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Quantum theory of a swept-gain laser amplifier

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The buildup of radiation from noise is considered in the case of the small-signal regime of a swept-gain amplifier with small Doppler width, of the sort indicated in recent schemes for an x-ray laser. The treatment uses the fully quantized electromagnetic field in order to take into account the spontaneous emission. The pulse is seen to grow nonexponentially with a rate that vanishes in the limit of large distances. The gain is generally much smaller than would be expected on the basis of the usual development. Many of the features normally associated with the small-signal regime of an amplifier, such as threshold conditions and spectral narrowing, are not seen. Instead we find spectral broadening, saturation, and the formation of steady states.

I. INTRODUCTION

In recent years there has been considerable interest in the development of lasers in the short-wavelength regime. Analysis of possible x-ray laser action was given by Duguay and Rentzepis,[1] and more recently by Lax and Guenther[2] and Bristow *et al.*[3] A scheme in which a high-current ion beam is swept at the speed of light along the length of an extended foil target, was proposed by McCorkle.[4] The ion-beam approach was used by Louisell, Scully, and McKnight[5,6] to propose a soft-x-ray laser which eliminates the Auger recombination problems. In addition they suggested the use of a gas target which allows a significant decrease in Doppler broadening. More recently a number of other schemes have been proposed which were discussed at length in a recent conference.[7]

The schemes involving the use of an ion beam[4,6] as the laser medium admit to detailed theoretical investigation. The ion-beam approach has two interesting features. First, it sweeps the excitation region at the speed of light so that the amplified spontaneous emission always encounters gain regions that have just been excited. This was noted from the beginning[1] to be a desirable feature of an x-ray laser scheme in view of the rapid decays that occur at short wavelengths. The second consequence is that the "thermal" velocity $\bar{u}$ can be quite small, especially when the ion beam is excited through collisions with a gaseous target.[6] This reduces the large Doppler widths and hence increases the gain.

When one decreases the Doppler width, one arrives at a circumstance that is not normally encountered in lasers and laser amplifiers. If the Doppler broadening is small, then the inverse bandwidth of the amplifiers goes like the spontaneous lifetime, which is the same as the decay time of the population inversion. In any laser or amplifier, the gain is determined directly by the polarization[8,9] of the medium and only indirectly by population inversion. The gain is given by the usual considerations only for times long compared to the rise time of the polarization (which is given by the inverse bandwidth). The gain formulas used in Refs. 1–6 are all variants of the usual expressions[9–11] and thus are suitable only if the Doppler broadening is large. In a recent publication[12] it was shown that the growth rate of pulses in the absence of Doppler broadening is less than one would expect on the basis of the conventional formulas. The reason for this is straightforward, namely, that, as the gain builds up, it encounters and follows the decay of the population inversion. If the rise and fall times are comparable, the result is a time-dependent gain function whose maximum is always smaller than the gain value estimated by taking the population inversion to be slowly varying compared to the rise time of the gain. This reduction in expected growth rates can range from factors of 3 to several orders of magnitude depending on the specific case. If one ignores extreme cases, then it is typically reduced by an order of magnitude.

In this paper we explore the small-signal regime of a traveling-wave, slow-rise-time amplifier in more detail than was done previously. In addition to the gain effects, one sees a number of other phenomena occurring. The usual "gain narrowing"[10] is greatly suppressed and is frequently replaced by a broadening of the pulse spectrum. This is significant in view of the discussion in Ref. 2 with respect to the importance of gain narrowing in the diagnostics of x-ray laser action.

In addition one sees the formation of coherent or bandwidth-limited pulses.[13] These temporal effects are associated with a nonexponential growth of energy that typifies these cases. The nonexponential growth is such that the growth rate decreases monotonically as a function of distance. In the presence of a loss this decrease causes the system to form steady states in the small-signal regime. In general then, the lethargic response of the atoms causes the small-signal regime of the amplifier to behave much like a nonlinear regime of an ordinary amplifier and to lose many of the properties normally associated with linear amplification. The amplifier does not, however, have all of the properties of nonlinear amplification. There is, for example, no nutation or ringing effects.[9] More importantly, there is no "coherence brightening,"[14] which appears in the nonlinear regime as a shift of the peak of the pulse and a buildup of peak power. In the earlier discussion of this problem,[12] the nonlinearities were retained, and it was seen that ordinary large-signal saturation effects are greatly suppressed by the coherence brightening. Thus, for sufficiently large amplification lengths, it is possible to obtain substantially the same output energy from these amplifiers as would be predicted on the basis of ordinary amplifier theory. The lowered gain in the small-signal regime is compensated for by the weakening of large signal saturation effects.

The investigation into this type of amplifier is made more complicated by the fact that none of the usual analytic techniques was applicable. Furthermore, this is a tedious problem to treat semiclassically, since there one is forced to include spontaneous emission in a heuristic fashion.[15] In the present case we choose to work directly with the fully quantum-mechanical field. We write the equations in the Heisenberg picture in the small-signal regime and solve them numerically using a procedure that parallels the numerical techniques that are applied to the semiclassical problem. In the small-signal regime such an approach is reasonably straightforward and involves solving numerically for the Green's function or propagator that expresses the operator at any space-time point in terms of the operators on the boundaries of the space-time region. The extension of these methods to the nonlinear regime is nontrivial and will be left for later consideration. After much of the work reported here was completed, we found that certain limiting cases could be solved analytically. The analytic solutions were used to qualitatively confirm the numerical results, but they are somewhat inpenetrable. We intend to present them in a later paper.

The paper is divided into six sections with this introduction being the first. In Sec. II we develop the field operator equations that we use in the later parts. We follow the methods discussed by Graham and Haken[16] in order to introduce the slowly varying amplitude and phase approximations[17] and the plane-wave approximation that are needed to make the problem tractable. In Sec. III we discuss how the numerical procedure works. We then go on and show a specific example which behaves in a fashion that represents all of the cases we discuss. In Sec. III C we show how "coherent" or bandwidth-limited pulses are formed in the amplifier. In Sec. IV A we discuss the issue of the gain in considerable detail. We are not interested here in the formula for the gain which arises in the detailed investigation of specific x-ray laser schemes. We define a quantity called the estimated gain (g_{est}) which is taken to be given by some other calculation. We show how it is possible to define a new parameter g_{eff}, which takes into account the lethargic rise time of the amplifier. This effective gain estimates the actual growth rate (g_{act}) more reliably than the conventional formula does. In Sec. IV B we do a parameter study of the general problem including the effects of Doppler broadening, finite duration of the excitation, and different level schemes. The effect of a finite loss is taken up in Sec. V as a special case since the proper understanding of the saturation effects and steady states requires extra care. The Sec. VI is a summary.

II. FIELD OPERATOR EQUATIONS OF MOTION

Present concepts of an x-ray "laser"[1-6] involve traveling-wave amplifiers rather than true laser oscillators. Thus they are similar to the H_2 and N_2 lasers.[18] From a theoretical viewpoint this means that the electromagnetic field cannot be simulated by a few discrete field modes, and it is necessary to take into account the continuous spectrum of the electromagnetic field in order to have a good description of the system.

Further, owing precisely to the fact that the process is a buildup from noise (amplified spontaneous emission),[19] we describe the electromagnetic field quantum mechanically in order to have the source terms (i.e., the contribution of the spontaneous emission) *a priori* included in the equations of motion. In order to simplify the problem and obtain tractable equations of motion, we shall limit our discussion to a unidirectional amplifier along the z axis, and we assume that the relevant part of the radiation field is along that axis only.[16] This ansatz is inappropriate in the case of spontaneous emission, which goes into all the modes space, which means that the magnitude of the source terms in the final equations are not necessarily correctly defined. However, we note that in a linear theory the magnitude of the field (which is proportional to the magnitude of the source terms) is unimportant, and that the only consequence of this difficulty is that all of our results will be given in some arbitrary set of units.

