Heisenberg equations of motion for the operators $R_m^{(0)}$ which are the interesting dynamical variables for fermion or boson systems. However, because of the presence of the interaction Hamiltonian λH_1, the time evolution of these operators is no longer simply exponential and is, in general, very complicated. Therefore, we propose to solve the Heisenberg equations of motion for another set of operators S_m whose time evolution is simply exponential, and to use these operators to find the time evolution of the operators $R_m^{(0)}$. The operators S_m satisfy equations similar to those satisfied by the operators $R_m^{(0)}$ [(2.6)–(2.8)], but with respect to the total Hamiltonian H. The operators S_m are defined by the equation[8]

$$\hat{L} S_m = \Omega_m S_m. \tag{2.11}$$

The operator frequencies Ω_m are chosen such that

$$\hat{L}\Omega_m = 0, \tag{2.12}$$

allowing the total time evolution of the operators S_m to be exponential:

$$S_m(t) = e^{i\Omega_m t} S_m(0). \tag{2.13}$$

We now look for approximate solutions for S_m and Ω_m in the form of a power series in the interaction strength λ:

$$S_m = R_m^{(0)} + \lambda R_m^{(1)} + \lambda^2 R_m^{(2)} \cdots \tag{2.14}$$

and

$$\Omega_m = \omega_m^{(0)} + \lambda\omega_m^{(1)} + \lambda^2\omega_m^{(2)} \cdots, \tag{2.15}$$

where each of the operators $R_m^{(j)}$ and $\omega_m^{(j)}$ for $j = 1, 2, 3 \ldots$, will be found as a series of basis operators $R_n^{(0)}$. If $\lambda = 0$, then $S_m = R_m^{(0)}$ and $\Omega_m = \omega_m^{(0)}$. The index j which appears in $R_m^{(j)}$ and $\omega_m^{(j)}$ indicates the power of λ which multiplies $R_m^{(j)}$ and $\omega_m^{(j)}$, and it will henceforth be called the order index. It is interesting to notice that we distinguish two separate perturbation expansions, one for the operator amplitudes and one for the operator frequencies. If we substitute the formal solutions (2.14) and (2.15) into (2.11) and (2.12), we obtain the equations

$$[\hat{L}_0 + \lambda\hat{L}_1 - \omega_m^{(0)} - \lambda\omega_m^{(1)} - \cdots][R_m^{(0)} + \lambda R_m^{(1)} + \cdots] = 0 \tag{2.16}$$

and

$$[\hat{L}_0 + \lambda\hat{L}_1][\omega_m^{(0)} + \lambda\omega_m^{(1)} + \cdots] = 0. \tag{2.17}$$

We can solve these equations for $R_m^{(j)}$ and $\omega_m^{(j)}$ by solving an equivalent set of equations which are found by setting the coefficients of equal powers of λ in (2.16) and (2.17) equal to zero:

$$[\hat{L}_0 - \omega_m^{(0)}]R_m^{(0)} = 0, \tag{2.18a}$$

$$[\hat{L}_0 - \omega_m^{(0)}]R_m^{(1)} + [\hat{L}_1 - \omega_m^{(1)}]R_m^{(0)} = 0, \tag{2.18b}$$

$$[\hat{L}_0 - \omega_m^{(0)}]R_m^{(2)} + [\hat{L}_1 - \omega_m^{(1)}]R_m^{(1)} - \omega_m^{(2)}R_m^{(0)} = 0, \tag{2.18c}$$

$$[\hat{L}_0 - \omega_m^{(0)}]R_m^{(3)} + [\hat{L}_1 - \omega_m^{(1)}]R_m^{(2)} - \omega_m^{(2)}R_m^{(1)} - \omega_m^{(1)}R_m^{(0)} = 0, \tag{2.18d}$$

etc., and

$$\hat{L}_0\omega_m^{(0)} = 0, \tag{2.19a}$$

$$\hat{L}_0\omega_m^{(1)} + \hat{L}_1\omega_m^{(0)} = 0, \tag{2.19b}$$

$$\hat{L}_0\omega_m^{(2)} + \hat{L}_1\omega_m^{(1)} = 0, \tag{2.19c}$$

$$\hat{L}_0\omega_m^{(3)} + \hat{L}_1\omega_m^{(2)} = 0, \tag{2.19d}$$

etc.

The formal similarity between Eqs. (2.11), (2.14)–(2.18), and those of the usual stationary-state perturbation theory in the Schrödinger picture can be easily seen by letting $\hat{L}_0 \to H_0$, $\omega_m^{(j)} \to E_m^{(j)}$, and $R_m^{(j)} \to |\psi_m^{(j)}\rangle$. However, we cannot solve Eqs. (2.18) by using Schrödinger-picture methods, because for interesting operators $R_m^{(0)}$, i.e., for boson creation and destruction operators, there is no well-defined scalar product in which $\hat{L}_0$ and $\hat{L}_1$ would be Hermitian super-operators, and because the frequencies $\omega_m^{(j)}$ are operators which must also satisfy Eqs. (2.19). Therefore, another method of solution for Eqs. (2.18) and (2.19) must be devised. The formal similarity between this operator perturbation method and stationary-state perturbation theory will be discussed further in Sec. III.

We define for an arbitrary operator function $G(\hat{L}_0)$ acting on the basis operators $R_m^{(0)}$ the eigenvalue equation

$$G(\hat{L}_0)R_m^{(0)} = G(\omega_m^{(0)})R_m^{(0)}, \tag{2.20}$$

which follows from the well-defined action of $\hat{L}_0$ on $R_m^{(0)}$ and $\omega_m^{(0)}$ [(2.18a), (2.19a)]. After remembering that the operators $R_m^{(j)}$ and $\omega_m^{(j)}$ are a series of basis operators such that the action of any operator function $G(\hat{L}_0)$ on $R_m^{(j)}$ and $\omega_m^{(j)}$ is also well defined, we can solve Eqs. (2.18) and (2.19) by operating from the left on Eqs. (2.18) with the operator $(\hat{L}_0 - \omega_m^{(0)})^{-1}$ and on Eqs. (2.19) with the operator $(\hat{L}_0)^{-1}$:

$$R_m^{(1)} = (\hat{L}_0 - \omega_m^{(0)})^{-1}[(\omega_m^{(1)} - \hat{L}_1)R_m^{(0)}] + C_{1,m}R_m^{(0)}, \tag{2.21a}$$

$$R_m^{(2)} = (\hat{L}_0 - \omega_m^{(0)})^{-1}[\omega_m^{(2)}R_m^{(0)} + (\omega_m^{(1)} - \hat{L}_1)R_m^{(1)}] + C_{2,m}R_m^{(0)}, \tag{2.21b}$$

$$R_m^{(3)} = (\hat{L}_0 - \omega_m^{(0)})^{-1}[\omega_m^{(3)}R_m^{(0)} + \omega_m^{(2)}R_m^{(1)} + (\omega_m^{(1)} - \hat{L}_1)R_m^{(2)}] + C_{3,m}R_m^{(0)}, \tag{2.21c}$$

etc., and

$$\omega_m^{(1)} = -(\hat{L}_0)^{-1}[\hat{L}_1\omega_m^{(0)}] + D_{1,m}, \tag{2.22a}$$

$$\omega_m^{(2)} = -(\hat{L}_0)^{-1}[\hat{L}_1\omega_m^{(1)}] + D_{2,m}, \tag{2.22b}$$

$$\omega_m^{(3)} = -(\hat{L}_0)^{-1}[\hat{L}_1\omega_m^{(2)}] + D_{3,m}, \tag{2.22c}$$

etc., where we have added to the particular solutions for $R_m^{(j)}$ and $\omega_m^{(j)}$ the homogeneous solutions $C_{j,m}R_m^{(0)}$ and $D_{j,m}$, respectively. $C_{j,m}$ and $D_{j,m}$ are arbitrary unperturbed constants of the motion which will be specified later.

The iteration procedure for solving Eqs. (2.21) and (2.22) begins with the substitution of $R_m^{(0)}$ and $\omega_m^{(0)}$ into (2.21a) and (2.22a), and after specifying $D_{1,m}$ and $C_{1,m}$ we obtain solutions for $R_m^{(1)}$ and $\omega_m^{(1)}$. These operator solutions are substituted into (2.21b) and (2.22b), and after specifying $D_{2,m}$ and $C_{2,m}$ we obtain solutions for $R_m^{(2)}$ and $\omega_m^{(2)}$. This step-by-step procedure is repeated until a desired order of approximation is reached.

In order to specify the unperturbed constants of the motion $D_{j,m}$ and $C_{j,m}$, we must look in more detail at Eqs. (2.21) and (2.22). Since the operators $R_m^{(j)}$, $\omega_m^{(j)}$, and the interaction Hamiltonian H_1 are written in terms of the basis set of operators $R_n^{(0)}$, we recognize that the operators in the brackets in (2.21) and (2.22) are only formal expressions for the basis operators $R_n^{(0)}$. Therefore, it is possible that the operators in these brackets may contain an operator unperturbed constant of the motion in (2.22) or an operator unperturbed constant of the motion multiplied by $R_m^{(0)}$ in (2.21) which will diverge after being operated on by the inverse operators $(\hat{L}_0)^{-1}$ in (2.22) and $(\hat{L}_0 - \omega_m^{(0)})^{-1}$ in (2.21). In Eqs. (2.21) we can eliminate divergent operators by simply specifying the arbitrary constant of the motion $D_{j,m}$.[9] For example, in Eq. (2.21b), if there is a part of the operator $[(\omega_m^{(1)} - \hat{L}_1)R_m^{(1)}]$ which is an unperturbed constant of the motion K multiplied by the operator $R_m^{(0)}$, then after substituting $\omega_m^{(2)}$ from (2.22b) into (2.21b) we can cancel this operator $KR_m^{(0)}$ by simply choosing $D_{2,m} = -K$. This method for choosing the operators $D_{j,m}$ uniquely specifies the operator frequencies $\omega_m^{(j)}$. It is important to notice, however, that if there exists an operator $R_k^{(0)}$ for $m \neq k$ such that $\omega_m^{(0)} = \omega_k^{(0)}$, and if the curly brackets in Eqs. (2.21) contain an unperturbed constant of the motion multiplying the operator $R_k^{(0)}$, then this term will also be divergent. However, it is not possible to eliminate this type of divergent term from (2.21) by simply specifying $D_{j,m}$. We will henceforth assume that if there are operators $R_k^{(0)}$ which are degenerate with the operators $R_m^{(0)}$, i.e., $\omega_m^{(0)} = \omega_k^{(0)}$ for $m \neq k$, then they must be *uncoupled* by this perturbation method. In other words, when we solve for the operators $R_m^{(j)}$, the operators $R_k^{(0)}$ will not appear in the square brackets. We will denote the perturbation method which we are developing in this section as nondegenerate operator perturbation theory. In Sec. V we will discuss degenerate operator perturbation theory.

In Eqs. (2.22) if the square brackets contain a divergent operator part, then we see that it cannot be removed, as there are no free parameters left in $\omega_m^{(j)}$ to be specified. The appearance of a divergent term would therefore imply that the constants of the motion simply cannot be expressed as a power series in the interaction strength, and another method of solution must be devised. We will continue the development of this method of nondegenerate operator perturbation theory under the assumption that all the constants of the motion can

be expressed as a power series in the interaction strength.

At this point, the arbitrary unperturbed constants of the motion $C_{j,m}$ are still unspecified. Since the operators S_m are only going to be used as a means of finding the time evolution of the operators $R_m^{(0)}$ and for this development have no real meaning, we simply set $C_{j,m}=0$. We should point out, however, that this choice does not have to be made. In particular, if these operators S_m for some problems do have real significance, then some other choice should be made.[10] For example, it may be important to require that the operators S_m satisfy the same algebra as their unperturbed operator counterparts $R_m^{(0)}$. In this case, we would choose the operators $C_{j,m}$ such that, order by order, the algebra of the operators S_m is the same as the algebra of the operators $R_m^{(0)}$.

