Parametric interactions of four electromagnetic waves due to relativistic and other nonlinear effects in plasmas

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(Received 20 May 1974; revised manuscript received 17 May 1975)

Parametric interactions between four electromagnetic waves subjected to a linear phase-matching condition have been considered in a plasma. The relativistic equation of motion of a single electron has been used. The evolution of two signal waves in time by two powerful pump fields has been studied. The largest contributions to frequency shifts and growth rates are found to come from the relativistic correction source. Moreover, in some cases of frequencies very large compared to the characteristic plasma frequency, a part of the relativistic effects totally cancels the other nonlinear effects. When the beat frequency of two of the waves equals the characteristic plasma frequency and in the neighborhood of this condition, the contributions from the nonrelativistic nonlinear sources are found to dominate. The frequency shifts and growth rates for high laser powers are considerable.

I. INTRODUCTION

The relativistic effect is known to be considerable in many cases of nonlinear parametric interactions; for example, this effect should not be ignored in studies of plasma beams, in plasmas having relativistic thermal velocities, or when ordered velocities due to very strong wave fields are relativistic. Some authors[1-6] have considered several cases of such relativistic problems.

Moreover, we find that parametric effects are also caused by the usual relativistic corrections in rarefied stationary plasmas under the influence of driver waves which should be transverse and not necessarily so strong that the corresponding ordered velocities are relativistic. In this paper we consider a simple problem of this type. The existence of such parametric interactions is pointed out below.

The unscreened self-magnetic interaction being, in fact, a relativistic process, the dispersive electromagnetic effects are relativistic (cf. Thompson[7]). This relativistic contribution even predominates over other nonlinear interactions of the same order in problems of wave propagation when the wave amplitude is finite. The evaluation of the nonlinear shift in frequency and wave number of a monochromatic wave of finite amplitude in a stationary plasma by Sluijter and Montgomery[8] shows that the contribution of the relativistic correction term is larger than that of the nonrelativistic part. Das[9] extended this work by studying the effects of two electromagnetic waves. These authors have used the relativistic equation of motion of a single electron in a Lorentz force field, in addition to the usual Maxwell equations. We have extended their work to the four-wave regime. The reason is that the expansion of the relativistic momentum $\vec{p}$ in powers of the velocity $\vec{v}$ shows that the first relativistic nonlinear correction is $m(v^2/2c^2)\vec{v}$. This quantity being cubic, and not quadratic, in $\vec{v}$, the effects of parametric interaction in plasmas without any steady streaming motion can only be determined if at least four electromagnetic waves are considered.

Our nonlinear sources are the force $-(e/mc)(\vec{v}\times\vec{H})$, the current $-en\vec{v}$, the nonlinear operator $\vec{v}\cdot\vec{\nabla}$ operating on $\vec{v}$, and the nonlinear relativistic relation between $\vec{p}$ and $\vec{v}$. The equilibrium-state field quantities bear the suffix 0. The linear or first approximations of $\vec{E}$, $\vec{H}$, $\vec{v}$, and n will be denoted by $\vec{E}_1$, $\vec{H}_1$, $\vec{v}_1$, and n_1, the next—that is, the second—approximations by $\vec{E}_2$, $\vec{H}_2$, $\vec{v}_2$, and n_2, the third by $\vec{E}_3$, $\vec{H}_3$, $\vec{v}_3$, and n_3, and so on. Some authors, on the contrary, call the linear approximation the zeroth or ground approximation, and in general call, $\vec{E}_n$ the $(n-1)$th approximation of $\vec{E}$.

The method of solution is the standard one and is discussed, for example, by Sagdeev and Galeev[10] and Nishikawa.[11] The known technique for parametric three-wave interactions could be easily extended to our four-wave problem. Here it may be mentioned that the small parameter for the perturbation theory is $|v_1|/c \sim e|E_1|/m\omega c$.

The simplest of the parametric four-wave interactions are those in which there are two powerful pump waves interacting with two signal waves. For further simplification, the study is confined to rectilinear propagation in an unmagnetized and homogeneous plasma. Our work also extends to the investigation of resonant interactions in which the difference between two of the four wave frequencies equals the plasma frequency.