In addition to that ansatz, we introduce an approximation which is the quantum-mechanical analog to the semiclassical slowly varying amplitude and phase approximation. As a result we shall obtain operator equations of motion which are, except for the source term, the exact replica of the semiclassical equations used in pulse-propagation theory.[9,20] We shall achieve this end by using a formalism developed to describe electromagnetic fields with continuous spectra centered around a central frequency ω_0 and a central wave number k.[16] If the significant part of the vector potential is sharply centered around frequency ν and wave vector k_0, it can be conveniently decomposed into a slowly varying part $\mathcal{A}^{\pm}(t,z)$ and a rapidly oscillating part as

$$\mathcal{A}^{\pm}(t,z)=\mathcal{A}\,e^{\pm i(k_0 z-\nu t)}, \tag{2.1}$$

where

$$\left|\frac{\partial \mathcal{A}^{\pm}(t,z)}{\partial t}\right| \ll \nu\,|\mathcal{A}^{\pm}(t,z)|,$$
$$\left|\frac{\partial \mathcal{A}^{\pm}(t,z)}{\partial z}\right| \ll k_0\,|\mathcal{A}^{\pm}(t,z)|. \tag{2.2}$$

In the frame of these approximations the slowly varying electric field operator is related to the slowly varying vector potential by the simple relation

$$\mathcal{E}^{\pm}(t,z)=\pm i(\nu/c)\mathcal{A}^{\pm}(t,z). \tag{2.3}$$

We describe the amplifier medium as a set of two-level systems of average frequency ν, located along the z axis. We will consider two different level schemes: In one of them, the lower level is a ground state and the upper level is allowed to decay into the ground state only (two-level amplifier), when in the other scheme, the lower level is also an excited state (three-level amplifier) (see Table I, Sec. IV B, for a sketch of these models). These two models differ mathematically by the fact that in the first one, the decay rate γ_b of the lower level $|b\rangle$ is equal to zero, and γ_a is the rate at which the upper level decays into the lower level, when in the latter one, γ_a and γ_b, which are taken to be the same in this derivation and will be denoted as $\gamma_a=\gamma_b=1/T_1$, give the rate of radiative decay into some distant ground state. We shall see that as far as light amplification is concerned, these two models have very different behavior, related to the fact that after a certain time the two-level amplifier becomes strongly absorbing, leading to a much lower gain than the three-level amplifier.

We first assume that the medium is homogeneously broadened and that all of the atoms at the position z are "excited" at the time $t=z/c$. The generalization to include inhomogeneous broadening and a finite temporal duration to the excitation are straightforward and will be presented at the end of this section. The Hamiltonian of the amplifying medium then reads

$$\mathcal{H}_A=\hbar\omega\sum_j \sigma_j^{+}\sigma_j, \tag{2.4}$$

where σ_j^{+},σ_j are the spin-flip operators of the jth atom. The inversion N_i of the ith atom is given by

$$N_i\equiv[\sigma_i^{+},\sigma_i]. \tag{2.5}$$

We assume that all of the atomic dipoles have the same direction $\vec{u}_d$, and from now on, $\mathcal{A}(t,z_j)$ will denote the scalar product of $\mathcal{A}$ with $\vec{u}_d$ evaluated at the location of the jth atom. In the dipole and rotating-wave approximations, the atom-field interaction reads[21,22]

$$\mathcal{H}_{AF}=\frac{\omega\wp}{ic}\sum_j(\sigma_j^{+}A^{+}-A^{-}\sigma_j^{-}). \tag{2.6}$$

z_j is the location of the jth atom, and $\wp$ is given by

$$\langle a_j|\frac{e}{m}\hat{p}_j|b_j\rangle=\frac{i}{\hbar}\langle a_j|[\mathcal{H}_A,e\vec{r}]|b_j\rangle$$
$$=i\omega\langle a_j|e\vec{r}|b_j\rangle=i\omega\wp\vec{u}_d, \tag{2.7}$$

where $\wp$ is real and the direction $\vec{u}_d$ of $\langle a_j|e\vec{r}|b_j\rangle$ is already included in the scalar operator $A(t,z_j)$. At this point, it is convenient to introduce the operators

$$\sigma(z,t)=\sum_j\sigma_j(t)\delta(z-z_j) \tag{2.8}$$

and

$$N(z,t)=\sum_j N_j\,\delta(z-z_j). \tag{2.9}$$

Further, we introduce the slowly varying spin-flip operator[17]

$$S^{+}(t,z)=\sigma^{+}(t,z)e^{i(k_0 z-\nu t)} \tag{2.10}$$

and its Hermitian conjugate $S(t,z)$. We note that the inversion operator $N(t,z)$ is intrinsically slowly varying. With this notation, the equations of motion for the scalar electric field operator $\mathcal{E}^{+}(t,z)$, $N(t,z)$, and $S^{\pm}(t,z)$ read, respectively,[23]

$$\frac{1}{c}\dot{\mathcal{E}}^{+}(t,z)+\frac{\partial}{\partial z}\mathcal{E}^{+}(t,z)+\kappa\mathcal{E}^{+}(t,z)=2\pi\frac{i\nu}{c}\wp S(t,z), \tag{2.11}$$

$$\dot{S}^{+}(t,z)=\left(i(\omega-\nu)-\frac{1}{T_2}\right)S^{+}(t,z)-\frac{\wp}{i\hbar}\,\mathcal{E}^{-}(t,z)N(t,z)\,, \tag{2.12}$$

$$\dot{N}(t,z)=-\frac{N(t,z)}{T_1}-[\mathfrak{N}_0/T_1]-\frac{2\wp}{i\hbar}\left[S^{+}(t,z)\mathcal{E}^{+}(t,z)-\mathcal{E}^{-}(t,z)S(t,z)\right]. \tag{2.13}$$

The equations of motion for $E^{-}(t,z)$ and $S(t,z)$ are the Hermitian conjugates of (2.11) and (2.12), respectively. The medium excitation is contained in the initial condition for $N(t,z)$, which is given at $t=z/c$ as $N(t,z)=\mathfrak{N}(z)$, where $\mathfrak{N}$ is the excitation density[9] in the amplifier (we follow Ref. 16 here in treating the inversion operator as a c number in the small-signal regime). The term in the square brackets is included only in the two-level model, and is appropriate for describing decays that go directly to the lower state. We note that the decay times T_1 and T_2 have been introduced phenomenologically in these equations, as well as the loss denoted as κ. They could have been obtained in a more rigorous manner by considering the coupling of the atomic system with heat baths[16,17,24] or by the usual Wigner-Weisskopf approach.[22] In the present work we are not interested in the detailed mechanisms of decay (from a microscopic viewpoint) and will adjust these constants to appropriate values.

In the present work we are interested in the linear regime of the amplifier only, i.e., the regime in which the population inversion is not modified by the action of the field. We thus neglect the last term in Eq. (2.13). We take the population inversion to be produced by a δ-function excitation swept at the speed of light along the (unidimensional) amplifier. In the case of a two-level laser system, the linear population inversion can then be written as

$$N(\mu,z)=\left(\frac{F+1}{F}\,e^{-\mu/T_1}-\frac{1}{F}\right)\mathfrak{N}(z)u(\mu)\,. \tag{2.14}$$

μ is the retarded time $\mu=t-z/c$, $u(\mu)$ is the unit step function, $T_1=\gamma_a^{-1}$ is the decay time of the upper level $|a\rangle$, and F is the fractional population inversion at time $\mu=0$.

$$F=\mathfrak{N}(z)/\mathfrak{M}(z)\,, \tag{2.15}$$

where $\mathfrak{M}(z)$ is the density of atoms in the amplifier. In the case of a three-level amplifier, both states $|a\rangle$ and $|b\rangle$ decay to distant ground states, and the linear population inversion reads

$$N(\mu,z)=\mathfrak{N}u(\mu)e^{-\mu/T_1}\,. \tag{2.16}$$

One then formally integrates Eq. (2.12) using the appropriate function $N(\mu)$, and substitutes that into Eq. (2.11) to get

$$\frac{\partial}{\partial z}E^{+}(\mu,z)=\alpha'\int_0^{\mu}d\mu'\,E^{+}(\mu',z)e^{-(\mu-\mu')/T_2}\frac{N(\mu,z)}{\mathfrak{N}}+2\pi\frac{i\nu_0\wp}{c}S(0,z)e^{-\mu/T_2}\,, \tag{2.17}$$

where $\alpha'=4\pi\wp^2\omega_0\mathfrak{N}/c\hbar$. Here the convective derivative $\partial/\partial z+(1/c)\partial/\partial t$ that appears in Eq. (2.11) simplifies to $\partial/\partial z$ by virtue of writing the equations in the time-retarded frame.[20]