C. Operator time evolution

The nth-order approximate operator solutions for S_m and Ω_m, which we define as $S_m^{(n)}$ and $\Omega_m^{(n)}$, respectively, have been found using the operator perturbation method of Sec. II B:

$$S_m^{(n)}=R_m^{(0)}+\sum_{l=1}^{n}\lambda^l R_m^{(l)} \tag{2.23}$$

and

$$\Omega_m^{(n)}=\omega_m^{(0)}+\sum_{l=1}^{n}\lambda^l\omega_m^{(l)}\,. \tag{2.24}$$

By substituting these approximate solutions (2.23) and (2.24) into Eq. (2.13) for S_m and Ω_m, we find the nth-order approximation to the time evolution of the operators $S_m(t)$:

$$S_m^{(n)}(t)=\exp(i\Omega_m^{(n)}t)S_m^{(n)}\,. \tag{2.25}$$

[Our notation is now chosen such that operators without any indicated time dependence, e.g., $S_m^{(n)}$ and $\Omega_m^{(n)}$ in (2.25), are operators evaluated at the initial time $t=0$.] The solution (2.25), however, is not "consistent" to nth order. From Eqs. (2.18), which we used to find (2.25), we see that each separate equation in (2.18) has an order in λ [(2.18a) – zeroth, (2.18b) – first, (2.18c) – second, etc.)], and that each separate term, for example in the nth-order equation, has this same order n indicated by the sum of the order indices of products like $\omega_m^{(i)}R_m^{(j)}$, where $i+j=n$. In particular, Eq. (2.18c) is of order 2 and has three terms, $[\hat{L}_0-\omega_m^{(0)}]R_m^{(2)}$, $[\hat{L}_1-\omega_m^{(1)}]R_m^{(1)}$, and $\omega_m^{(2)}R_m^{(0)}$, in each of which the sum of the order indices is 2. However, in the solution (2.25), the exponential which contains frequencies $\omega_m^{(l)}$ is multiplied by operators $R_m^{(j)}$ where $l+j$ can be greater than n. These terms are not defined by the perturbation method to nth order and should be neglected in (2.25). The "consistent" nth-order solution becomes

$$S_m^{(n)}(t)=\sum_{l=0}^{n}\lambda^l\exp(i\Omega_m^{(n-l)}t)R_m^{(l)}\,, \tag{2.26}$$

where for each additional power of λ in the amplitude, the frequency operator contains one less power of λ such that term by term the sum of the order indices in the frequency and amplitude operators is n. The "consistency" of solutions of this form, which we will explicitly show in Sec. IV, is that the algebra of the operators $R_m^{(0)}$ is preserved in an order-by-order manner for all times, where the order in this context refers to the power of λ appearing in the amplitude. We should also notice that in the limit $n\to\infty$, the form of the approximate operator solution (2.26), which for any finite order n is a power series of operators with exponential time dependence, becomes a single operator with exponential time dependence (2.13).

The time evolution of the operators $S_m^{(k)}(t)$ can now be used to find the time evolution of the operators $R_l^{(0)}(t)$. After inverting the expansion (2.23) of the operators $S_m^{(n)}$ such that we obtain an expansion of the operators $R_l^{(0)}$ in terms of the operators $S_m^{(k)}$, we can simply substitute the solution (2.26) for the time evolution of the operators $S_m^{(k)}(t)$ into this expansion, thus finding the time evolution of the operators $R_l^{(0)}(t)$.

The expansion for the operators $R_l^{(0)}$ in terms of the operators $S_m^{(k)}$ can be found by inverting Eqs. (2.23) in a step-by-step manner. We first bring the second term on the right-hand side of (2.23) to the left-hand side of (2.23):

$$R_m^{(0)}=S_m^{(n)}-\sum_{l=1}^{n}\lambda^l R_m^{(l)}\,. \tag{2.27}$$

Since the operators $R_m^{(l)}$ are functions of the basis operators $R_j^{(0)}$, we can iterate Eq. (2.27), eliminating order by order the operators $R_m^{(l)}$ from the right-hand side of (2.27) and replacing them order by order with a series of operators $S_k^{(n-l)}$. For example, if the operators $R_m^{(l)}$ are only in the form of a sum of operators $R_j^{(0)}$,

$$R_m^{(l)}=\sum_{\{j\}}d_j^{ml}R_j^{(0)}\,, \tag{2.28}$$

where d_j^{ml} are known constants, then after substituting (2.28) into (2.27), we find

$$R_m^{(0)}=S_m^{(n)}-\sum_{l=1}^{n}\lambda^l\sum_{\{j\}}d_j^{ml}R_j^{(0)}\,, \tag{2.29}$$

which can be rewritten as

$$R_m^{(0)} = S_m^{(n)} - \lambda \sum_{\{j\}} d_j^{m1} R_j^{(0)} - \sum_{l=2}^{n} \lambda^l \sum_{\{j\}} d_j^{ml} R_j^{(0)} . \tag{2.30}$$

Since in the second term on the right-hand side of (2.30) $R_j^{(0)}$ is multiplied by λ, we only substitute the $(n-1)$th-order solution

$$R_j^{(0)} = S_j^{(n-1)} - \sum_{k=1}^{n-1} \lambda^k \sum_{\{i\}} d_i^{jk} R_i^{(0)} \tag{2.31}$$

into (2.30) in order to iterate (2.30) to first order in λ:

$$R_m^{(0)} = S_m^{(n)} - \lambda \sum_{\{j\}} d_j^{m1} \left(S_j^{(n-1)} - \sum_{k=1}^{n-1} \lambda^k \sum_{\{i\}} d_i^{jk} R_i^{(0)} \right) - \sum_{l=2}^{n} \lambda^l \sum_{\{j\}} d_j^{ml} R_j^{(0)} . \tag{2.32}$$

By simply interchanging dummy indices in Eq. (2.32), the first-order iterated solution for $R_m^{(0)}$ becomes

$$R_m^{(0)} = S_m^{(n)} - \lambda \sum_{\{j\}} d_j^{m1} S_j^{(n-1)} + \sum_{l=2}^{n} \lambda^l \sum_{\{j\}} E_j^{ml} R_j^{(0)} , \tag{2.33}$$

where the constants E_j^{ml} are defined as

$$E_j^{ml} = \sum_{\{i\}} (d_i^{m1} d_j^{il-1}) - d_j^{ml} . \tag{2.34}$$

This step-by-step iteration procedure is continued until the operators $R_m^{(0)}$ are written exclusively in terms of the operators $S_j^{(k)}$:

$$R_m^{(0)} = \sum_{l=0}^{n} (-\lambda)^l \sum_{\{j\}} \alpha_j^{ml} S_j^{(n-l)} , \tag{2.35}$$

where the constants α_j^{ml} have been found during the iteration procedure. After evolving to time t, we can simply substitute the solution for the time evolution of the operators $S_j^{(n-l)}(t)$ from Eq. (2.26) into Eq. (2.35) giving us the time-dependent solution

$$R_m^{(0)}(t) = \sum_{l=0}^{n} (-\lambda)^l \sum_{\{j\}} \alpha_j^{ml} \sum_{k=0}^{n-l} \lambda^k \exp(i\Omega_j^{(n-l-k)} t) R_j^{(k)} . \tag{2.36}$$

In the case where the expansion of the operators $R_j^{(k)}$ consists also of products of operators $R_i^{(0)}$, it will become necessary for consistency to eliminate some terms which are of an order higher than n. These terms can be removed in a manner analogous to that used in going from Eq. (2.25) to Eq. (2.26).

III. RELATION TO TIME-INDEPENDENT STATIONARY-STATE PERTURBATION THEORY

The relation between the operator perturbation method of Sec. II and the usual stationary-state perturbation method will be shown by considering the general basis set of operators

$$R_{ij}^{(0)} = |\psi_i^{(0)}\rangle\langle\psi_j^{(0)}| , \tag{3.1}$$

which can be chosen as the starting point for the perturbation method of Sec. II. Since the action of these operators (3.1) takes us from state $|\psi_j^{(0)}\rangle$ to state $|\psi_i^{(0)}\rangle$, we shall call them "transition operators." These operators satisfy the eigenvalue equation

$$\hat{L}_0 R_{ij}^{(0)} = \omega_{ij}^{(0)} R_{ij}^{(0)} , \tag{3.2}$$

where

$$\omega_{ij}^{(0)} = E_i^{(0)} - E_j^{(0)} . \tag{3.3}$$

Similarly, we can define transition operators for the total Hamiltonian $H = H_0 + \lambda H_1$:

$$S_{ij} = |\psi_i\rangle\langle\psi_j| , \tag{3.4}$$

satisfying

$$\hat{L} S_{ij} = \Omega_{ij} S_{ij} , \tag{3.5}$$

where

$$\Omega_{ij} = E_i - E_j . \tag{3.6}$$

In stationary-state perturbation theory it is assumed that the eigenstates $|\psi_i\rangle$ and eigenvalues E_i can be expanded as a power series in the interaction strength:

$$|\psi_i\rangle = \sum_{n=0} \lambda^n |\psi_i^{(n)}\rangle \tag{3.7}$$

and

$$E_i = \sum_{n=0} \lambda^n E_i^{(n)} . \tag{3.8}$$

Therefore, we find after substituting (3.7) and (3.8) into (3.4) and (3.6),

$$S_{ij} = \sum_{n=0} \lambda^n \left(\sum_{l=0}^{n} |\psi_i^{(l)}\rangle\langle\psi_j^{(n-l)}| \right) = \sum_{n=0} \lambda^n R_{ij}^{(n)} \tag{3.9}$$

and

$$\Omega_{ij} = \sum_{n=0} \lambda^n (E_i^{(n)} - E_j^{(n)}) = \sum_{n=0} \lambda^n \omega_{ij}^{(n)} , \tag{3.10}$$

where we have used Eqs. (2.14) and (2.15) in defining $R_{ij}^{(n)}$ and $\omega_{ij}^{(n)}$. However, in order to solve for $R_{ij}^{(n)}$ and $\omega_{ij}^{(n)}$, it is not necessary to work with the Schrödinger-picture variables $|\psi_j^{(n)}\rangle$ and $E_j^{(n)}$,

because an analogous Hilbert-space formalism, which works directly with the variables $R_{ij}^{(n)}$ and $\omega_{ij}^{(n)}$, can be developed. In this formalism the operators $R_{ij}^{(0)}$, $\omega_{ij}^{(0)}$, and $\hat{L}$ play a role which corresponds to the role played by the state vectors $|\psi_i^{(0)}\rangle$, by the energies $E_i^{(0)}$, and by the Hamiltonian H, respectively.

We will now define the underlying Hilbert-space structure for the operator formalism which will allow us to solve Eqs. (3.9) and (3.10). The Hilbert-space $\mathcal{H}$ contains all the operators A satisfying the condition

$$\mathrm{Tr}\,[A^\dagger A]<\infty \ . \tag{3.11}$$

For every $A, B\in\mathcal{H}$, the scalar product is defined by

$$(A|\,B)=\mathrm{Tr}[A^\dagger B] \ . \tag{3.12}$$

It should be noticed that the creation and destruction operators for the harmonic oscillator do not belong to $\mathcal{H}$. However, we easily observe that the transition operators belong to $\mathcal{H}$ and satisfy the orthogonality-normalization condition

$$(R_{ij}^{(0)}\,|\,R_{kl}^{(0)})=\delta_{ik}\delta_{jl} \tag{3.13}$$

and the completeness relation

$$\sum_{i,j}(R_{ij}^{(0)}|A)R_{ij}^{(0)}=A \tag{3.14}$$

for all $A\in\mathcal{H}$. Therefore, the transition operators $R_{ij}^{(0)}$ form an orthonormal basis in $\mathcal{H}$. The Liouville operator $\hat{L}$ is Hermitian in $\mathcal{H}$:

$$\begin{aligned}(A|\,\hat{L}B)&=\mathrm{Tr}\,[A^\dagger(\hat{L}B)]=\mathrm{Tr}(A^\dagger[H,B])\\&=\mathrm{Tr}[([H,A])^\dagger B]=(\hat{L}A|\,B)\ ,\end{aligned} \tag{3.15}$$

where we have used the definition of $\hat{L}$ and the cyclic symmetry property of the trace.