The growth rates and frequency shifts found here are very pronounced, and the threshold values of the pump-wave intensities are not very high. The effects of the relativistic correction terms are predominant in all the nonresonant cases. But in the near-resonant interactions, in which the difference between two of the frequencies is close to the characteristic plasma frequency ω_p, the relativistic terms can be totally neglected in the estimation of the enhanced growth.

In the resonant interactions, in which this difference of frequencies exactly equals ω_p, the equations for the parametric effects contain secularly growing sources. The relativistic effect is also negligible in the resonant interactions.

Sugihara[12] and Sitenko[13] have considered some cases of four-wave interactions; Wilhelmsson and Pavlenko[14] have investigated the interaction of five waves for explosive instabilities. But the problem worked out here is outside the subject matter of these studies.

II. FIELD EQUATIONS AND EQUATIONS FOR FOUR-WAVE INTERACTIONS

Following Sluijter and Montgomery[8] and Das,[9] we start with the self-consistent system of relativistic equations for electromagnetic waves in cold plasmas:

$$\left[\frac{\partial}{\partial t}+(\vec{v}\cdot\vec{\nabla})\right]\frac{\vec{v}}{(1-v^2/c^2)^{1/2}}=-\frac{e\vec{E}}{m}-\frac{e}{mc}(\vec{v}\times\vec{H}), \tag{2.1}$$

$$\text{curl}\vec{E}=-\frac{1}{c}\frac{\partial\vec{H}}{\partial t}, \tag{2.2}$$

$$\text{curl}\vec{H}=\frac{1}{c}\frac{\partial\vec{E}}{\partial t}-\frac{4\pi e}{c}(n_0+n)\vec{v}, \tag{2.3}$$

$$\text{div}\vec{E}=-4\pi e n, \quad \text{div}\vec{H}=0. \tag{2.4}$$

[By $\vec{E}_1$ we now denote the vector sum of the four elliptically polarized electromagnetic waves in the medium of the linear approximation. Assuming that all of them are propagating parallel to $0z$ we have $n_1=0$ and

$$\vec{E}_1=\vec{e}_x\sum_{i=1}^{4}a_i\cos\theta_i+\vec{e}_y\sum_{i=1}^{4}b_i\sin\theta_i, \tag{2.5}$$

where $\theta_i=k_iz-\omega_it$, $k_i^2c^2=\omega_i^2-\omega_p^2$, $\omega_p^2=4\pi n_0e^2/m$, and $\vec{e}_x$, $\vec{e}_y$, and $\vec{e}_z$ are the unit vectors parallel to $0x$, $0y$, and $0z$, respectively. Hence

$$\vec{v}_1=\frac{e}{m}\left(\vec{e}_x\sum\frac{a_i}{\omega_i}\sin\theta_i-\vec{e}_y\sum\frac{b_i}{\omega_i}\cos\theta_i\right). \tag{2.6}$$

The second-order quantities are given by

$$\ddot{\vec{v}}_2+\omega_p^2\vec{v}_2=\frac{e^2\vec{e}_z}{2m^2}\sum_{i\neq j}\left(\frac{2k_i}{\omega_i}(a_i^2-b_i^2)\cos2\theta_i+\frac{(a_ia_j-b_ib_j)(\omega_i+\omega_j)(k_i+k_j)}{\omega_i\omega_j}\cos(\theta_i+\theta_j)-\frac{(a_ia_j+b_ib_j)(\omega_i-\omega_j)(k_i-k_j)}{\omega_i\omega_j}\cos(\theta_i-\theta_j)\right); \tag{2.7}$$

$$\dot{\vec{v}}_2=-\frac{e\vec{E}_2}{m}-\frac{e}{mc}(\vec{v}_1\times\vec{H}_1), \quad \vec{E}_2=4\pi e n_0\vec{v}_2, \quad \vec{H}_2=0, \quad \text{div}\vec{E}_2=-4\pi e n_2; \tag{2.8}$$

$$n_2=-\frac{\omega_p^2}{8\pi m}\sum\left(\frac{(a_ia_j-b_ib_j)(k_i+k_j)^2\cos(\theta_i+\theta_j)}{\omega_i\omega_j[(\omega_i+\omega_j)^2-\omega_p^2]}+\frac{2k_i^2(a_i^2-b_i^2)\cos2\theta_i}{\omega_i^2(4\omega_i^2-\omega_p^2)}-\frac{(a_ia_j+b_ib_j)(k_i-k_j)^2\cos(\theta_i-\theta_j)}{\omega_i\omega_j[(\omega_i-\omega_j)^2-\omega_p^2]}\right). \tag{2.9}$$