We now proceed to generalize the above equations for the case of an inhomogeneously broadened medium and of a pump having a finite temporal extension T_p, and characterized by a pump function $P(\mu_0)$, such that $\int d\mu_0 P(\mu_0)=1$. The inclusion of the Doppler broadening is obtained by going back to the atomic Hamiltonian (2.4) and replacing ω by ω_j (including it inside the sum). Then, instead of defining $\sigma(z,t)$ as in (2.10), we define

$$\sigma(z,t,\omega)=\sum_j^{(\omega)}\sigma_j\delta(z-z_j)\,, \tag{2.18}$$

where the sum is restricted to the ensemble of atoms having the same frequency ω. $N(z,t,\omega)$ is defined in the same way. The net result is then obtained by summing the contributions of the different frequencies over the Doppler distribution $\sigma(\omega)$. Further, in order to include the effects of an extended pump, it is necessary to indicate at which time μ_0 a given atom is "excited" (or created). Thus

$$\sigma(z,t,\omega)\to\sigma(z,t,t_0,\omega)\,. \tag{2.19}$$

The slowly varying spin-flip operators (2.10) are labeled by four parameters. Defining (in the retarded frame)

$$S(\mu=\mu_0,z,\mu_0,\omega)\equiv S(z,\mu_0,\omega)\,, \tag{2.20}$$

we obtain

$$\frac{\partial\mathcal{E}^{+}}{\partial z}(\mu,z)=\alpha\int_{\mu_0}^{\mu}d\mu'\,\mathcal{E}^{+}(\mu',z)D(\mu-\mu')e^{-(\mu-\mu')/T_2}\int_0^{\mu'}d\mu_0\,P(\mu_0)\,\frac{N(\mu'-\mu_0)}{\mathfrak{N}}+2\pi i\nu_0\frac{\wp}{c}\int d\omega\,\sigma(\omega)\int_0^{\mu}d\mu_0\,P(\mu_0)S(z,\mu_0,\omega)e^{-(\mu-\mu_0)/T_2}\,, \tag{2.21}$$

where $\alpha = \alpha' \sqrt{\pi} T_2^*$, and T_2^* is the dephasing time of the atomic ensemble. $D(\mu - \mu')$ is the Fourier transform of the Doppler distribution $\sigma(\omega)$ and is explicitly given by[9]

$$D(\mu - \mu') = \frac{1}{\sqrt{\pi}\, T_2^*}\, e^{-(\mu-\mu')^2/T_2^{*2}} . \tag{2.22}$$

Depending on which model we are interested in, we use Eq. (2.21) in conjunction with Eq. (2.14) or Eq. (2.16).

III. NATURE OF SOLUTION

In this section we discuss the various aspects of the solution of the operator equations in order to establish a foundation for the parameter study in Sec. IV. In the first part of this section we discuss the numerical solution and show how the quantum-mechanical and semiclassical solutions are related. In the second subsection we present a particular calculation in some detail to show the general behavior of pulses in these amplifiers. In Sec. III C we discuss the formation of bandwidth-limited pulses in the amplifier.

A. Numerical-solution method

We know of no manner by which the field operator equations developed in Sec. III can be solved analytically except in certain limiting cases, which will be discussed in a subsequent publication. Up to the present time we have exploited the numerical solution of a semiclassical model to discuss these problems.[12] In that case the spontaneous emission is treated with an ansatz involving appropriately chosen stochastic functions.[15] In the present treatment we use a direct numerical solution of the operator equations. This technique is considerably more satisfactory than the semiclassical methods and is also competitive in time and effort involved when one takes into account the necessity in the semiclassical approach of taking an average over many numerical calculations.

The solution of the operator equation is based on the same numerical procedure as in the semiclassical case. The space-time region over which the solution is taken is gridded into finite steps. Standard numerical techniques,[26] in this case the trapezoidal rule for integration in time and second-order predictor corrector in the spatial dimension, are used in the numerical procedure. These reduce the integro-differential equation to a series of recursive algebraic steps in the numerical code. The operators are algebraic rather than numerical objects, so when the numerical procedure calls for the addition (or multiplication, which is not needed in the present case except for multiplying operators by c numbers), one must use the appropriate algebraic rules. The numerical method winds up constructing two propagators G_1 and G_2 which express the operator at any time-space point in terms of the operators on the spatial and temporal boundaries. That is,

$$\mathcal{E}^+(\mu, z) = \int d\mu_0\, d\omega\, dz'\, G_1(\mu, z; z', \omega, \mu_0) S(z', \omega, \mu_0) + \int d\mu'\, G_2(\mu, z; \mu') \mathcal{E}^+(\mu', 0) . \tag{3.1}$$

The method we use then is equivalent (in the case of a linear integro-differential equation) to a Green's-function approach. Since this approach is more widely known than the numerical techniques themselves, we would like to continue the discussion as if we had used the Green's function from the beginning. Before proceeding, we note that there is a great deal of either useless or redundant information in Eq. (3.1). In particular, the wave function in the present case is the vacuum insofar as the light is concerned, so that all expressions containing $\mathcal{E}^+(\mu, 0)$ or $\mathcal{E}(\mu, 0)$ or their normally ordered products will vanish, when expectation values are taken. Thus the term involving $G_2(\mu, z'; \mu')$ can be discarded from the solution from the beginning. We then look for a solution of the general equation

$$\frac{\partial \mathcal{E}^+}{\partial z}(\mu, z) = \int_0^\mu d\mu'\, K(\mu, \mu') \mathcal{E}^+(\mu', z) + \int H(\mu, q) S(z, q)\, dq - \tfrac{1}{2}\kappa \mathcal{E}^+(\mu, z) , \tag{3.2}$$

where we have used the single variable q to represent μ_0 and ω, and K and H are general functions of the indicated variables, of the form

$$\mathcal{E}^+(\mu, z) = \int dq\, dz'\, G(\mu, z; z', q) S(z', q) . \tag{3.3}$$

One then proceeds by substituting Eq. (3.3) into Eq. (3.2). The operator $S(z, q)$ is an algebraic entity, so that terms involving $S(z, q)$ can be added to those involving $S(z', q')$ if and only if $z = z'$, $q = q'$. Thus the operators can be viewed as being orthogonal to each other from the standpoint of projecting out of the resulting equation those terms which contribute to the z', q' piece of the Green's function. One then gets

$$\frac{\partial G(\mu, z; z', q)}{\partial z} = \int_0^\mu d\mu'\, K(\mu, \mu') G(\mu', z; z', q) + H(\mu, q)\delta(z - z') - \tfrac{1}{2}\kappa G(\mu, z; z', q) . \tag{3.4}$$

This is now a purely numerical equation which can, in principle, be integrated numerically. In practice the computer time and storage requirements for G in the form given above would be unmanageable. There is, however, a considerable amount of redundancy left in this equation. In particular, G is a function of $z-z'$. One makes use of this fact by noting that any information regarding $z'\neq 0$ can be constructed from the case $z'=0$. Hence we solve only for the function $G^0(\mu, z; q)$, where

$$G^0(\mu, z; q)=G(\mu, z; 0, q) \tag{3.5}$$

and

$$\frac{\partial G^0(\mu, z; q)}{\partial z}=\int_0^\mu d\mu' K(\mu, \mu')G^0(\mu', z; q) -\tfrac{1}{2}\kappa G^0(\mu, z; q)+H(\mu, q)\delta(z). \tag{3.6}$$

The inhomogeneous term then appears only in the first step of the integration procedure (i.e., $z=0$), and plays a role similar to an initial condition. One needs to be careful in properly introducing it into the calculation, but once that is done, the rest is the same as the usual methods that are applied to the (noiseless) semiclassical pulse-propagation equations. We note that if one replaces the inhomogeneous term with an initial condition, and $G^0(\mu, z; q)$ with the semiclassical amplitude $E(\mu, z)$, then Eq. (3.6) is the equation of motion describing the semiclassical problem that is the analog of the quantum-mechanical one, and the semiclassical codes can be modified in a straightforward manner to carry out the calculation.