At this point, the operator perturbation method can be developed exactly in the same manner as the standard Schrödinger perturbation theory. The solutions to Eqs. (2.18) become

$$\omega_{ij}^{(1)}=(R_{ij}^{(0)}\,|\,\hat{L}_1R_{ij}^{(0)})\ , \tag{3.16a}$$

$$\omega_{ij}^{(2)}=\sum_{kl\neq ij}\frac{|(R_{ij}^{(0)}|\hat{L}_1R_{kl}^{(0)}|^2}{\omega_{ij}^{(0)}-\omega_{kl}^{(0)}}\ , \tag{3.16b}$$

etc., and

$$R_{ij}^{(1)}=\sum_{kl\neq ij}\frac{(R_{kl}^{(0)}|\hat{L}_1R_{ij}^{(0)})}{\omega_{ij}^{(0)}-\omega_{kl}^{(0)}}R_{kl}^{(0)}\ , \tag{3.17a}$$

$$\begin{aligned}R_{ij}^{(2)}=\sum_{mn\neq ij}\Bigg(&\sum_{kl\neq ij}\frac{(R_{mn}^{(0)}|\hat{L}_1R_{kl})(R_{kl}^{(0)}|\hat{L}_1R_{ij}^{(0)})}{(\omega_{ij}^{(0)}-\omega_{kl}^{(0)})(\omega_{ij}^{(0)}-\omega_{mn}^{(0)})}\\&-\frac{(R_{ij}^{(0)}|\hat{L}_1R_{ij}^{(0)})(R_{mn}^{(0)}|\hat{L}_1R_{ij}^{(0)})}{(\omega_{ij}^{(0)}-\omega_{mn}^{(0)})^2}\Bigg)R_{mn}^{(0)}\ ,\end{aligned} \tag{3.17b}$$

etc., where $(R_{ij}^{(0)}|R_{ij}^{(n)})=0$ for $n=1,2,3,\ldots$, and the sum $\sum_{kl\neq ij}$ means that if $l=j$ $(k=i)$, then $k\neq i$ $(l\neq j)$. By substituting the definitions of $\hat{L}_1$, $R_{ij}^{(0)}$, and $\omega_{ij}^{(0)}$ [Eqs. (2.4), (3.1), (3.3)] into Eqs. (3.16) and (3.17), we can rewrite the solutions $R_{ij}^{(n)}$ and $\omega_{ij}^{(n)}$ in the form defined by Eqs. (3.9) and (3.10) and thereby illustrate the connection between the operator and state-vector solutions. For example,

$$\omega_{ij}^{(1)}=(R_{ij}^{(0)}|\hat{L}_1R_{ij}^{(0)})=\mathrm{Tr}\,(R_{ij}^{\dagger(0)}[H_1,R_{ij}^{(0)}])=\langle\psi_i^{(0)}|H_1|\,\psi_i^{(0)}\rangle-\langle\psi_j^{(0)}|H_1|\,\psi_j^{(0)}\rangle=E_i^{(1)}-E_j^{(1)} \tag{3.18}$$

and

$$\begin{aligned}R_{ij}^{(1)}&=\sum_{kl\neq ij}\frac{(R_{kl}^{(0)}|\hat{L}_1R_{ij}^{(0)})R_{kl}^{(0)}}{\omega_{ij}^{(0)}-\omega_{kl}^{(0)}}\\&=\sum_{kl\neq ij}\frac{\mathrm{Tr}\,(R_{kl}^{\dagger(0)}[H_1,R_{ij}^{(0)}])R_{kl}^{(0)}}{\omega_{ij}^{(0)}-\omega_{kl}^{(0)}}\\&=\sum_{k\neq i}\frac{\langle\psi_k^{(0)}|H_1|\,\psi_i^{(0)}\rangle|\psi_k^{(0)}\rangle\langle\psi_j^{(0)}|}{E_i^{(0)}-E_k^{(0)}}+\sum_{l\neq j}\frac{\langle\psi_j^{(0)}|H_1|\,\psi_l^{(0)}\rangle|\psi_i^{(0)}\rangle\langle\psi_l^{(0)}|}{E_j^{(0)}-E_l^{(0)}}\\&=|\,\psi_i^{(1)}\rangle\langle\psi_j^{(0)}|+|\,\psi_i^{(0)}\rangle\langle\psi_j^{(1)}|\ .\end{aligned} \tag{3.19}$$

For many systems there are an infinite number of transition operators $R_{ij}^{(0)}$. We find for a system which consists of a finite number of different harmonic oscillators, however, that all the transition operators can be written as a series of creation and destruction operators which, like the transition operators, will have c-number frequencies. Therefore, the basis set of operators $R_m^{(0)}$ which is

used to begin the operator perturbation method in this case can consist of only a finite number of harmonic-oscillator creation and destruction operators instead of an infinite number of transition operators. These operators, and not the transition operators, are the physically interesting operators for most problems.

This very convenient simplification of the basis set of operators, however, forces us to solve Eqs. (2.18) using the method developed in Sec. II, and not the Hilbert-space formalism developed in this section. In addition, we will, in general, find that the frequencies $\omega_m^{(n)}$ become operators. For example, the harmonic oscillator which has equal spacing of its energy levels $(E_{j+1}^{(0)} - E_j^{(0)} = \omega$, for $j = 0, 1, 2, \ldots)$ has creation and destruction operators of the form

$$a^\dagger = \sum_{j=0} (j+1)^{1/2} R_{j+1,j}^{(0)} \tag{3.20}$$

and

$$a = \sum_{j=0} j^{1/2} R_{j-1,j}^{(0)} \ . \tag{3.21}$$

If we act with $\hat{L}_0$, in particular, on (3.20), we find

$$\begin{aligned} \hat{L}_0 a^\dagger &= \sum_{j=0} (j+1)^{1/2} \hat{L}_0 R_{j+1,j}^{(0)} \\ &= \sum_{j=0} (j+1)^{1/2} (E_{j+1}^{(0)} - E_j^{(0)}) R_{j+1,j}^{(0)} \\ &= \omega \sum_{j=0} (j+1)^{1/2} R_{j+1,j}^{(0)} = \omega a^\dagger \ . \end{aligned} \tag{3.22}$$

Under the influence of the perturbation, Eq. (3.20) becomes

$$S = \sum_{j=0} (j+1)^{1/2} S_{j+1,j} \ , \tag{3.23}$$

where S_{ij} is defined by Eq. (3.4). However, the action of $\hat{L}$ on S may no longer result in simply a c number multiplying S if the interaction disturbs the equal spacing of the energy levels of the total Hamiltonian $(E_{j+1} - E_j \neq \text{const}$, for $j = 0, 1, 2, \ldots)$:

$$\begin{aligned} \hat{L}S &= \sum_{j=0} (j+1)^{1/2} \hat{L} S_{j+1,j} \\ &= \sum_{j=0} (j+1)^{1/2} (E_{j+1} - E_j) S_{j+1,j} \\ &\neq \text{const} \times S \ . \end{aligned} \tag{3.24}$$

Therefore, the equation $\hat{L}S = \Omega S$ requires Ω to be an operator.

IV. JAYNES-CUMMINGS MODEL

A. Preliminary remarks

We will illustrate the method of operator perturbation theory developed in Sec. II by choosing a model problem which shows the interesting and important features of the method and is exactly solvable. In addition, we want the unperturbed Hamiltonian to be represented by fermion and boson creation and destruction operators. The simplest model which satisfies these criteria is the model considered by Jaynes and Cummings: a single two-level atom coupled to a single quantized field mode.[7]

The Hamiltonian for this model in the rotating-wave approximation is

$$H = \tfrac{1}{2}\omega_0 \sigma_3 + \omega(a^\dagger a + \tfrac{1}{2}) + \lambda(a^\dagger \sigma_- + \sigma_+ a) \ , \tag{4.1}$$

where we require $\lambda |\omega - \omega_0|^{-1} \ll 1$. The two-level atom, which has energy-level separation ω_0, is represented by the usual Pauli matrices σ_j, $j = 1, 2, 3$, where

$$[\sigma_j, \sigma_k] = 2i\sigma_l \epsilon_{jkl} \ , \tag{4.2a}$$

$$\{\sigma_j, \sigma_k\} = 2\delta_{jk} \ , \tag{4.2b}$$

and

$$\sigma_\pm = \tfrac{1}{2}(\sigma_1 \pm i\sigma_2) \ . \tag{4.3}$$

The single quantized field mode, which has energy-level separation ω, is represented by simple harmonic-oscillator creation and destruction operators $a^\dagger$ and a, where

$$[a, a^\dagger] = 1 \ . \tag{4.4}$$

The eigenvalues and eigenstates of the unperturbed Hamiltonian are

$$E_{\pm,n}^{(0)} = \pm \tfrac{1}{2}\omega_0 + \omega(n + \tfrac{1}{2}) \tag{4.5}$$

and

$$|\psi_{\pm,n}^{(0)}\rangle = |\pm, n\rangle \ . \tag{4.6}$$

The basis operators $R_m^{(0)}$ and frequencies $\omega_m^{(0)}$ which satisfy Eqs. (2.6) and (2.7) are simply

$$R_m^{(0)} = \{\sigma_+, a^\dagger, \sigma_3\} \tag{4.7}$$

and

$$\omega_m^{(0)} = \{\omega_0, \omega, 0\} \ , \tag{4.8}$$

where we have not included the operators a and σ_- in (4.7) since they are simply the Hermitian conjugate operators of $a^\dagger$ and σ_+. σ_3, which equals $2\sigma_+\sigma_- - 1$, can obviously be computed from a knowledge of the operators σ_+ and σ_-, and has only been included in (4.7) for convenience.

B. Perturbation method

We will begin the perturbation method by calculating the operators S_m and Ω_m to third order in λ. It will be sufficient for this example to calculate only the operator $S_1^{(3)}$, where $R_1^{(0)}=\sigma_+$, and to simply state the results for $S_2^{(3)}$ and $S_3^{(3)}$, where $R_2^{(0)}=a^\dagger$ and $R_3^{(0)}=\sigma_3$.