For the computation of the third-order effects we note that

$$-(\vec{v}_2\cdot\vec{\nabla})\vec{v}_1-(e/mc)(\vec{v}_2\times\vec{H}_1)=0, \tag{2.10}$$

which means that two of the third-order sources in the momentum transfer equation (2.1) cancel each other. Hence the third-order electric field $\vec{E}_3$ is actually determined from the equation

$$\ddot{\vec{E}}_3-c^2\nabla^2\vec{E}_3+\omega_p^2\vec{E}_3=\frac{m\omega_p^2}{e}\left(-\frac{\partial}{\partial t}\frac{v_1^2\vec{v}_1}{2c^2}+\frac{1}{n_0}\frac{\partial}{\partial t}(n_2\vec{v}_1)\right). \tag{2.11}$$

The first term inside the large parentheses comes from the expansion of the relativistic momentum in powers of $\vec{v}_1$.

If this nonlinear equation is solved for secularity-free behavior by the method of Bogoliubov, Krilov, and Mitropolsky (cf. Montgomery and Tidman[15] and Tidman and Stainer[16]), or of Lindstedt (cf. Bellman[17]), then the results of Sluijter and Montgomery[8] for a single wave and of Das[9] for two waves can be recovered.

The investigation of the four-wave interactions will be restricted by the following linear phase-matching conditions:

$$\omega_4=\omega_1+\omega_2+\omega_3, \quad k_4=k_1+k_2+k_3. \tag{2.12}$$

In the general case, any further restrictions are excluded, and so we have

$$(\omega_i-\omega_j)^2\neq\omega_p^2, \quad i,j=1,2,3,4. \tag{2.13}$$

We now use (2.6) and (2.9) in (2.11) and then put

$$\theta_4 = \theta_1 + \theta_2 + \theta_3 \tag{2.14}$$

in the resulting relation. It is necessary to single out only the vectors $\vec{e}_x \cos\theta_i$ and $\vec{e}_y \sin\theta_i$ and some of their coefficients, namely, only those relevant coefficients which are obtained from the mutual interaction of the four waves, subject to (2.14). The coefficients of $\cos\theta_i$ and $\sin\theta_i$ for $i=2,3,4$ can be easily guessed from those of $\cos\theta_1$ and $\sin\theta_1$ by following a rule of cyclic rotation of 1, 2, 3, and 4 for the numerical suffixes of a and b. So, mentioning only the relevant coefficients of $\cos\theta_1$ and $\sin\theta_1$ on the right-hand side of (2.11), we write

$$\begin{aligned}\ddot{\vec{E}}_3 - c^2\nabla^2\vec{E}_3 + \omega_p^2\vec{E}_3 = \frac{e^2\omega_p^2}{4m^2c^2}\Bigg\{ & \vec{e}_x\cos\theta_1\Bigg[a_1\left(\frac{a_3^2+b_3^2}{\omega_3^2}+\frac{a_4^2+b_4^2}{\omega_4^2}\right) \\ & -\frac{\omega_1}{\omega_2\omega_3\omega_4}\bigg((2a_2a_4+2b_2b_4)a_3+(a_2a_3-b_2b_3)a_4-\frac{(a_2a_3-b_2b_3)(k_2+k_3)^2c^2}{(\omega_2+\omega_3)^2-\omega_p^2}a_4 \\ & -\frac{(a_2a_4+b_2b_4)(k_4-k_2)^2c^2}{(\omega_4-\omega_2)^2-\omega_p^2}2a_3\bigg)\Bigg] \\ & +\vec{e}_y\sin\theta_1\Bigg[b_1\left(\frac{a_3^2+b_3^2}{\omega_3^2}+\frac{a_4^2+b_4^2}{\omega_4^2}\right) \\ & -\frac{\omega_1}{\omega_2\omega_3\omega_4}\bigg(-2(a_2a_4+b_2b_4)b_3+(a_2a_3-b_2b_3)b_4-\frac{(a_2a_3-b_2b_3)(k_2+k_3)^2c^2}{(\omega_2+\omega_3)^2-\omega_p^2}b_4 \\ & +\frac{(a_2a_4+b_2b_4)(k_4-k_2)^2c^2}{(\omega_4-\omega_2)^2-\omega_p^2}2b_3\bigg)+\cdots\Bigg]\Bigg\}\ .\end{aligned} \tag{2.15}$$

The remaining terms $(+\cdots)$ can moreover be determined with the help of (2.19) and (2.20) and similar relations for $i=3,4$.