The interesting physical quantities are the expectation values of the field intensity, the power spectrum, and the resulting energies, widths, etc. The field intensity is given by

$$\langle I(\mu, z)\rangle=\langle \mathcal{E}^-(\mu, z)\mathcal{E}^+(\mu, z)\rangle \tag{3.7}$$

and, by Eq. (3.3),

$$\langle I(\mu, z)=\int_0^z dq\,dq'\,dz'\,dz''[G^0(\mu, z'; q)]^*G^0(\mu, z'', q') \times \langle S^+(z', q)S(z'', q')\rangle . \tag{3.8}$$

In the present case we have that

$$\langle S^+(z', q)S(z'', q')\rangle=\mathfrak{N}|a|^2\delta(q-q')\delta(z-z'), \tag{3.9}$$

where $|a|^2$ is the probability of finding the atom in the upper state. In the present case it is proper to take $|a|^2$ to be independent of μ_0, z, and ω. Then

$$\langle I(\mu, z)\rangle=\mathfrak{N}|a|^2\int_0^z dz'\int dq\,|G^0(\mu, z'; q)|^2 . \tag{3.10}$$

The pulse energy is then

$$\langle \mathcal{T}(z)\rangle=\int_0^\infty d\mu\,\langle I(\mu, z)\rangle . \tag{3.11}$$

Similarly one gets the power spectrum as

$$\langle \tilde{I}(\delta, z)\rangle =\mathfrak{N}|a|^2\int_0^z dz'\int dq\left|\int d\delta e^{-i\delta\mu}G^0(\mu, z'; q)\right|^2 . \tag{3.12}$$

The spectral and temporal widths are taken to be the full width at half-maximum of the functions $\langle I(\mu, z)\rangle$ and $\langle \tilde{I}(\omega, z)\rangle$, respectively. One could go on and discuss the higher moments of the field as well, but the statistics are Gaussian and the higher moments have a simple relationship to the quantities constructed here.

B. Example of homogeneously broadened three-level amplifier

We now proceed to present the physical results obtained by the numerical computation. We shall first limit our discussion to the case of an amplifier consisting of homogeneously broadened atoms which decay to a distant ground state (three-level model), and shall assume for now that the pump consists in a true δ-function excitation swept at the velocity of light ($T_p=0$).[4,6] For the present case we have taken $T_1=T_2$, and have chosen the gain parameter α' and length z to be such that $2\alpha'T_2L=32.6$, where the conventional gain calculation in a homogeneously broadened medium gives $g_{\text{est}}=2\alpha'T_2$. In Fig. 1(a) we show the development of the shape of the pulse as it grows and propagates along the amplifier. For short distances ($z\sim 0$), the field is due to the spontaneous emission exclusively, and its shape exhibits a simple exponential decay. The shape halfway down the amplifier and at the output ($z=L$) shows the characteristic reshaping that occurs in the presence of gain. By the end of the amplifier the pulse energy has grown to about 100 times the value it would have in the absence of gain. Beyond this the pulse continues to reshape, but much less drastically. We have usually confined our investigation to the sort of gain lengths shown in Fig. 1, since one learns little about the pulse properties from extending the calculation still further.

In Fig. 1(b) we show the spectrum at the two ends of the amplifier. One notes that except for a suppression of the wings of the spectrum, there is little change due to the amplification. In particular, unlike ordinary amplification processes, there is no "gain narrowing."[10] This occurs because the rapid decay processes limit the time duration of the gain. This constrains the pulse width and hence the spectral width.

One of the problems that will occur throughout this investigation is the difficulty of defining gain. In Fig. 2(a) we show the energy on a logarithmic scale as a function of distance for the example given in Fig. 1. One sees the usual buildup from noise followed by a relatively flat portion indicating the usual sort of growth in the pulse energy. The slope of this curve is smaller than what one expects on the basis of the usual gain estimate by about a factor of 6. It is clear that the conventional gain estimate should not be used in this problem since it is both inaccurate and imprecise. When one tries to find an appropriate gain parameter, one runs into a difficulty which is illustrated in Fig. 2(b). There the energy is plotted for a length 20 times as long as in 2(a). One sees that there is a small but persistent curvature to the energy curve. This shows that the growth of the energy is not exponential in the small-signal regime. In the insert in Fig. 2(b), we plot g_{act}, where

$$g_{\text{act}} = \frac{1}{\mathcal{T}(z)} \frac{d\mathcal{T}(z)}{dz} \tag{3.13}$$

is the slope of the energy curve as a function of $\ln\mathcal{T}$. In an ordinary amplifier,[9-11] this curve would be flat, except for the "infinity" at the left edge due to the buildup from noise; but in the present case, it is a monotonically decreasing function of distance, tending to zero in the limit of infinite

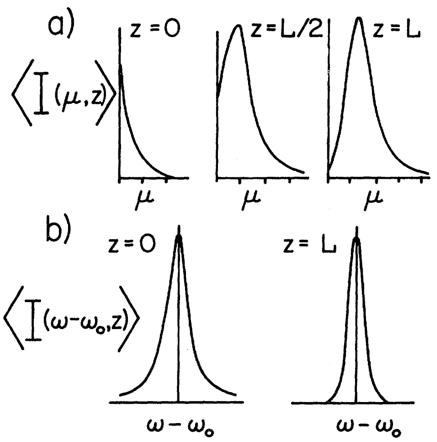


FIG. 1. (a) Plot of pulse intensity $\langle I(\mu,z)\rangle$ vs the retarded time μ, for three positions $z=0$, $L/2$, L in the amplifier. (b) Plot of power spectrum $\langle I(\omega-\omega_0,z)\rangle$ vs $\omega-\omega_0$ for two positions $z=0$, L in the amplifier. In all the plots the vertical scales are different and are given in arbitrary units. The calculations performed here were done for the case of a homogeneously broadened medium with $T_1=T_2$, $T_p=0$, and in which the decay processes go to distant ground states.

z. There is then no well-defined gain for this problem. There is, however, an energy $\mathcal{T}$ above which the nonlinearities will become important and the value of the linear gain is no longer pertinent. As shown in an earlier publication,[12] one must not try to interpret the behavior of the pulse in the nonlinear regime in terms of the growth rate in the linear regime. This sets a practical upper limit to the value of the energy beyond which g_{act} is irrelevant. With the exception of very short distances, one can take the distance-dependent gain to be defined to within about a factor of 2. The vertical bar in the small insert in Fig. 2(b) represents the point $z=L$ in Fig. 1(a). This length represents a reasonably average value for g_{act}, and whenever numbers are presented, they will be given for this length.

C. Formation of bandwidth-limited pulses

One of the characteristics of pulse amplification in the case of rapid gain decay is the formation of bandwidth-limited pulses. In semiclassical terms, this means that there are no temporal fluctuations in the amplitude and phase of the pulse. Since all the examples given in this paper evolve in essentially the same way, we discuss only one example in detail.

The notion of bandwidth-limited pulse is a relatively straightforward (although occasionally imperfect) test to tell whether all of the spectral components found in the spectrum are accounted for in the temporal fluctuations implied by the dependence of $\langle I(\mu,z)\rangle$. If this is the case, the spectrum will be accounted for by the Fourier transform of the field amplitude, which is given by

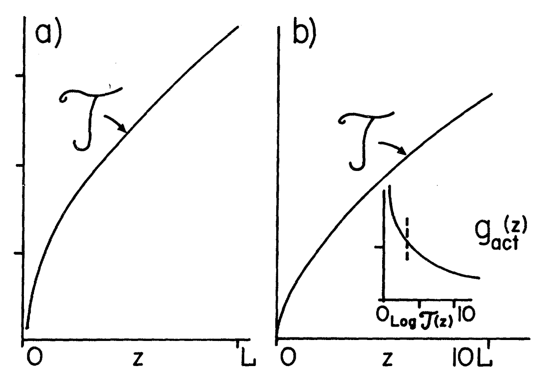


FIG. 2. (a) Energy vs distance on a logarithmic scale for the case shown in Fig. 1. (b) Energy vs distance on a logarithmic scale for the same case except that much larger distances are considered. The slope of the curve in (3b) called g_{act} is plotted as a function of $\ln\mathcal{T}$. The vertical line is the position $z=L$, which is the value used whenever g_{act} is given as a number.

$\langle I(\mu, z)\rangle^{1/2}$. In nonlinear problems one must worry about the possibility of nonrandom 180° phase changes[9, 20] such as occur in "zero" π pulses which require sign changes in the amplitude. Fortunately, these phase flips play no role in the small-signal regime of an amplifier and can be ignored here. One can then construct a test spectrum $I_{\text{test}}(\delta, z)$ such that

$$I_{\text{test}}(\delta, z) = \left| \int d\mu\, e^{-i\delta\mu} \langle I(\mu, z)\rangle^{1/2} \right|^2 . \tag{3.14}$$

A pulse is considered to be bandwidth limited or "coherent" in the language of semiclassical pulse propagation if I_{test} is the same as $\langle I(\delta, z)\rangle$.