We substitute the operator $R_1^{(0)}$ and the frequency $\omega_1^{(0)}$ into Eqs. (2.21a) and (2.22a), finding

$$R_1^{(1)}=(\hat{L}_0-\omega_0)^{-1}\{\omega_1^{(1)}\sigma_+ +a^\dagger\sigma_3\}+C_{1,1}\sigma_+ \tag{4.9}$$

and

$$\omega_1^{(1)}=D_{1,1}\ . \tag{4.10}$$

Since $(\hat{L}_0-\omega_0)^{-1}$ acting on the operator $a^\dagger\sigma_3$ in (4.9) is well defined, we choose $D_{1,1}=0$. The arbitrary unperturbed constant of the motion $C_{1,1}$ is set equal to zero. The first-order solutions now become

$$R_1^{(1)}=(\omega-\omega_0)^{-1}\{a^\dagger\sigma_3\} \tag{4.11}$$

and

$$\omega_1^{(1)}=0\ . \tag{4.12}$$

We substitute these operator solutions into Eqs. (2.21b) and (2.22b), finding

$$R_1^{(2)}=(\hat{L}_0-\omega_0)^{-1}\times\{\omega_1^{(2)}\sigma_+-(\omega-\omega_0)^{-1}[2\sigma_-(a^\dagger)^2-(2a^\dagger a+1)\sigma_+]\}+C_{2,1}\sigma_+ \tag{4.13}$$

and

$$\omega_1^{(2)}=D_{2,1}\ . \tag{4.14}$$

The action of $(\hat{L}_0-\omega_0)^{-1}$ on the operator $\sigma_-(a^\dagger)^2$ in (4.13) is well defined. However, the action of $(\hat{L}_0-\omega_0)^{-1}$ on the operator $(2a^\dagger a+1)\sigma_+$ in (4.13) is divergent and can be eliminated by choosing $D_{2,1}=-(\omega-\omega_0)^{-1}(2a^\dagger a+1)$. The arbitrary unperturbed constant of the motion $C_{2,1}$ is set equal to zero. The second-order solutions now become

$$R_1^{(2)}=-(\omega-\omega_0)^{-2}\sigma_-(a^\dagger)^2 \tag{4.15}$$

and

$$\omega_1^{(2)}=-(\omega-\omega_0)^{-1}(2a^\dagger a+1)\ . \tag{4.16}$$

We substitute these operator solutions into Eqs. (2.21c) and (2.22c), finding

$$R_1^{(3)}=(\hat{L}_0-\omega_0)^{-1}\{\omega_1^{(3)}\sigma_+-(\omega-\omega_0)^{-2}(2a^\dagger a+1)a^\dagger\sigma_3+(\omega-\omega_0)^{-2}(1+\sigma_3a^\dagger a)a^\dagger\}+C_{3,1}\sigma_+ \tag{4.17}$$

and

$$\omega_1^{(3)}=(\hat{L}_0)^{-1}2(\omega-\omega_0)^{-1}(\sigma_+a-a^\dagger\sigma_-)+D_{3,1}\ . \tag{4.18}$$

Since the action of $(\hat{L}_0-\omega_0)^{-1}$ on the operator $-(2a^\dagger a+1)a^\dagger\sigma_3+(1+\sigma_3a^\dagger a)a^\dagger$ in (4.17) is well defined, we choose $D_{3,1}=0$. The arbitrary unperturbed constant of the motion $C_{3,1}$ is set equal to zero. The action of $(\hat{L}_0)^{-1}$ on the operator $(\sigma_+a-a^\dagger\sigma_-)$ in (4.18) is well defined. The third-order solutions now become

$$R_1^{(3)}=-(\omega-\omega_0)^{-3}a^\dagger a\sigma_3a^\dagger \tag{4.19}$$

and

$$\omega_1^{(3)}=-2(\omega-\omega_0)^{-2}(\sigma_+a+a^\dagger\sigma_-)\ , \tag{4.20}$$

where we have substituted (4.20) into (4.17) and combined terms.

The third-order solutions for $S_1^{(3)}$ and $\Omega_1^{(3)}$ are easily found by combining equations (4.11), (4.12), (4.15), (4.16), (4.19), (4.20), and the definitions $R_1^{(0)}=\sigma_+$ and $\omega_1^{(0)}=\omega_0$:

$$S_1^{(3)}=\sigma_+ +\alpha a^\dagger\sigma_3-\alpha^2\sigma_-(a^\dagger)^2-\alpha^3a^\dagger a\sigma_3a^\dagger \tag{4.21}$$

and

$$\Omega_1^{(3)}=\omega_0-\lambda\alpha(2a^\dagger a+1)-2\lambda\alpha^2(a^\dagger\sigma_-+\sigma_+a)\ , \tag{4.22}$$

where

$$\alpha=\lambda(\omega-\omega_0)^{-1}\ . \tag{4.23}$$

By following similar procedures for the operators S_2 and S_3 we can find their third-order operator solutions:

$$S_2^{(3)}=a^\dagger+\alpha\sigma_+ +\alpha^3[2\sigma_-(a^\dagger)^2-(2a^\dagger a+1)\sigma_+]\ , \tag{4.24}$$

$$\Omega_2^{(3)}=\omega-\lambda\alpha\sigma_3+\lambda\alpha^2 2(a^\dagger\sigma_-+\sigma_+a)\ , \tag{4.25}$$

$$S_3^{(3)}=\sigma_3-2\alpha(a^\dagger\sigma_-+\sigma_+a)\ , \tag{4.26}$$

$$\Omega_3^{(3)}=0\ . \tag{4.27}$$

C. Operator time evolution

The time evolution of the operator $S_1^{(3)}(t)$ can be found by simply substituting the solutions (4.11), (4.15), (4.19), and (4.22) into Eq. (2.26):

$$S_1^{(3)}(t)=\exp(i\omega_0t)\{\exp[-i(\lambda\alpha(2a^\dagger a+1)+2\lambda\alpha^2(a^\dagger\sigma_-+\sigma_+a))t]\sigma_+ +\alpha\exp[-i\lambda\alpha(2a^\dagger a+1)t]a^\dagger\sigma_3-\alpha^2\sigma_-(a^\dagger)^2-\alpha^3a^\dagger a\sigma_3a^\dagger\}\ . \tag{4.28}$$

In the remainder of this section it will be sufficient for our purpose of illustration to simply work to second order in λ. The time evolution of the operators $S_2^{(2)}(t)$ and $S_3^{(2)}(t)$ is

$$S_2^{(2)}(t)=\exp[i(\omega-\lambda\alpha\sigma_3)t]a^\dagger+\alpha\exp(i\omega t)\sigma_+ , \tag{4.29}$$

$$S_3^{(2)}(t)=\sigma_3-2\alpha(a^\dagger\sigma_-+\sigma_+a) , \tag{4.30}$$

where in calculating (4.29) and (4.30) we have followed procedures similar to those used in calculating (4.28).

After finding the expansion for the operators $\sigma_+(t)$ in terms of the operators $S_m^{(k)}(t)$, we can simply substitute the time evolution of the operators from (4.28)–(4.30) into this expansion, obtaining the time evolution of the operator $\sigma_+(t)$. The expansion for the operator σ_+ in terms of the operators $S_m^{(k)}$ can be found by an iteration procedure after inverting Eqs. (4.21), (4.24), and (4.26). We invert these equations by bringing all the terms from the right-hand side of these equations to the left-hand side with the exception of the first term:

$$\sigma_+=S_1^{(2)}-\alpha a^\dagger\sigma_3+\alpha^2\sigma_-(a^\dagger)^2 , \tag{4.31}$$

$$a^\dagger=S_2^{(2)}-\alpha\sigma_+ , \tag{4.32}$$

$$\sigma_3=S_3^{(2)}+2\alpha(a^\dagger\sigma_-+\sigma_+a) . \tag{4.33}$$

The first-order iteration of Eq. (4.31) is found by substituting Eqs. (4.32) and (4.33) to first order in λ into the term proportional to λ in (4.31):

$$\sigma_+=S_1^{(2)}-\alpha(S_2^{(1)}-\alpha\sigma_+)[S_3^{(1)}+2\alpha(a^\dagger\sigma_-+\sigma_+a)]+\alpha^2\sigma_-(a^\dagger)^2 . \tag{4.34}$$

By combining some terms and dropping other third-order terms, Eq. (4.34) becomes

$$\sigma_+=S_1^{(2)}-\alpha S_2^{(1)}S_3^{(1)}+\alpha^2[\sigma_+S_3^{(0)}-2S_2^{(0)}(a^\dagger\sigma_-+\sigma_+a)+\sigma_-(a^\dagger)^2] . \tag{4.35}$$

The second-order iteration of Eq. (4.31) is found by substituting Eqs. (4.31) and (4.32) to zeroth order in λ into the term proportional to λ^2 in (4.35):

$$\sigma_+=S_1^{(2)}-\alpha S_2^{(1)}S_3^{(1)}+\alpha^2[S_1^{(0)}S_3^{(0)}-2S_2^{(0)}(S_2^{(0)}S_1^{\dagger(0)}+S_1^{(0)}S_2^{\dagger(0)})+S_1^{\dagger(0)}(S_2^{(0)})^2] . \tag{4.36}$$

At time t, Eq. (4.36) becomes simply

$$\sigma_+(t)=S_1^{(2)}(t)-\alpha S_2^{(1)}(t)S_3^{(1)}(t)+\alpha^2\{S_1^{(0)}(t)S_3^{(0)}(t)-2S_2^{(0)}(t)[S_2^{(0)}(t)S_1^{\dagger(0)}(t)+S_1^{(0)}(t)S_2^{\dagger(0)}(t)]+S_1^{\dagger(0)}(t)[S_2^{(0)}(t)]^2\} . \tag{4.37}$$

In a similar manner, we find the second-order iteration of Eqs. (4.32) and (4.33) at time t:

$$a^\dagger(t)=S_2^{(2)}(t)-\alpha S_1^{(1)}(t)+\alpha^2S_3^{(0)}(t)S_2^{(0)}(t) , \tag{4.38}$$

$$\sigma_3(t)=S_3^{(2)}(t)+2\alpha[S_2^{(1)}(t)S_1^{\dagger(1)}(t)+S_1^{(1)}(t)S_2^{\dagger(1)}(t)]-4\alpha^2[S_1^{(0)}(t)S_1^{\dagger(0)}(t)+S_2^{(0)}(t)S_2^{\dagger(0)}(t)S_3^{(0)}(t)] . \tag{4.39}$$

The time evolution of the operator $\sigma_+(t)$ to second order in λ is found by substituting the solutions (4.28)–(4.30) into Eq. (4.37):

$$\begin{aligned}\sigma_+(t)=&(\exp\{i[\omega_0-\lambda\alpha(2a^\dagger a+1)]t\}\sigma_+ +\alpha e^{i\omega_0t}a^\dagger\sigma_3-\alpha^2e^{i\omega_0t}\sigma_-(a^\dagger)^2)\\&-\alpha\{(e^{i\omega t}a^\dagger+\alpha e^{i\omega t}\sigma_+)[\sigma_3-2\alpha(a^\dagger\sigma_-+\sigma_+a)]\}\\&+\alpha^2\{(e^{i\omega_0t}\sigma_+)\sigma_3-2(e^{i\omega t}a^\dagger)[(e^{i\omega t}a^\dagger)(e^{-i\omega_0t}\sigma_-)+(e^{i\omega_0t}\sigma_+)(e^{-i\omega t}a)]+(e^{-i\omega_0t}\sigma_-)(e^{i\omega t}a^\dagger)^2\} .\end{aligned} \tag{4.40}$$

By combining terms of similar order in λ and by dropping terms of third order, Eq. (4.40) becomes

$$\sigma_+(t)=e^{i\omega_0t}\{\exp[-i\lambda\alpha(2a^\dagger a+1)t]\sigma_+ +\alpha(1-e^{i(\omega-\omega_0)t})a^\dagger\sigma_3-\alpha^2[(1-2e^{i(\omega-\omega_0)t}+e^{2i(\omega-\omega_0)t})\sigma_-(a^\dagger)^2+(1-e^{i(\omega-\omega_0)t})(2a^\dagger a+1)\sigma_+]\} . \tag{4.41}$$

In a similar manner we find the second-order time evolution of the operators $a^\dagger(t)$ and $\sigma_3(t)$:

$$a^\dagger(t)=e^{i\omega t}[e^{-i\lambda\alpha\sigma_3t}a^\dagger+\alpha(1-e^{-i(\omega-\omega_0)t})\sigma_+ +\alpha^2(1-e^{-i(\omega-\omega_0)t})\sigma_3a^\dagger] , \tag{4.42}$$

$$\sigma_3(t)=\sigma_3-2\alpha[(1-e^{i(\omega-\omega_0)t})a^\dagger\sigma_-+\text{H.c.}]+2\alpha^2[(e^{i(\omega-\omega_0)t}+e^{-i(\omega-\omega_0)t}-2)(\sigma_+\sigma_-+a^\dagger a\sigma_3)] . \tag{4.43}$$

The solutions (4.41)–(4.43) evaluated at the initial time $t=0$ satisfy the proper boundary conditions. In addition, the initial-time commutation relations (4.2) and (4.4) of the operator solutions (4.41)–(4.43) are preserved for all times to second order in the interaction strength λ (see the Appendix). The order-by-order preservation of the commutation relations is a very attractive and important feature of the operator solutions found by this perturbation method, since it means that the operators' time evolution is unitary order by order. In this context, as we have pointed out in Sec. II C, the order refers to the power of λ appearing in the amplitude.

D. Exact solution

In order to clarify the sense in which the approximate operator solutions found using this perturbation method represent the exact operator solutions, a comparison of them will be made. For this model, we should compare the approximate and exact solutions for the operators $\sigma_+(t)$ and $a^\dagger(t)$. Before making this comparison we must calculate the exact operator solutions.[11]

We begin by rewriting the Hamiltonian (4.1) as

$$H=\omega N-\gamma\sigma_3+\lambda(a^\dagger\sigma_-+\sigma_+a)\ , \tag{4.44}$$

where the total excitation number N and the detuning γ are defined, respectively, as

$$N=a^\dagger a+\sigma_+\sigma_-\ , \tag{4.45}$$

$$\gamma=\tfrac{1}{2}(\omega-\omega_0)\ . \tag{4.46}$$

Two constants of the motion, N and C, can be found from the Hamiltonian (4.44) by simply noticing that $[N,C]=0$, where

$$C=-\gamma\sigma_3+\lambda P\ , \tag{4.47}$$

$$P=a^\dagger\sigma_-+\sigma_+a\ . \tag{4.48}$$

The Heisenberg equations of motion for the operators $a^\dagger$ and σ_+ are obtained by using the Hamiltonian (4.1):

$$\left(i\frac{d}{dt}+\omega\right)a^\dagger=-\lambda\sigma_+\ , \tag{4.49}$$

$$\left(i\frac{d}{dt}+\omega+2C\right)\sigma_+=\lambda a^\dagger\ . \tag{4.50}$$

Since C is a constant of the motion, we can solve (4.49) and (4.50) as two first-order linear coupled differential equations with constant coefficients. We must be very careful, however, since we are dealing with operators. In (4.50), the constant of the motion C appears to the left of σ_+. We therefore look for solutions of the form

$$\sigma_+(t)=e^{i\beta+t}s_++e^{i\beta-t}s_-\ , \tag{4.51}$$

$$a^\dagger(t)=e^{i\beta+t}d_++e^{i\beta-t}d_-\ , \tag{4.52}$$

where $\beta_\pm$, $d_\pm$, and $s_\pm$ are initial time operators. In Eqs. (4.51) and (4.52), we write the same exponential time dependence for both σ_+ and $a^\dagger$, since they both satisfy the same second-order differential equation

$$\left(i\frac{d}{dt}+\omega\right)\left(i\frac{d}{dt}+\omega+2C\right)\begin{pmatrix}\sigma_+\\ a^\dagger\end{pmatrix}=-\lambda^2\begin{pmatrix}\sigma_+\\ a^\dagger\end{pmatrix}\ . \tag{4.53}$$

By substituting a trial solution of the form $e^{i\beta t}$ into (4.53), we obtain the auxiliary equation for β,

$$\beta^2-2\beta(\omega+C)+\omega(\omega+2C)+\lambda^2=0\ , \tag{4.54}$$

which, if we assume $[\beta,C]=0$, has two roots:

$$\beta_\pm=\omega+r_\pm\ , \tag{4.55}$$

where

$$r_\pm=C\pm[\lambda^2(N-1)+\gamma^2]^{1/2}\ . \tag{4.56}$$

The assumption $[\beta,C]=0$ is easily satisfied by (4.55).

In order to determine the operator constants $s_\pm$ and $d_\pm$, we must apply the boundary conditions at $t=0$. By simply evaluating (4.51) and (4.52) at $t=0$, we find

$$\sigma_+=s_++s_-\ , \tag{4.57}$$

$$a^\dagger=d_++d_-\ . \tag{4.58}$$

If we substitute the solutions (4.51) and (4.52) into (4.49) and (4.50) and subsequently evaluate these equations at $t=0$, then we find that $s_\pm$ and $d_\pm$ also satisfy the equations

$$r_+d_++r_-d_-=\lambda(s_++s_-)=\lambda\sigma_+\ , \tag{4.59}$$

$$r_-s_++r_+s_-=\lambda(d_++d_-)=\lambda a^\dagger\ . \tag{4.60}$$

After solving the algebraic equations (4.57)–(4.60) for $d_\pm$ and $s_\pm$, we can find the solutions to Eqs. (4.49) and (4.50):

$$\sigma_+(t)=e^{i\omega t}[\,e^{ir_+t}(r_+-r_-)^{-1}(r_+\sigma_+-\lambda a^\dagger)+e^{ir_-t}(r_+-r_-)^{-1}(-r_-\sigma_++\lambda a^\dagger)]\ , \tag{4.61}$$

$$a^\dagger(t)=e^{i\omega t}[\,e^{ir_+t}(r_+-r_-)^{-1}(-r_-a^\dagger+\lambda\sigma_+)+e^{ir_-t}(r_+-r_-)^{-1}(r_+a^\dagger-\lambda\sigma_+)]\ . \tag{4.62}$$

Equations (4.61) and (4.62) are the exact operator solutions for the basic Jaynes-Cummings variables.

In order to compare the exact solutions (4.61) and (4.62) with the approximate second-order solutions (4.41) and (4.42), found using the perturbation method of Sec. II, we should expand the

exact solutions to second order in λ. We can find these solutions by dropping terms in (4.61) and (4.62) of order higher than second in the same manner as that used to reduce (2.25) to (2.26). By substituting the expansions

$$(r_+ - r_-)^{-1} \sim (2\gamma)^{-1}[1 - 2\alpha^2(N-1)] , \tag{4.63}$$

$$r_\pm \sim 2\gamma[C/2\gamma \pm \tfrac{1}{2} \pm \alpha^2(N-1)] \tag{4.64}$$

into Eqs. (4.61) and (4.62), and using the algebraic properties (4.2) and (4.4), we find

$$\sigma_+(t) = e^{i\omega t}(\exp\{i2\gamma[C/2\gamma - \tfrac{1}{2} - \alpha^2(N-1)]t\}\sigma_+ - \alpha e^{iCt}(2i\sin\gamma t)a^\dagger\sigma_+\sigma_- + \alpha^2(1 - e^{-i2\gamma t})(N-1)\sigma_+) , \tag{4.65}$$

$$a^\dagger(t) = e^{i\omega t}(\exp\{i2\gamma[C/2\gamma + \tfrac{1}{2} + \alpha^2(N-1)]t\}\tfrac{1}{2}(1+\sigma_3)a^\dagger + \exp\{i2\gamma[C/2\gamma - \tfrac{1}{2} - \alpha^2(N-1)]t\}\tfrac{1}{2}(1-\sigma_3)a^\dagger$$
$$- \alpha[e^{iCt}(2i\sin\gamma t)(Pa^\dagger - \sigma_+)] + \alpha^2[(e^{-i\gamma(1+\sigma_3)t} - e^{i\gamma(1-\sigma_3)t})(N-1)\sigma_3 a^\dagger]) . \tag{4.66}$$

The solutions (4.65) and (4.66) do not look the same as the solutions (4.41) and (4.42).

Before commenting on this fact, let us look more closely at the operator e^{iCt} which appears in (4.65) and (4.66). By using the relation $C^2 = \lambda^2 N + \gamma^2$ and the usual Taylor-series expansions for the sine and cosine functions, the exponential operator e^{iCt} can be rewritten as

$$e^{iCt} = \cos Ct + i\sin Ct = \cos\gamma(1+4\alpha^2 N)^{1/2}t + i[C/\gamma(1+4\alpha^2 N)^{1/2}]\sin\gamma(1+4\alpha^2 N)^{1/2}t . \tag{4.67}$$

If we expand (4.67) to second order in λ, we find

$$e^{iCt} \sim e^{i\gamma(1+2\alpha^2 N)t}\tfrac{1}{2}(1-\sigma_3) + e^{-i\gamma(1+2\alpha^2 N)t}\tfrac{1}{2}(1+\sigma_3) + \alpha(2i\sin\gamma t)P + \alpha^2(2i\sin\gamma t)\sigma_3 N , \tag{4.68}$$

where we have used (4.7) and (4.48). If we substitute (4.68) into (4.65) and (4.66), and neglect terms of order higher than second, then after combining terms in equal powers of λ and making use of the operator algebra (4.2) and (4.4), we recover the operator solutions (4.41) and (4.42).

We now see that in working with operator solutions we should not expect that the form of any one solution is unique, since the operator algebra allows the solutions to be written in many different ways. For example, in (4.13) we made the choice $\omega_1^{(2)} = -(\omega - \omega_0)^{-1}(2a^\dagger a + 1)$, but instead we could have chosen $\omega_1^{(2)} = -(\omega - \omega_0)^{-1}(2N - 1)$, since the operator $(2a^\dagger a + 1)\sigma_+$ could have been written as $(2N-1)\sigma_+$ in (4.13). If we had made this new choice for $\omega_1^{(2)}$, then the operator form of $\omega_1^{(3)}$ and $R_1^{(3)}$ would not be the same as (4.19) and (4.20). In addition, we should point out that the solutions (4.65) and (4.66), which appeared to be of second order actually contained higher-order terms hidden in the operator e^{iCt} [see (4.67) and (4.68)]. Therefore, in reducing the approximate solutions (4.65) and (4.66) to the form of the solutions (4.41) and (4.42), we have illustrated that different-looking operator solutions, which are calculated to the same order n, are equal to order n. For this reason either of the second-order operator solutions (4.41), (4.42) or (4.65), (4.66) can be used for second-order calculations.

E. Correlation functions

The advantage of having computed the time evolution of operators is that it is now easy to calculate important quantities like correlation functions which are difficult to calculate in the Schrödinger picture. For example, let us write, using (4.42) and its Hermitian conjugate, the correlation function for the destruction of a photon at time t' and the creation of a photon at time t:

$$\langle a^\dagger(t)a(t')\rangle = \langle[e^{-i\lambda\alpha\sigma_3 t}a^\dagger + (\alpha\sigma_+ + \alpha^2\sigma_3 a^\dagger)(1 - e^{-i(\omega-\omega_0)t})][e^{i\lambda\alpha\sigma_3 t'}a + (\alpha\sigma_- + \alpha^2\sigma_3 a)(1 - e^{i(\omega-\omega_0)t'})]\rangle e^{i\omega(t-t')} . \tag{4.69}$$

After specifying the initial state we then must only calculate the expectation values of the initial time operators in (4.69) in order to find the correlation function. The complication of finding the correlation function in the Schrödinger picture can be seen if we rewrite this same correlation function in the form

$$\langle a^\dagger(t)a(t')\rangle = \langle\psi(0)|e^{iHt}a^\dagger e^{-iHt}e^{iHt'}ae^{-iHt'}|\psi(0)\rangle$$
$$= \sum_i \langle\psi(t)|a^\dagger|\psi_i(t)\rangle\langle\psi_i(t')|a|\psi(t')\rangle , \tag{4.70}$$

where

$$\sum_i |\psi_i(0)\rangle\langle\psi_i(0)| = 1 .$$

F. Long-time evolution

In order to evaluate the long-time validity of the time evolution of the operators calculated in Sec. IV C, we will, for convenience and simplicity, focus our attention on one particular quantity of physical interest—the probability of emission, P_E, or absorption, P_A, of a single photon. This probability was calculated by Jaynes and Cummings after simply diagonalizing the Hamiltonian:

$$P_A=|\langle +,n-1|U(t)|-,n\rangle|^2=P_E=|\langle -,n|U(t)|+,n-1\rangle|^2$$
$$=\frac{[2\sqrt{n}\,\lambda/(\omega-\omega_0)]^2}{1+[2\sqrt{n}\,\lambda/(\omega-\omega_0)]^2}\sin^2\left\{\frac{\omega-\omega_0}{2}\left[1+\left(\frac{2\sqrt{n}\,\lambda}{\omega-\omega_0}\right)^2\right]^{1/2}t\right\}. \tag{4.71}$$