The example given in the previous subsection is not a particularly illuminating case, since it is bandwidth limited throughout its evolution. We give, instead, a case discussed in detail later on, which is the same as the one in Sec. III B except that some Doppler broadening is included ($T_2^* = 0.3T_1$). In Fig. 3 the spectrum (solid line) and test spectrum (dashed curve) are shown as a function of distance. The scales of the spectra have been adjusted to have equal peaks for ease in presentation. The case $z=0$ shows a substantial difference between the two spectra. Thus there are substantial fluctuations within the over-all pulse width. At $z=L$ the two spectra are essentially the same, and the pulse is bandwidth limited.

In ordinary amplification processes (i.e., $T_1 \gg T_2$ or T_2^*), such pulses do not form in the linear regime of the amplifier since the width of the pulse and the time scale of the fluctuations tend to increase at the same time. In the presence of rapid gain decay the pulse widths are limited in duration and the substructure can be eliminated.

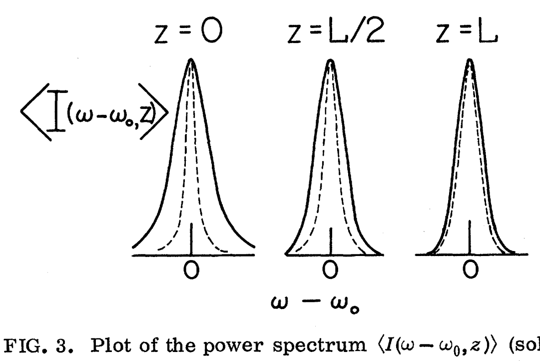


FIG. 3. Plot of the power spectrum $\langle I(\omega-\omega_0, z)\rangle$ (solid curves) and the test spectrum described in Eq. (3.14) (broken curves) vs frequency $\omega - \omega_0$. The plots are given at $z=0$, $L/2$, L. The vertical scales are arbitrary and are adjusted for display purposes to have the same heights at the maxima. The curves are given for an inhomogeneously broadened medium with $T_2^* = 0.3T_1$, with the other properties the same as in the case of Fig. 1.

IV. PARAMETER STUDY

In this section we discuss how the properties of pulse amplification vary with the parameters that define the active medium. The most important feature of this study is that the growth rates vary wildly from one case to another, and, as shown in Sec. III B, these rates are not well estimated by the conventional formula. In addition, the rates are distance dependent and hence not well defined. In Sec. IV A we discuss the issue of growth rates and show how the discrepancy between the actual rate and the rate given by the conventional formula, varies with changing parameters. In Sec. IV B we keep the growth rates fixed as well as we can, and show how the temporal and spectral widths vary with changing parameters.

A. Gain considerations

The issue of gain has received exhaustive attention in the past under conditions in which the small-signal growth is characterized by Beer's Law, i.e., in which

$$\frac{d\langle I(\delta, z)\rangle}{dz} = g(\delta)\langle I(\delta, z)\rangle . \tag{4.1}$$

This formula then leads to the "gain-narrowing" process and allows one to define a gain for the amplifier. The Beer's-law formula, or some variation of it, follows from Eq. (2.22) whenever the small-signal ensemble average population inversion varies slowly in time compared to the rise time of the gain. This is guaranteed in all types of amplifiers that have been considered up until now by one of the following conditions being true: $T_1 \gg T_2^*$, $T_1 \gg T_2$, $T_p \gg T_1$. In this paper we are discussing the behavior of the amplifier when the rise time of the gain is comparable to the decay of the inversion, and, as was shown in Sec. III B, the pulse behaves in a fashion that is incompatible with Beer's law.

In order to discuss the subject of gain, it is useful to treat the system semiclassically.[9] Thus we take Eq. (3.6) to be the equation of motion for the pulse itself. We replace $G^0(\mu, z; q)$ with a semiclassical amplitude $\mathcal{E}(\mu, z)$, and replace the inhomogeneous term with a suitable initial condition, i.e.,

$$\frac{\partial \mathcal{E}}{\partial z} = \int_0^\mu d\mu' K(\mu, \mu')\mathcal{E}(\mu') . \tag{4.2}$$

Here $K(\mu, \mu')$ represents all of the parts of the homogeneous term in the right-hand side of Eq. (2.21) except for the field operator. The discussion of the derivation of the gain coefficient from the time-domain description in Eq. (4.2) is equivalent to the one in the frequency domain, using a two-step procedure. One removes that part of $K(\mu, \mu')$ that has its origin in the behavior of the inversion from inside the integral and evaluates it at $\mu = 0$. Then one removes the amplitude $\mathcal{E}(\mu', z)$ from the integral (this step follows from the previous one) and evaluates the remaining integral in the limit $\mu \to \infty$. This leads to an estimated gain, which in the present notation reads

$$g_{\text{est}} = 2\alpha \int_0^\infty d\mu\, e^{-\mu/T_2} D(\mu). \tag{4.3}$$

We have specifically excluded possible effects of the finite excitation [i.e., $P(\mu_0)$] from this formula, since it does not appear in the expressions for gain currently used in the ion-beam schemes.

In the cases we are concerned with here, the approximations involved in obtaining g_{est} are invalid. In order to obtain an alternative estimate of the gain we proceed in the following manner. For any given field $\mathcal{E}(\mu, z)$, we can rewrite Eq. (4.2) as

$$g\frac{\partial \mathcal{E}}{\partial z} = \frac{g_{\mathcal{E}}(\mu)}{2}\mathcal{E};$$

$$g_{\mathcal{E}}(\mu) = \frac{1}{\mathcal{E}(\mu, z)} \int_0^\mu d\mu' K(\mu, \mu') \mathcal{E}(\mu', z), \tag{4.4}$$

where $g_{\mathcal{E}}(\mu)$ is a functional of $\mathcal{E}$. This function $g_{\mathcal{E}}(\mu)$ describes the instantaneous growth of the pulse. In practice we see that the field changes [see Fig. 1(a)] and hence the gain changes. We choose to investigate the simplest case, namely, that of a constant field $\mathcal{E}_0$. We write $g(\mu) \equiv g_{\mathcal{E}_0}(\mu)$ as the gain function that arises from the constant field. We then define the effective gain parameter as the maximum value of $g(\mu)$, i.e.,

$$g_{\text{eff}} = [g(\mu)]_{\max}. \tag{4.5}$$

In general the function of $g_{\mathcal{E}}(\mu)$ turns out to be sensitive primarily to the leading edge of the pulse, and thus a constant field and an exponential have similar growth rates. As a result, g_{eff} tends to characterize well the initial (short z) amplification of the spontaneous emission. However, as seen in Fig. 2(b), the growth rates rapidly fall off as the pulse reshapes. Thus, for larger z, g_{eff} tends to overestimate the growth rates by factors of between 2 and 4.

The calculation of g_{eff} is tedious in general because of the Gaussian dependence of the inhomogeneous broadening. We therefore consider in detail only the case $T_2^* \gg T_1, T_2$. For a three-level amplifier with an extended pump, we take $T_1 = T_2$ and the pump to be constant over the interval $[0, T_p]$. It is then straightforward to show

$$g(\mu) = \frac{2\alpha'}{T_p} \int_0^{\min(\mu, T_p)} d\mu_0\, e^{\mu_0/T_1} e^{-\mu/T_1} (\mu - \mu_0), \tag{4.6}$$

where $\min(\mu, T_p)$ indicates the smallest of μ and T_p. The maximum value of $g(\mu)$ is then

$$g_{\text{eff}} = g_{\text{est}} \left(\frac{e^{T_p/T_1} - 1}{T_p/T_1} \right) \exp\left(\frac{-T_p/T_1 e^{T_p/T_1}}{e^{T_p/T_1} - 1} \right). \tag{4.7}$$

For $T_p \ll T_1$, we get $g_{\text{eff}} = g_{\text{est}}/e$ which, with the additional factor of 2 as just discussed, gives a growth rate $g_{\text{est}}/6$, which is what occurred in the calculation in Sec. III B. For $T_p \gg T_1$, one has $g_{\text{eff}} = g_{\text{est}} T_1/T_p$. We note that with our present convention for defining g_{est}, the gain is seen to drop below that of the conventional estimate as T_p becomes large. In the conventions used in cw excitation, the factor T_1/T_p is incorporated into the inversion density and g_{eff} and g_{est} are the same.

In the case of a two-level amplifier, i.e., one in which the lower state is the ground state of the atom and the decay goes directly from the upper state to the lower state, one takes $T_2 = 2T_1$, which is appropriate in this case, and for $T_p \ll T_1$ gets

$$g_{\text{eff}} = \frac{1}{4}\left(\frac{F}{F+1}\right) g_{\text{est}}, \tag{4.8}$$

where $g_{\text{est}} = 2\alpha' T_2$ as before. In the best case (complete inversion at $\mu = 0$), the effective gain parameter g_{eff} is smaller than g_{est} by a factor of 8. For weak inversions, for example, $F = 0.05$, the actual gain can be smaller than the estimation of the ordinary theory by two orders of magnitude. The reason that the two-level model suffers a much more severe loss in gain than the three-level case is due to the fact that the population inversion persists for a time that is substantially shorter than T_2.