It is interesting to notice that the expansion parameter $\lambda|\omega-\omega_0|^{-1}\ll 1$, discussed in Sec. IV A, appears modified by the factor $2\sqrt{n}$ in the exact solution (4.71). We therefore restrict the excitation number n such that

$$2\sqrt{n}\,\lambda/(\omega-\omega_0)=\Delta\ll 1\ . \tag{4.72}$$

In the Heisenberg picture, we can calculate the quantity (4.71) by looking at the time evolution of the expectation value of the photon number operator, where absorption [emission] of a photon corresponds to the choice of initial state $|\psi(0)\rangle=|-,n\rangle$ $[|\psi(0)\rangle=|+,n-1\rangle]$. The expectation value of the photon number can be easily found from the correlation function (4.69) by setting $t'=t$:

$$\langle -,n|a^\dagger(t)a(t)|-,n\rangle=n-\Delta^2\sin^2[\tfrac{1}{2}(\omega-\omega_0)]t\ , \tag{4.73}$$

$$\langle +,n-1|a^\dagger(t)a(t)|+,n-1\rangle$$
$$=n-1+\Delta^2\sin^2[\tfrac{1}{2}(\omega-\omega_0)]t\ . \tag{4.74}$$

In (4.73), the atom is initially in the ground state such that only absorption of a photon is possible. The expectation value of $a^\dagger(t)a(t)$ is therefore always less than or equal to the number of photons at time $t=0$. In (4.74), the atom is initially excited such that only emission of a photon is possible. The expectation value of $a^\dagger(t)a(t)$ is therefore always greater than or equal to the number of photons at time $t=0$. The probability of absorption or emission can be identified as the magnitude of the second term on the right-hand side of Eqs. (4.73) and (4.74). If we expand (4.71) to second order in λ, then (4.71) reduces to the probability contained in (4.73) and (4.74).

If we calculate the probability of emission or absorption of a photon $P(t)$ using the Dyson expansion, then we find to second order

$$P(t)=\Delta^2\sin^2[\tfrac{1}{2}(\omega-\omega_0)]t\ , \tag{4.75a}$$

and, in general, to order $4k+2$,

$$P(t)=\Delta^2\sum_{m=0}^{k}F_m(t)\left[\Delta^2\left(\frac{\omega-\omega_0}{2}\right)t\right]^{2m}\ , \tag{4.75b}$$

where the time evolution of the functions $F_m(t)$ can be seen from (4.71) to be in the form of products of $\sin[\frac{1}{2}(\omega-\omega_0)]t$ and $\cos[\frac{1}{2}(\omega-\omega_0)]t$. Corrections of order Δ also appear in $F_m(t)$, but they are unimportant for this discussion. A rough estimate of the relative error made in using a finite number of terms k in (4.75b) is

$$\frac{|P^{(2k+2)}-P^{(2k)}|}{P^{(2k)}}\sim\left[\Delta^2\left(\frac{\omega-\omega_0}{2}\right)t\right]^{2k+2}\ . \tag{4.76}$$

The time t at which the relative error in any order is proportional to Δ is

$$\frac{\omega-\omega_0}{2}\ t\sim\frac{\Delta^{1/(2k+2)}}{\Delta^2}\lesssim\frac{1}{\Delta^2}\ . \tag{4.77}$$

The probability calculated to *any* arbitrarily high order is not valid for a time t longer than $1/2\pi\Delta^2$ periods of oscillation.

The probability of emission or absorption of a photon $P(t)$ using the method developed in this paper is, to second order,

$$P(t)=\Delta^2\sin^2[\tfrac{1}{2}(\omega-\omega_0)]t\ , \tag{4.78a}$$

and is, in general, to order $2k+2$,

$$P(t)=\Delta^2\sum_{m=0}^{k}(-\Delta^2)^m\sin^2\left[\left(\frac{\omega-\omega_0}{2}\right)\xi^{(2k-2m)}t\right]\ , \tag{4.78b}$$

where

$$\xi^{(2m)}=\sum_{l=0}^{m}{}_{1/2}C_l(\Delta^2)^l\ , \tag{4.79}$$

$${}_{1/2}C_0=1\ , \tag{4.80a}$$

and

$${}_{1/2}C_k=[(\tfrac{1}{2})(\tfrac{1}{2}-1)\cdots(\tfrac{1}{2}-k+1)]/k! \tag{4.80b}$$

for $k=1,2,\ldots$. The binomial coefficients ${}_{1/2}C_k$ in (4.79) will henceforth be neglected for convenience. Let us again compute the time t at which the relative error in any order is proportional to Δ. It can be shown that the relative error in the amplitude is proportional to $\Delta^2\ll\Delta$. Therefore, the most important contribution to the error occurs when the phase of the approximate solution (4.78b) eventually becomes $\frac{1}{2}\pi$ out of phase with the exact solution (4.71):

$$(\xi^{(2k+2)}-\xi^{(2k)})[\tfrac{1}{2}(\omega-\omega_0)]t=\tfrac{1}{2}\pi\ . \tag{4.81}$$

If we simply require that the phase difference should never be larger than Δ, i.e., the relative error should be less than or proportional to Δ, then we find

$$(\Delta^2)^{k+1}[\tfrac{1}{2}(\omega-\omega_0)]t\sim\Delta\ , \tag{4.82}$$

which can be rewritten as

$$[\tfrac{1}{2}(\omega-\omega_0)]t\sim 1/\Delta^{2k+1}\ . \tag{4.83}$$

As a result, the probability calculated to order k is valid for $1/\Delta^2$ periods of oscillation longer than the probability calculated to order $k-2$.

We should point out that, to second order, the solutions (4.78a) and (4.75a) are the same and both valid for

$$[\tfrac{1}{2}(\omega-\omega_0)]t \sim 1/\Delta \ . \tag{4.84}$$

However, the Dyson expansion computed to any higher finite order can only be improved to the point where

$$[\tfrac{1}{2}(\omega-\omega_0)]t \sim 1/\Delta^2 \ . \tag{4.85}$$

Thus, the solution computed by this perturbation method, which distinguishes expansions in amplitude and frequency separately, is significantly improved order by order. For example, the fourth-order solution is valid for times t such that

$$[\tfrac{1}{2}(\omega-\omega_0)]t \sim 1/\Delta^3 \ , \tag{4.86}$$

which is already an improvement over (4.85).[12]

V. DEGENERATE PERTURBATION METHOD

A. Formalism

In Sec. II we presented a method of nondegenerate operator perturbation theory in the Heisenberg picture. In order to begin this perturbation method, the degenerate operators in the basis set were required to be uncoupled. In this section we will develop a perturbation method for treating the case where the degenerate operators in the basis set are coupled. This method will simply consist of a procedure for decoupling degenerate operators such that the method of Sec. II can be used.

For notational convenience, we will write the set of basis operators $R_m^{(0)}$ as $R_{mi}^{(0)}$, where the index i denotes different operators which have the same frequency $\omega_m^{(0)}$ such that

$$\hat{L}_0 R_{mi}^{(0)} = \omega_m^{(0)} R_{mi}^{(0)} \tag{5.1}$$

and

$$\omega_m^{(0)} \neq \omega_n^{(0)} \tag{5.2}$$

for all $n \neq m$. Since we usually do not know if the degenerate operators in the basis set are coupled or uncoupled, we will simply begin the perturbation method of Sec. II by substituting the operators $R_{mi}^{(0)}$ and frequencies $\omega_m^{(0)}$ into Eqs. (2.21a) and (2.22a):

$$R_{mi}^{(1)} = (\hat{L}_0 - \omega_m^{(0)})^{-1}[(\omega_{mi}^{(1)} - \hat{L}_1)R_{mi}^{(0)}] \ , \tag{5.3}$$

$$\omega_{mi}^{(1)} = D_{1,m,i} \ , \tag{5.4}$$

where the homogeneous part of the solution in Eq. (5.3) is unimportant and has been set equal to zero. We now assume that the action of $\hat{L}_1$ on $R_{mi}^{(0)}$ generates an operator $R_{mj}^{(0)}$, degenerate with $R_{mi}^{(0)}$, where $i \neq j$. (It is interesting to note that the interaction may couple $R_{mi}^{(0)}$ to an operator which is degenerate with it and which is not contained in the basis set. In this case, we would simply expand the original basis set to include this new operator. As we will see in Sec. V B, this is typically the case.) Other operators that are nondegenerate with $R_{mi}^{(0)}$ may also be generated, but since they are well defined, after being operated on with the operator $(\hat{L}_0 - \omega_0)^{-1}$, we can temporarily neglect them in (5.3). Since the operator $R_{mj}^{(0)}$, after being operated on with the operator $(\hat{L}_0 - \omega_m^{(0)})^{-1}$, is not well defined, and since this type of divergence cannot be removed by simply specifying $D_{1,m,i}$, we are not able to continue the perturbation method.

Therefore, we shall begin the perturbation method again by defining a new basis operator with frequency $\omega_m^{(0)}$, which is an arbitrary linear combination of these two degenerate operators:

$$R_{m\{k\}}^{(0)} = \sum_{l \in \{k\}} C_l^m R_{ml}^{(0)} \tag{5.5}$$

where the set $\{k\}$ contains two elements i and j. By substituting (5.5) and the frequency $\omega_m^{(0)}$ into Eqs. (2.21a) and (2.22a), we find

$$R_{m\{k\}}^{(1)} = (\hat{L}_0 - \omega_m^{(0)})^{-1}[(\omega_{m\{k\}}^{(1)} - \hat{L}_1)R_{m\{k\}}^{(0)}] \tag{5.6}$$

and

$$\omega_{m\{k\}}^{(1)} = D_{1,m,\{k\}} \ . \tag{5.7}$$

The action of the operator $\hat{L}_1$ on $R_{m\{k\}}^{(0)}$ may generate another operator $R_{mq}^{(0)}$ for $q \neq i,j$, which is degenerate with $R_{m\{k\}}^{(0)}$. In this case we must begin the perturbation method once again, by defining a new operator with frequency $\omega_m^{(0)}$ which is an arbitrary linear combination of these degenerate operators:

$$R_{m\{k\}}^{(0)} = \sum_{l \in \{k\}} C_l^m R_{ml}^{(0)} \ , \tag{5.8}$$

where the set $\{k\}$ now contains three elements, i, j, and q. Eventually, however, after repeating this procedure as many times as is necessary, we will find an operator $R_{m\{k\}}^{(0)}$ which, when operated on by $\hat{L}_1$, will not generate any operators $R_{mp}^{(0)}$ for $p \notin \{k\}$ and which, as a result, is decoupled from all other operators $R_{mp}^{(0)}$.

At this stage, the decoupled basis set of operators $R_{m\{k\}}^{(0)}$ remain unspecified, since in (5.8) C_l^m are arbitrary constants. However, before specifying these constants, we will define the action of $\hat{L}_1$ on the operators $R_{m\{k\}}^{(0)}$:

$$\hat{L}_1 R^{(0)}_{m\{k\}} = \sum_{l\in\{k\}} C^m_l \hat{L}_1 R^{(0)}_{ml} = \sum_{l\in\{k\}} C^m_l \left(\sum_{r\in\{k\}} E^r_l R^{(0)}_{mr} + \sum_{s\neq m} \sum_r F^s_r R^{(0)}_{sr} \right) , \tag{5.9}$$

where E^r_l and F^s_r are known constants and, for definiteness, we have specified that $\hat{L}_1 R^{(0)}_{ml}$ does not generate any products of basis operators. The first term on the right-hand side of (5.9) represents the part of the operator $\hat{L}_1 R^{(0)}_{m\{k\}}$ which is divergent after being operated on by $(\hat{L}_0 - \omega^{(0)}_m)^{-1}$, and the second term represents the part which is finite.