If one includes a finite $T_p \gtrsim T_1$, then one finds that

$$g_{\text{eff}} = \frac{g_{\text{est}} T_1}{F T_p} \left(\frac{F(F+2)}{(F+1)} - 2\ln(F+1) \right). \tag{4.9}$$

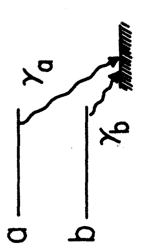
For small fractional inversions ($F \ll 1$), the two expressions inside the large parentheses cancel to first and second order in the value of F, giving a quadratic dependence on F. Thus for $F \ll 1$, the gain falls off very rapidly with decreasing values of F.

The calculations performed in this publication, unless otherwise indicated, were carried out for the same value of g_{eff} ($g_{\text{eff}} L = 12$). The values of g_{est} needed to achieve this value and the resulting observed gains g_{act} are shown in Table I. The various groupings in the table correspond to the

TABLE I. Summary of the values of the estimated (g_{est}), effective (g_{eff}), and actual (g_{act}) gains calculated in the course of this investigation. The actual gains are calculated at $z=L$. The upper case is for a "two-level system" whose level structure is given in the attached figure. The lower case is for a "two-level system" with the level structure and decays shown. The quantities varied are T_2^*, defined in Eq. (2.22) as the dephasing time of an atomic ensemble due to the Doppler broadening; T_p, used in Eq. (2.21) as the temporal duration of the excitation; F, defined in Eq. (2.15) as the fractional population inversion.

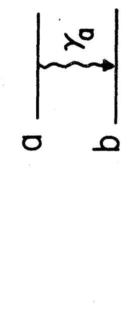
Three-level system

T_p/T_1	0	0	0	0	1	2.5	7.5
T_2^*/T_1	∞	1	0.3	0.1	∞	∞	∞
$g_{est}L$	32.6	22.4	16.7	14.2	33.9	41.0	88.7
$g_{eff}L$	12.0	12.0	12.0	12.0	12.0	12.0	12.0
$g_{act}L$	4.88	5.0	5.25	5.72	4.71	4.82	5.52



Two-level system

F	1	0.3	0.1	0.66	0.66	0.66	0.66	1	0.3	0.1
T_p/T_1	0	0	0	0	0	0	0	1	1	1
T_2^*/T_1	∞	∞	∞	∞	1	0.4	0.2	∞	∞	∞
$g_{est}L$	96.2	209	528	120	42.5	26.3	19.5	120	605	4230
$g_{eff}L$	12	12	12	12	12	12	12	12	12	12
$g_{act}L$	3.59	3.33	3.23	3.48	3.63	3.99	4.38	3.01	2.84	2.66



parameter studies performed in Sec. V. First, it is seen that the values of g_{act} are all fairly similar, so that g_{eff} is at least a reasonably precise estimate of the gain. Generally, the two-level cases have somewhat smaller gains, but on the whole the variations within the values of g_{act} are comparable to the amount by which g_{act} itself can vary as a function of distance. In the first series we see the behavior of the gain as a function of T_2^*. The value of g_{est} for large values of T_2^* is close to g_{act}, which is in accord with the earlier discussion. This convergence should not be misconstrued as indicating that the gain increases as T_2^* becomes smaller. We have had to increase the number of atoms in order to keep the gain the same. If all parameters of the problem are fixed except T_2^* then both g_{est} and g_{eff} are monotonically decreasing functions of T_2^*, which converge in the limit that $T_2^* << T_1$. The next sequence shows the effect of an extended pump on the gain. As indicated earlier, the difference between the two estimates of the gain diverge from each other for large T_p. This is due to the fact that the gain estimate is made assuming an instantaneous pumping. If one includes T_p in g_{est} then the system would behave with respect to increasing T_p in the same way that it responds to smaller values of T_2^*. With that convention both g_{est} and g_{act} would monotonically decrease as T_p becomes large, and would converge on each other in that limit. The two-level systems are characterized by a much wider discrepancy between the actual and estimated gains, with very large drops in gain occurring in weakly inverted systems (small F). This decrease is due to the extremely short duration of the gain pulse, which becomes still shorter as F becomes small. Since the pulse is so short, the convergence of g_{est} and g_{eff} occurs for very large T_2^*. This is shown in the second sequence. The behavior of the gain in a two-level system with respect to changing T_p is the same as for a three-level system, in that the gain falls off as T_p^{-1}, as T_p gets large. We have considered, instead, the more interesting case where T_p is fixed and F is varied. One sees that in the most extreme case there is a three-orders-of-magnitude difference between the expected and actual gain.

B. Pulse properties

In this subsection we discuss how the spectral widths $\Delta\omega$ and temporal widths Δt of the pulses (always full widths at half-maximum) vary with the parameters. In order to eliminate, as much as possible, the effects of differential growth rates, we calculate each case for a length L such that $g_{eff}L=12$, resulting in actual amplification factors of between 20 and 200 above the level of the noise.

In Fig. 4, we show the results for the three-level amplifier depicted in the upper part of Table I. In the first (second) column we show the temporal (spectral) widths as a function of distance z, measured in terms of $g_{\mathrm{act}}(L)$. In all cases we have set $T_1 = T_2$. In Fig. 4(a) we change the Doppler broadening, with $T_2^* = \infty$, T_1, $0.3T_1$, and $0.1T_1$, keeping $T_p = 0$. In Fig. 4(b) we change the temporal duration of the excitation, with $T_p = 0$, T_1, $2.5T_1$, and $7.5T_1$, and set $T_2^* = \infty$ (no Doppler broadening). In virtually all cases the pulse widths are seen to stabilize quite rapidly, with a width that correlates with the width of the gain function $g(\mu)$ defined in the previous subsection. This stabilization is a property of the field dependence of the gain function in Eq. (4.4). If $g_{\mathcal{E}}(\mu)$ were independent of $\mathcal{E}$, which is to some degree the case for small T_2^* and large T_p, the pulse will contract in time rather than stabilize.

In the spectral behavior one sees that, by and large, there is no decrease in the spectral width in the cases shown here, i.e., there is no gain narrowing.[10] In the case of extreme Doppler broadening, the rate of narrowing is about a factor of 2 less than one would expect on the basis of Eq. (4.1). In the case $T_p \gg T_1$, the rate is initially about the same as given by Beer's law but it abruptly terminates. In both cases these anomalies are due to the fact that the pulse is contracting in time rather than broadening the way it would if T_1 were long.

In Fig. 5 we show the results for the two-level system depicted in the bottom half of Table I. In each case we have $T_1 = 2T_2$, which is the appropriate value for this case. In Fig. 5(a) we keep $T_p = 0$, $T_2^* = \infty$, and set $F = 1$, 0.3, and 0.1. In Fig. 5(b) we fix $F = \frac{2}{3}$, $T_p = 0$, and change T_2^* such that $T_2^* = \infty$, T_1, $0.4T_1$, and $0.2T_1$. We do not show explicitly the case for varying F with $T_p = T_1$, since the results are nearly identical to the case in 5(a). As with the cases in Fig. 4, the pulse widths Δt tend to stabilize, but the transient behavior is now associated with an initial contraction of the pulse. The reason for this contraction is that the gain persists for times that are of the order $FT_1/(1+F)$, which can be quite short compared

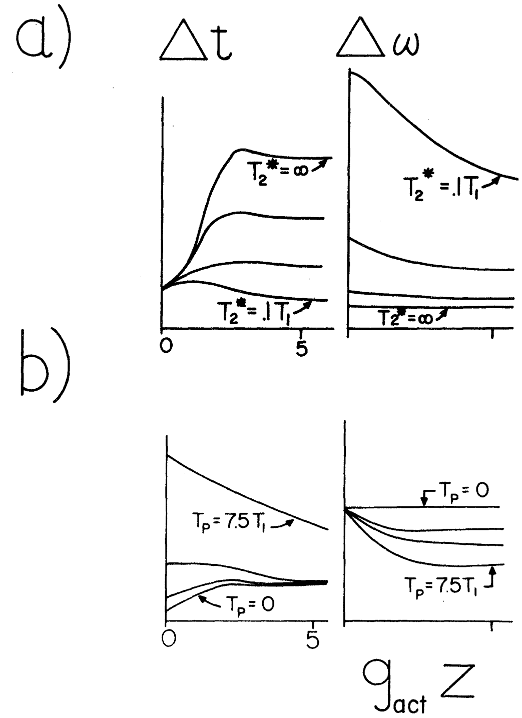


FIG. 4. (a) Plot of the spectral width $\Delta\omega$ and temporal width Δt vs $g_{\mathrm{act}}z$ for a three-level amplifier with $T_1 = T_2$, $T_p = 0$, and $T_2^* = \infty$, T_1, $0.3T_1$, and $0.1T_1$. (b) Same as in (a) except that $T_2^* \gg T_1$ in all cases, and the parameter varied is the temporal duration of the pump T_p, which has the values $T_p = 0$, T_1, $2.5T_1$, and $7.5T_1$. In both cases, and in Fig. 6, the curves are monotonic in T_p, so only the outer two are labeled.

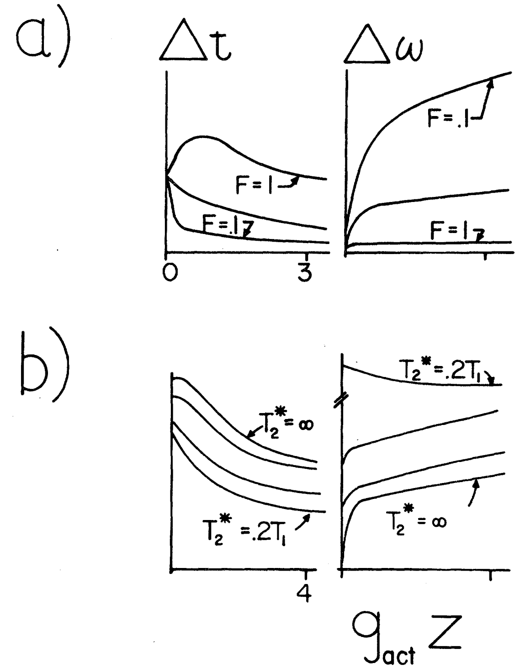


FIG. 5. (a) Same as Fig. 4, except that a "two-level" scheme is used. The parameters for this case are $T_p = 0$, $T_2 = 2T_1$, $T_2^* \gg T_p$. The quantity varied is the fractional population inversion F, where $F = 1$, 0.3, 0.1. (b) Same as above except $F = 0.66$ and $T_2^* = \infty$, and $T_2^* = T_1$, $0.4T_1$, $0.2T_1$. The scale in the curve for $\Delta\omega$ is broken, with the upper portion being a factor of 2 larger than the break point.

to T_1. As a result of the short pulse widths, one gets a corresponding sharp increase in the spectral width. In the case $F=0.1$ the spectral width increases by an order of magnitude for $g_{act}z \sim 0.5$. In Fig. 5(b), we show that one has to go to extremely large Doppler broadening in this case before one finds any spectral narrowing.

V. EFFECTS OF A FINITE LOSS

In this section we consider the effect of the inclusion of an unsaturable loss term in the calculation.[27] The effects of a small loss are straightforward in practice. The loss merely lowers the gain in an appropriate fashion, but does not greatly alter the effects noted in Sec. IV. If, however, the losses are substantial, the behavior of the amplifier is radically altered. In particular, it is possible to see saturation effects and the formation of apparent steady states in the small-signal regime of the amplifier. These effects make it next to impossible to meaningfully define a threshold for this problem.

The difficulties which arise when a loss is included have their origin in the shape dependence of the gain which was discussed in Sec. IIIB. This shape dependence causes the gain to decrease monotonically as a function of distance as shown in Fig. 2(b). One can then readily imagine that, for sufficiently large z, interesting effects can occur in the presence of a loss. In Fig. 6 we consider the development of the pulse energy as a function of distance for various values of the loss κ. We have used the three-level homogeneously broadened model for the amplifier with $T_1 = T_2$, $T_2^* = \infty$, $T_p = 0$, and $g_{eff} = 0.48$. In each case the value of the length L is adjusted so that $(g_{eff} - \kappa)L$ is fixed at 20. One notes immediately that the curves vary greatly from one another, which shows that scaling of the distance using $(g_{eff} - \kappa)z$ is inappropriate. It would be preferable to scale the distance by $(g_{act} - \kappa)z$. Unfortunately, such a scaling is impossible at least in principle, since for large z, the growth rate always vanishes (i.e., $g_{act} - \kappa = 0$). For the largest value of the loss presented in Fig. 6, one can show that the pulse evolution will drop the gain [see Fig. 2(b)] well below the loss over the distances shown here. The evolution in that case is representative of a laser amplifier below threshold. For very low losses ($\kappa = 0.12$), the amplifier behaves like a system above threshold. However, a disturbing feature appears in the case $\kappa = 0.18$. In that case, even though something like a small signal gain is well established for intermediate z, as z gets large, one sees the onset of what appears to be a saturation. This particular case is shown in more detail in Fig. 7, where the energy vs distance is plotted on a linear scale. One sees the build up from the initial noise in the insert, which shows the small-z regime in greater detail. The dashed curve indicates how the energy would develop if there were no gain. The large-z regime shows the "saturation" and formation of a "steady state." The "steady-state pulse" which results is shown in Fig. 7(b). From a phenomenological standpoint, the amplifier goes through all the normal stages of pulse development except that it does so entirely without benefit of nonlinearities.

What is happening here is not saturation as understood in a normal laser. For one, the output obeys Gaussian statistics and thus there are substantial fluctuations. These fluctuations occur both from one firing of the amplifier to the other, and also as a function of z (hence the quotes around steady state). The output pulse similarly bears no relation to the usual steady-state pulses[27] that have been discussed in the past. The usual semiclassical steady-state pulse (i.e., as described in a system without spontaneous emission) can be shown to be identically zero in this case. The saturation in this case, like the case of a large loss, behaves much more like a laser below threshold.

The reasons for this behavior are essentially the same as those responsible for the nonexponential growth of the pulse. However, up until now we have discussed the issue of gain with a semiclassical interpretation, and in order to properly understand what happens in the presence of the loss, one must view the growth of the pulse in the context of the quantum-mechanical model. Let us reconsider then the nature of the solution as discussed in Sec. III A. We take the solution formed from the Green's function in the form presented in Eq. (3.3) and neglect the coordinates involving the excitation and Doppler broadening. We set $|a|^2 = 1$ for convenience, and we then write the expression for the intensity following the steps indicated in Sec. III [see Eqs. (3.2)–(3.10)]:

$$\langle I(\mu, z)\rangle = \int_0^z dz' \, |G_\kappa^0(\mu, z')|^2 . \tag{5.1}$$

We label the Green's function with κ to indicate that it is calculated for a finite value of the loss. The Green's function is the quantity that obeys the semiclassical equation of motion for this case, and the quantum-mechanical solution $\langle I(\mu, z)\rangle$ is the incoherent superposition of many semiclassical solutions. In the lossless case the intensity at a distance z is dominated by the values of $G_{\kappa=0}^0$ for $z - z' \gg 0$; i.e., the amplified spontaneous emission (ASE) from the "input" end is much larger than that from other parts of the amplifier. Thus the solution is dominated by a small number of semi-

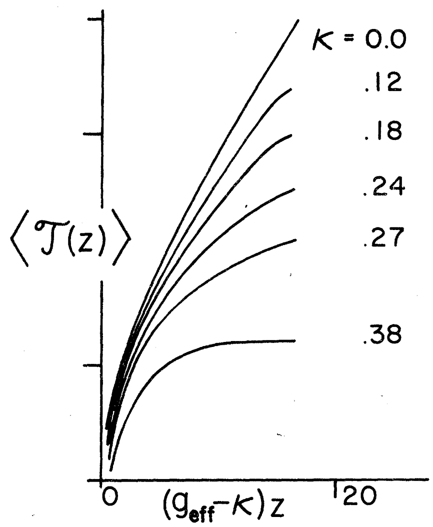


FIG. 6. Energy on a logarithmic scale vs distance, which is given as $(g_{\rm eff}-\kappa)z$. The parameter varied is the loss κ, which takes the values $\kappa=0$, 0.12, 0.18, 0.24, 0.27, and 0.38. The curves are labeled with their values of κ as indicated. In all cases $g_{\rm eff}=0.48$.