After substituting (5.9) into (5.6), we find that in order for the operator $R^{(1)}_{m\{k\}}$ to be well defined, we must choose $D_{1,m,\{k\}}$ such that

$$D_{1,m,\{k\}} R^{(0)}_{m\{k\}} - \sum_{l\in\{k\}} C^m_l \sum_{r\in\{k\}} E^r_l R^{(0)}_{mr} = 0 . \tag{5.10}$$

In particular, if in (5.9) $F^s_r = 0$ for all r, s, then (5.10) reduces to the eigenvalue equation

$$(\hat{L}_1 - D_{1,m,\{k\}}) R^{(0)}_{m\{k\}} = 0 . \tag{5.11}$$

The solutions to Eqs. (5.10) and (5.11) are a set of decoupled degenerate operators $R^{(0)}_{m\{k\}b}$ and their corresponding frequencies $\omega^{(1)}_{m\{k\}b}$. The index b distinguishes for each m and $\{k\}$ all the different solutions; i.e., for the solutions to (5.11), the index b distinguishes all the different eigenfrequencies and eigenoperators. From (5.7), we see that $\omega^{(1)}_{m\{k\}b}$ represents the various solutions for $D_{1,m,\{k\}}$. If the interaction removes the degeneracy to first order, then each frequency $\omega^{(1)}_{m\{k\}b}$ will be different. If the interaction removes the degeneracy but not to first order, then each frequency $\omega^{(1)}_{m\{k\}b}$ will be the same and it will become necessary to follow a procedure similar to the one developed here, but in some higher order of the perturbation method, in order to remove the degeneracy. In addition, since D_1, i.e., $\omega^{(1)}_{m\{k\}b}$ for each b, is an unperturbed constant of the motion, i.e., it may be an operator, the method for solving Eqs. (5.10) and (5.11) is not well established. Therefore, we cannot specify a general method of solution here, but will illustrate one particular method in Sec. V B. The set of first-order operator solutions $R^{(1)}_{m\{k\}}$ which are found from Eqs. (5.6) and (5.9) are

$$R^{(1)}_{m\{k\}} = \sum_{l\in\{k\}} C^m_l \sum_{s\neq m} \sum_r (\omega^{(0)}_s - \omega^{(0)}_m)^{-1} F^s_r R^{(0)}_{sr} , \tag{5.12}$$

where the set of constants C^m_l are obtained from the eigenvalue equations (5.10) and (5.11), and they are different for different b such that $R^{(1)}_{m\{k\}} \to R^{(1)}_{m\{k\}b}$. The procedure for continuing the perturbation method to the point where we obtain approximate time-dependent solutions for the basis operators $R^{(0)}_{mi}(t)$ is at this point the same as that described in Sec. II.

B. Jaynes-Cummings model

We would like to illustrate the degenerate operator perturbation method developed in Sec. V A using the Jaynes-Cummings model of Sec. IV. In order for the model to be degenerate, we will set $\omega = \omega_0$ in the Hamiltonian (4.1):

$$H = \omega(\sigma_+\sigma_- + a^\dagger a) + \lambda(a^\dagger \sigma_- + \sigma_+ a) , \tag{5.13}$$

where we have used the operator identity $\sigma_+\sigma_- = \frac{1}{2}(1+\sigma_3)$. The basis operator set $R^{(0)}_{mi}$ defined in (4.7) now contains two degenerate operators $R^{(0)}_{11} = \sigma_+$ and $R^{(0)}_{12} = a^\dagger$ with frequency $\omega^{(0)}_1 = \omega$.

We begin the perturbation method of Sec. II by substituting the operator $R^{(0)}_{11}$ and frequency $\omega^{(0)}_1$ into (2.21a) and (2.22a):

$$R^{(1)}_{11} = (\hat{L}_0 - \omega)^{-1}(\omega^{(1)}_{11}\sigma_+ + \sigma_3 a^\dagger) , \tag{5.14}$$

$$\omega^{(1)}_{11} = D_{1,1,1} . \tag{5.15}$$

Since the action of $(\hat{L}_0 - \omega)^{-1}$ on $\sigma_3 a^\dagger$ is divergent, and since this divergence cannot be removed by specifying $D_{1,1,1}$, we must define a new basis operator $R^{(0)}_{1\{k\}}$ with frequency ω, which is an arbitrary linear combination of the two degenerate operators $R^{(0)}_{11}$ and $R^{(0)}_{13} = \sigma_3 a^\dagger$:

$$R^{(0)}_{1\{k\}} = \sum_{l\in\{k\}} C^1_l R^{(0)}_{1l} , \tag{5.16}$$

where the set $\{k\}$ contains two elements, 1 and 3. In addition, we have increased the size of the basis set by including the operator $\sigma_3 a^\dagger$.

We begin the perturbation method again by substituting the operator $R^{(0)}_{1\{k\}}$ from (5.16) and the frequency $\omega^{(0)}_1$ into (2.21a) and (2.22a):

$$R^{(1)}_{1\{k\}} = (\hat{L}_0 - \omega)^{-1}[\omega^{(1)}_{1\{k\}} R^{(0)}_{1\{k\}} + C^1_1 \sigma_3 a^\dagger - C^1_3(2P\sigma_3 a^\dagger + \sigma_+)] , \tag{5.17}$$

$$\omega^{(1)}_{1\{k\}} = D_{1,1,\{k\}} , \tag{5.18}$$

where P, which equals H_1, is defined in (4.48). Since the action of $(\hat{L}_0 - \omega)^{-1}$ on $2P\sigma_3 a^\dagger$ is divergent, and since this divergence cannot be removed by specifying $D_{1,1,\{k\}}$, we must define a new basis operator $R^{(0)}_{1\{k\}}$ with frequency ω, which is an arbitrary linear combination of these three degenerate operators $R^{(0)}_{11}$, $R^{(0)}_{13}$, and $R^{(0)}_{14} = P\sigma_3 a^\dagger$:

$$R^{(0)}_{1\{k\}} = \sum_{l\in\{k\}} C^1_l R^{(0)}_{1l} , \tag{5.19}$$

where the set $\{k\}$ contains contains three elements, 1, 3, and 4. In addition, we have again increased the size of the basis set by including the operator $P\sigma_3 a^\dagger$.

Since $\hat{L}_1 P = 0$ and $\hat{L}_1 \sigma_3 a^\dagger = 2P\sigma_3 a^\dagger + \sigma_+$, we can see, after substituting the operator $R^{(0)}_{1\{k\}}$ from (5.19) and the frequency ω into (2.21a) and (2.22a), that the action of $\hat{L}_1$ on $R^{(0)}_{1\{k\}}$ gives rise to two new degenerate operators, $P^2\sigma_3 a^\dagger$ and $P\sigma_+$, which contain only higher powers of P multiplying σ_+ and $\sigma_3 a^\dagger$. By continuing this procedure, we find that only higher powers of P multiplying σ_+ and $\sigma_3 a^\dagger$ are generated. Therefore, the new basis operator with frequency ω becomes

$$R^{(0)}_{1\{k\}} = \sum_{l=0}^{\infty} (C^1_l P^l \sigma_+ + E^1_l P^l \sigma_3 a^\dagger) , \tag{5.20}$$

where the set $\{k\}$ contains all the operators $P^l\sigma_+$ and $P^l\sigma_3 a^\dagger$, where $l = 0, 1, 2, \ldots$.

Before specifying the arbitrary constants C^1_l and E^1_l, we will calculate the action of $\hat{L}_1$ on $R^{(0)}_{1\{k\}}$ from (5.20):

$$\begin{aligned}\hat{L}_1 R^{(0)}_{1\{k\}} &= \sum_{l=0}^{\infty} (C^1_l P^l \hat{L}_1 \sigma_+ + E^1_l P^l \hat{L}_1 \sigma_3 a^\dagger) \\ &= \sum_{l=0}^{\infty} [C^1_l P^l (-\sigma_3 a^\dagger) + E^1_l P^l (2P\sigma_3 a^\dagger + \sigma_+)] \\ &= (2PE - \mathfrak{c})\sigma_3 a^\dagger + E\sigma_+ , \end{aligned} \tag{5.21}$$

where we have used the definitions and properties

$$E \equiv \sum_{l=0}^{\infty} E^1_l P^l , \tag{5.22a}$$

$$\mathfrak{c} \equiv \sum_{l=0}^{\infty} C^1_l P^l , \tag{5.22b}$$

$$[P, \mathfrak{c}] = [P, E] = [\mathfrak{c}, E] = 0 . \tag{5.23}$$

After substituting (5.21) into (5.6) for $m = 1$, we find that in order for the operator $R^{(1)}_{1\{k\}}$ to be well defined, we must choose $D_{1,1,\{k\}}$ such that it satisfies (5.11):

$$(2PE - \mathfrak{c})\sigma_3 a^\dagger + E\sigma_+ = D_{1,1,\{k\}}(\mathfrak{c}\sigma_+ + E\sigma_3 a^\dagger) , \tag{5.24}$$

where $R^{(0)}_{1\{k\}} = \mathfrak{c}\sigma_+ + E\sigma_3 a^\dagger$. We can solve the eigenvalue equation (5.24) by equating the operator coefficients of the operators σ_+ and $\sigma_3 a^\dagger$, obtaining two equations:

$$(2P - D_{1,1,\{k\}})E = \mathfrak{c} \tag{5.25a}$$

and

$$E = D_{1,1,\{k\}}\mathfrak{c} . \tag{5.25b}$$

By substituting (5.25b) into (5.25a) for E, we obtain

$$[(2P - D_{1,1,\{k\}})D_{1,1,\{k\}} - 1]\mathfrak{c} = 0 , \tag{5.26}$$

which for $\mathfrak{c} \neq 0$ can be satisfied by setting

$$(2P - D_{1,1,\{k\}})D_{1,1,\{k\}} - 1 = 0 . \tag{5.27}$$

If we assume $[D_{1,1,\{k\}}, P] = 0$, then we can simply solve (5.27) as a quadratic equation. The solution is

$$D_{1,1,\{k\}} = D^{\pm}_1 = P \pm \sqrt{N-1} , \tag{5.28}$$

where $P^2 = N$ and, obviously, the assumption $[D_{1,1,\{k\}}, P] = 0$ is satisfied. In addition, $D^{\pm}_1$ from (5.28) should be an unperturbed constant of the motion, and this condition is also satisfied since $\hat{L}_0 P = 0$. By substituting (5.28) into (5.25b), we find the relation between E and $\mathfrak{c}$ for the two roots (5.28):

$$E^{\pm} = (P \pm \sqrt{N-1})\mathfrak{c}^{\pm} , \tag{5.29}$$

where we have defined the new variables $E^{\pm}$ and $\mathfrak{c}^{\pm}$ distinguishing the two solutions which correspond to the two roots $D^{\pm}_1$. The two solutions for $R^{(0)}_{1\{k\}}$, which we define to be $R^{(0)}_{1\{k\}\pm}$, are

$$R^{(0)}_{1\{k\}\pm} = \mathfrak{c}^{\pm}[\sigma_+ + (P \pm \sqrt{N-1})\sigma_3 a^\dagger] , \tag{5.30}$$

where the operators $\mathfrak{c}^{\pm}$ are unspecified functions of P. The first-order operators $R^1_{1\{k\}\pm}$ and $\omega^{(1)}_{1\{k\}\pm}$ are found from (5.28) and (5.30):

$$R^{(1)}_{1\{k\}\pm} = 0 , \tag{5.31}$$

$$\omega^{(1)}_{1\{k\}\pm} = P \pm \sqrt{N-1} . \tag{5.32}$$

By substituting these solutions into (2.21b) and (2.22b), we find that $R^{(2)}_{1\{k\}\pm}$ and $\omega^{(2)}_{1\{k\}\pm}$ are zero, which means that all higher-order terms identically vanish. The operator solutions $S_{1\{k\}\pm}$ and $\Omega_{1\{k\}\pm}$, where S_m and Ω_m are defined by (2.14) and (2.15), become

$$S_{1\{k\}\pm} = \mathfrak{c}^{\pm}[\sigma_+ + (P \pm \sqrt{N-1})\sigma_3 a^\dagger] \tag{5.33}$$

and

$$\Omega_{1\{k\}\pm} = \omega + \lambda[P \pm \sqrt{N-1}] . \tag{5.34}$$

Using these operators (5.33) and (5.34), we would like to find the time evolution of the operators $\sigma_+(t)$ and $a^\dagger(t)$. We will first look for $\sigma_+(t)$ by choosing

$$\mathfrak{c}^{\pm} = T^{\mp}/(T^+ - T^-) , \tag{5.35}$$

where $T^{\pm} = P \pm \sqrt{N-1}$. Since $T^+T^- = T^-T^+ = 1$, we can write (5.33) as

$$S_{1\pm} = (T^+ - T^-)^{-1}(T^{\mp}\sigma_+ + \sigma_3 a^\dagger) . \tag{5.36}$$