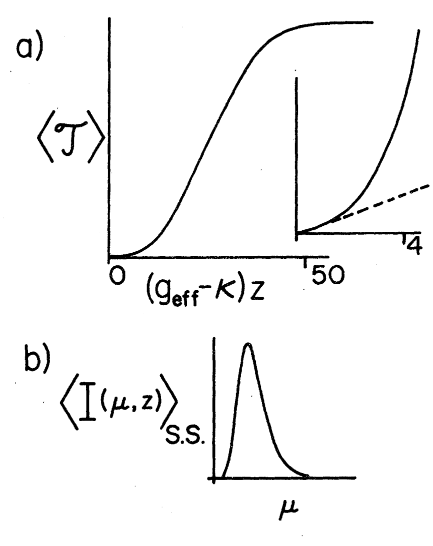


FIG. 7. Energy vs distance on a linear scale for the case $g_{\rm eff}=0.48$, $\kappa=0.18$. Both plots are given as a function of $(g_{\rm eff}-\kappa)z$, with the insert being a greatly expanded picture of the curve for $0 \lesssim (g_{\rm eff}-\kappa)z \lesssim 4$. The dashed line is included to emphasize the initial straight-line nature of the buildup of the pulse from spontaneous emission.

classical contributions, and a straightforward semiclassical interpretation of the gain is appropriate. In the presence of a loss, however, the Green's function G_κ^0 vanishes as $z-z'$ becomes very large. In other words, if one goes for sufficiently large z, the contribution from the input end is small. The pulse in the loss case as a function of z is thus dominated by different contributions from the ASE. The quantum-mechanical intensity $\langle I(\mu,z)\rangle$ is a composite of different semiclassical pieces, each of which then vanish as z gets bigger and are replaced by other pieces. As long as one keeps the composite nature of the pulse in mind, it is then reasonable to say that the pulse evolves into a shape for which $g_{\rm act}=\kappa$. In this sense the phenomenon is indeed a saturation, but because it is a shape effect, it is of little value insofar as predicting the output energy is concerned. An alternative interpretation of this effect is to note that in the absence of true exponential growth of the pulse, the loss will invariably overcome the gain, and one must regard this problem from the beginning as analogous to a laser below threshold.[28] One can then construct a crude solution in the following way. We rewrite Eq. (5.1) as

$$\langle I(\mu,z)\rangle = \int_0^z dz' \, |G^0_{\kappa=0}(\mu, z-z')|^2 e^{-\kappa(z-z')} . \tag{5.2}$$

It is straightforward to show from Eq. (3.3) that a factorization of this sort is possible, and that $G^0_{\kappa=0}$ is in fact a function only of the difference between the position variables. If $G^0_{\kappa=0}$ were constant as a function of $z-z'$, then this would be the description of a system without gain (e.g., a laser well below threshold). For the purpose of discussion, let us treat $G^0_{\kappa=0}$ as if it were constant and evaluate it at $z-z'=\kappa^{-1}$ to give

$$\langle I(\mu,z)\rangle \approx (1/\kappa)|G^0_{\kappa=0}(\mu,\kappa^{-1})|^2 . \tag{5.3}$$

Although this is hardly an accurate formula, it is more useful than the semiclassical interpretation in understanding the behavior indicated in Fig. 6 for large z. One regards the steady state as being due to a system without gain, driven by a "spontaneous emission" that becomes large as $\kappa \to 0$. Thus, for example, it is clear with this description why the output obeys Gaussian statistics, and why the output is a "steady state" only in the average sense, but actually has large fluctuations.

In conclusion then, when one includes a loss, the nonexponential growth of the pulse causes the system to appear to go through all the normal stages of laser development, i.e., buildup from noise, exponential (approximately) growth, saturation, and finally, the formation of a steady state, all without benefit of nonlinearities. If the loss is small, the pulse will become so large that one must take the nonlinearities into account and adjust the interpretation of the effects accordingly. Thus, insofar as the small-signal regime is concerned, a small loss has no particularly interesting consequences. The strange behavior in the large-loss case is related to the impossibility of defining a threshold in the absence of exponential gain. Thus one can consider the steady state either as a shape-dependent saturation, or as the result of the system being below threshold. These interpretations are largely equivalent, but one may be more useful than the other depending on which facet of the problem one is interested in. The amplifier configuration could well be called a "noise-driven amplifier" to differentiate it from an ordinary "noise amplifier."[19,25] In the ordinary case the spontaneous emission is needed only to get the process started, whereas here one needs a constant infusion of spontaneous emission in order to sustain the optical pulse.

VI. CONCLUSION

In this paper we have considered the growth of an optical pulse from noise in the small-signal regime of a swept-gain amplifier as proposed in recent x-ray laser schemes.[4-6] We are especially interested in the case where the laser medium is an ion beam so that there is little random motion to the atoms and the Doppler broadening is small. In that limit one cannot use either rate equations[11] or Fourier transforms[10] to solve the problem. Thus, except in certain limiting cases which will be discussed in a subsequent publication, the equations do not appear to admit an analytic solution. We have used instead a numerical calculation of the Green's function that is appropriate for the Heisenberg equations. We have used this solution to construct the intensities and power spectra that result from the amplification process. We have not considered higher moments[29] of the field, since in view of the Gaussian statistics that characterize this system they can be deduced in a straightforward manner from the features that are discussed here.

In the circumstance we are considering here, the inverse of the bandwidth of the amplifier is a time of the order of, and occasionally considerably longer than, the decay time of the population inversion. The most obvious consequence of this fact is that the actual growth rates of the pulse are less than would be expected on the basis of the conventional gain calculation. The reduction in growth rate compared to what is expected on the basis of the usual gain formula varies considerably from one case to another, and ranges from factors of 3 to 10^3 over the parameter range considered here. There are in addition several other interesting features that appear. There is a general absence of features that usually[10] characterize the linear regime of an amplifier. First the growth of the pulse energy is nonexponential, and the gain is not well defined. In other cases where nonexponential growth can occur, the growth rate (which we call $g_{\rm act}$) increases monotonically to a well-defined constant. Here, in contrast, $g_{\rm act}$ is usually seen to decrease monotonically to zero. Another feature of ordinary small-signal amplification that is largely absent here is the narrowing of the width of the power spectrum that occurs in most lasers and laser amplifiers.[9] Instead the spectrum is either fairly static or broadens as a function of distance. The amplification process thus leads either to little change or to a decrease in the classical coherence of the light. The absence of gain narrowing is an important consideration since spectral narrowing is frequently taken to be a definitive experimental proof[2,30] of the occurrence of amplification. This sort of proof will have to be substantially modified for the amplifiers considered here. Along with an absence of the typical behavior associated with amplification of weak signals, one has a number of effects that are normally associated with the large-signal regime of a laser amplifier.[9] One sees the formation of bandwidth-limited or coherent pulses in nearly all cases. In the presence of losses, the gain saturates and one sees what appears to be steady states.[27] This latter is associated with the difficulty in defining thresholds in this problem.

These various odd effects can all be largely understood by interpreting the result semiclassically. One considers $\langle I(\mu, z)\rangle$ to be the result of a semiclassical field amplitude from which one can construct a time-dependent gain function. The finite width of the gain then accounts for the absence of gain narrowing, the presence of spectral broadening (when the gain is of very short duration), and the formation of bandwidth-limited pulses. The gain function constructed above is dependent on the shape of the field amplitude, which is the cause of the nonexponential gain. This nonexponential behavior is closely related to the apparent saturation and the formation of steady states. The latter, however, cannot be discussed on a purely semiclassical basis. Such an approach would demand that the steady-state pulse energy would vanish. In many respects it is more useful to view such steady states as being due to a laser below threshold.

Of the various effects noted here, the lowering of the gain is the most serious, since it is an unpleasant obstacle to achieving laser action at low wavelengths. The lowered gain is due to the fact that the gain function we construct is always smaller than the value of the estimated gain. In order to have a basis for the discussion of the gain, we calculate the gain function for the special case of a constant field and use it to define an effective gain g_{eff} which can be obtained by an analytical procedure. This gain reduces to the conventional formula in all cases in which that formula is appropriate. This can be expected to occur whenever the product of the bandwidth of the amplifier and the time duration of the gain is much greater than unity. In other cases g_{eff} is a considerably more accurate value for the gain than the usual formula, even though it still overestimates the growth rate by factors of 2 to 4. The degree to which the gain is reduced is dependent on the atomic level scheme used. The most serious reductions occur when the ground state is the lower level of the transition.

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