The operator σ_+ can now be written as

$$\sigma_+ = S_{1-} - S_{1+} . \tag{5.37}$$

After evolving to time t and using the solutions

$$S_{1\pm}(t) = e^{i(\omega+\lambda T^{\pm})t} S_{1\pm} , \tag{5.38}$$

we find

$$\sigma_+(t) = e^{i\omega t}[e^{i\lambda T^- t}(T^+ - T^-)^{-1}(T^+\sigma_+ + \sigma_3 a^\dagger) - e^{i\lambda T^+ t}(T^+ - T^-)^{-1}(T^-\sigma_+ + \sigma_3 a^\dagger)] . \tag{5.39}$$

By choosing

$$c^{\pm} = (T^{\mp})^2(T^+ - T^-)^{-1} , \tag{5.40}$$

and by rewriting

$$\begin{aligned}\sigma_+ + T^{\pm}\sigma_3 a^\dagger &= \sigma_+ + T^{\pm}(a^\dagger - 2P\sigma_+) \\ &= (1 - 2PT^{\pm})\sigma_+ + T^{\pm}a^\dagger \\ &= -(T^{\pm})^2\sigma_+ + T^{\pm}a^\dagger ,\end{aligned} \tag{5.41}$$

Eq. (5.33) becomes

$$S_{1\pm} = (T^+ - T^-)^{-1}(T^{\mp}a^\dagger - \sigma_+) . \tag{5.42}$$

The operator $a^\dagger$ can now be written as

$$a^\dagger = S_{1-} - S_{1+} . \tag{5.43}$$

After evolving to time t and using the solutions

$$S_{1\pm}(t) = e^{i(\omega+\lambda T^{\pm})t} S_{1\pm} , \tag{5.44}$$

we find

$$a^\dagger(t) = e^{i\omega t}[e^{i\lambda T^- t}(T^+ - T^-)^{-1}(T^+ a^\dagger - \sigma_+) - e^{i\lambda T^+ t}(T^+ - T^-)^{-1}(T^- a^\dagger - \sigma_+)] . \tag{5.45}$$

We would like to compare the solutions (5.39) and (5.45) with the solutions (4.61) and (4.62). For convenience, we will rewrite solution (5.39) by using the relation

$$T^{\pm}\sigma_+ + \sigma_3 a^\dagger = T^{\pm}\sigma_+ + a^\dagger - 2P\sigma_+ = -T^{\mp}\sigma_+ + a^\dagger . \tag{5.46}$$

Equation (5.39) becomes

$$\sigma_+(t) = e^{i\omega t}[-e^{i\lambda T^- t}(T^+ - T^-)^{-1}(T^-\sigma_+ - a^\dagger) + e^{i\lambda T^+ t}(T^+ - T^-)^{-1}(T^+\sigma_+ - a^\dagger)] . \tag{5.47}$$

By restricting Eqs. (4.61) and (4.62) to the case of resonance ($\omega = \omega_0$), we find that $r_\pm \to \lambda T^{\pm}$, and the solutions (4.61) and (4.62) become equal to the solutions (5.45) and (5.47), as they should.

Since we have not explicitly calculated the exact operator solution for $\sigma_3(t)$ in Sec. IV D, and since it can be calculated from a knowledge of $\sigma_+(t)$ from (5.47) and its Hermitian conjugate, we will not calculate it here. In addition, we would like to point out that we are able to find exact operator solutions relatively easily for this model only because the operator $P = H_1$ is an unperturbed and an exact constant of the motion. For most problems, exact operator solutions will not be found so easily.

VI. REMARKS

In this paper we have developed a systematic method of operator perturbation theory in the Heisenberg picture which is suitable for finding the time evolution of systems characterized by the Hamiltonian $H = H_0 + \lambda H_1$. We have required that H_0 consists only of a finite number of different fermion and/or boson creation and destruction operators. The significance of this requirement is that the time evolution of the solutions is exponential in form and characteristically periodic. It would be desirable to generalize the perturbation method to include models where H_0 consists of an infinite continuum of different fermion and/or boson creation and destruction operators, since a large number of physical models have a continuous spectrum. The difficulties which occur in generalizing the perturbation method are due primarily to two factors: (a) the quasidegenerate nature of the usual model Hamiltonians, and (b) the power-series expansion in the coupling constant λ which cannot be exactly written as two separate power-series expansions, one in amplitude and one in frequency. The quasidegenerate nature of the Hamiltonian leads to nonperiodic time evolution, which is usually approximately in the form of exponential decay for long periods of time. The nonexistence of separate power-series expansions in λ of the amplitude and frequency leads to the inability of this perturbation method's solutions to preserve the unitarity of the time evolution order by order in λ. In fact, for the case of a harmonic oscillator interacting with all modes of the electromagnetic field, it can be seen that terms in every order of λ are needed in order to preserve the unitarity of the time evolution.[13] Nonetheless, since the interacting time evolution, which is typical of most systems with a continuous spectrum, is exponential but with frequencies which have both a real and imaginary part, we suspect that a generalized perturbation method can be developed which will almost consist of two separate power-series expansions, one in amplitude and one in frequency. This generalized perturbation method for treating systems where H_0 has a continuous spectrum is presently being formulated.[14]

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APPENDIX

It is important to demonstrate that the canonical commutation relations (4.2) and (4.4) are preserved for all time by the solutions (4.41)–(4.43) to second order in λ. We will not compute the commutation relations (4.2) directly, however, but will verify the equivalent set of conditions

$$\sigma_-(t)\sigma_+(t)=\tfrac{1}{2}[1-\sigma_3(t)]\ , \tag{A1}$$

$$\sigma_+(t)\sigma_-(t)=\tfrac{1}{2}[1+\sigma_3(t)]\ , \tag{A2}$$

$$\sigma_3(t)\sigma_+(t)=\sigma_+(t)\ , \tag{A3}$$

$$\sigma_+(t)\sigma_3(t)=-\sigma_+(t)\ , \tag{A4}$$

$$[\sigma_+(t)]^2=0\ . \tag{A5}$$

In addition, we will verify the commutation relations

$$[a(t),a^\dagger(t)]=1\ , \tag{A6}$$

$$[\sigma_+(t),a^\dagger(t)]=0\ , \tag{A7}$$

$$[\sigma_+(t),a(t)]=0\ . \tag{A8}$$

We begin by substituting Eq. (4.41) and its Hermitian conjugate into the left-hand side of Eq. (A1):

$$\sigma(t)\sigma_+(t)=\{\sigma_-e^{i\lambda\alpha(2a^\dagger a+1)t}+\alpha\sigma_3 a(1-e^{-i(\omega-\omega_0)t})-\alpha^2[a^2\sigma_+(1-e^{-i(\omega-\omega_0)t})^2+\sigma_-(2a^\dagger a+1)(1-e^{-i(\omega-\omega_0)t})]\}$$
$$\times\{e^{-i\lambda\alpha(2a^\dagger a+1)t}\sigma_++\alpha(1-e^{i(\omega-\omega_0)t})a^\dagger\sigma_3-\alpha^2[(1-e^{i(\omega-\omega_0)t})^2\sigma_-(a^\dagger)^2+(1-e^{i(\omega-\omega_0)t})(2a^\dagger a+1)\sigma_+]\}\ . \tag{A9}$$

By combining terms in equal powers of λ, by neglecting terms of order higher than second, and by using conditions (A1)–(A8) at the initial time $t=0$, Eq. (A9) becomes

$$\sigma_-(t)\sigma_+(t)=\tfrac{1}{2}(1-\sigma_3)+\alpha[(1-e^{i(\omega-\omega_0)t})a^\dagger\sigma_-+\text{H.c.}]-\alpha^2[(e^{i(\omega-\omega_0)t}+e^{-i(\omega-\omega_0)t}-2)(\sigma_+\sigma_-+a^\dagger a\sigma_3)]=\tfrac{1}{2}[1-\sigma_3(t)]\ , \tag{A10}$$

where we have used Eq. (4.43) for $\sigma_3(t)$.

In order to verify Eq. (A7) we substitute Eqs. (4.41) and (4.42) into the left-hand side of (A7):

$$[\sigma_+(t),a^\dagger(t)]=e^{i(\omega+\omega_0)t}\{e^{-i\lambda\alpha(2a^\dagger a+1)t}\sigma_++\alpha(1-e^{i(\omega-\omega_0)t})a^\dagger\sigma_3$$
$$-\alpha^2[(1-e^{i(\omega-\omega_0)t})^2\sigma_-(a^\dagger)^2+(1-e^{i(\omega-\omega_0)t})(2a^\dagger a+1)\sigma_+]\}$$
$$\times[e^{-i\lambda\alpha\sigma_3 t}a^\dagger+\alpha(1-e^{-i(\omega-\omega_0)t})\sigma_++\alpha^2(1-e^{-i(\omega-\omega_0)t})\sigma_3a^\dagger]$$
$$-e^{i(\omega+\omega_0)t}[e^{-i\lambda\alpha\sigma_3 t}a^\dagger+\alpha(1-e^{-i(\omega-\omega_0)t})\sigma_++\alpha^2(1-e^{-i(\omega-\omega_0)t})\sigma_3a^\dagger]$$
$$\times\{e^{-i\lambda\alpha(2a^\dagger a+1)t}\sigma_++\alpha(1-e^{i(\omega-\omega_0)t})a^\dagger\sigma_3-\alpha^2[(1-e^{i(\omega-\omega_0)t})^2\sigma_-(a^\dagger)^2+(1-e^{i(\omega-\omega_0)t})(2a^\dagger a+1)\sigma_+]\}\ . \tag{A11}$$

By combining terms in equal powers of λ, by neglecting terms of order higher than second, and by using conditions (A1)–(A8) at the initial time $t=0$, Eq. (A11) becomes

$$[\sigma_+(t),a^\dagger(t)]=e^{i(\omega+\omega_0)t}(e^{-i\lambda\alpha(2a^\dagger a+1)t}\sigma_+e^{-i\lambda\alpha\sigma_3 t}a^\dagger$$
$$-e^{-i\lambda\alpha\sigma_3 t}a^\dagger e^{-i\lambda\alpha(2a^\dagger a+1)t}\sigma_+)$$
$$=0\ , \tag{A12}$$

where we find that the frequency operators in the exponentials cannot be trivially commuted with the amplitude operators, as was the case in verifying (A1). The time-dependent operator coefficients of each power of λ in amplitude are zero for all time.

In order to verify the remaining relations (A2)–(A6) and (A8), we must simply follow the same procedures as those used in verifying (A1) and (A7). These two relations were chosen as they explicitly illustrate different aspects of these types of calculations: the commutator equals an operator, the commutator equals a c number, and the frequency operators in the exponentials cannot always be trivially commuted with the amplitude operators.

We therefore find that to second order in λ the operator algebra, represented by equations (A1)–(A8), is preserved for all times.

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