Let the powerful driver waves be those having the amplitudes a_3, b_3, a_4, and b_4, and let a_1, b_1, a_2, and b_2 be the amplitudes of the signal waves. To study the slow evolution of the signal waves in time, we regard a_1, b_1, a_2, and b_2 of (2.5) to be slowly varying functions of time, such that

$$|\ddot{a}_i| \ll |\omega_i \dot{a}_i|\ , \quad |\ddot{b}_i| \ll |\omega_i \dot{b}_i|\ . \tag{2.16}$$

Hence, retaining $\dot{a}_j$ and $\dot{b}_j$ only, and expressing $\sin\theta_j$ and $\cos\theta_j$ in exponential form, we get

$$\ddot{\vec{E}}_3 - c^2\nabla^2\vec{E}_3 + \omega_p^2\vec{E}_3 = -\sum_{j=1}^{4}\omega_j[e^{i\theta_j}(i\vec{e}_x\dot{a}_j + \vec{e}_y\dot{b}_j) + \text{c.c.}] \tag{2.16a}$$

When the derived right-hand-side result is equated to the right-hand side of (2.15), an equation is obtained which is apparently free of $\vec{E}_3$, but which has the potential of yielding the temporal evolution of the amplitudes of the four waves of $\vec{E}_1$ given by (2.5). Since we are interested only in the temporal evolution of a_1, b_1, a_2, and b_2, we retain on the right-hand side of (2.16a) the coefficients of $\vec{e}_x$ and $\vec{e}_y$ which contain these amplitudes and their first-order time derivatives. Hence, equating from both sides of the equation thus obtained from (2.15) the coefficients of $\cos\theta_1$, $\sin\theta_1$, $\cos\theta_2$, and $\sin\theta_2$, the following four linear, mutually connected, first-order differential equations for a_1, b_1, a_2, and b_2 are obtained:

$$\begin{aligned}\frac{\partial a_1}{\partial t} - \frac{ie^2\omega_p^2 a_1}{8m^2c^2\omega_1}\left(\frac{a_3^2+b_3^2}{\omega_3^2}+\frac{a_4^2+b_4^2}{\omega_4^2}\right) = -\frac{ie^2\omega_p^2}{8m^2c^2\omega_2\omega_3\omega_4}\bigg(& 3a_2a_3a_4+2b_2a_3b_4-b_2b_3a_4-\frac{(a_2a_3-b_2b_3)(k_2+k_3)^2c^2}{(\omega_2+\omega_3)^2-\omega_p^2}a_4 \\ & -\frac{2(a_2a_4+b_2b_4)(k_1+k_3)^2c^2}{(\omega_1+\omega_3)^2-\omega_p^2}a_3\bigg)\ ,\end{aligned} \tag{2.17}$$

$$\begin{aligned}\frac{\partial b_1}{\partial t} - \frac{ie^2\omega_p^2 b_1}{8m^2c^2\omega_1}\left(\frac{a_3^2+b_3^2}{\omega_3^2}+\frac{a_4^2+b_4^2}{\omega_4^2}\right) = \frac{ie^2\omega_p^2}{8m^2c^2\omega_2\omega_3\omega_4}\bigg(& 2a_2b_3a_4+3b_2b_3b_4-a_2a_3b_4+\frac{(a_2a_3-b_2b_3)(k_2+k_3)^2c^2}{(\omega_2+\omega_3)^2-\omega_p^2}b_4 \\ & -\frac{2(a_2a_4+b_2b_4)(k_1+k_3)^2c^2}{(\omega_1+\omega_3)^2-\omega_p^2}b_3\bigg)\ ,\end{aligned} \tag{2.18}$$

$$\frac{\partial a_2}{\partial t}-\frac{ie^2\omega_p^2 a_2}{8m^2c^2\omega_2}\left(\frac{a_3^2+b_3^2}{\omega_3^2}+\frac{a_4^2+b_4^2}{\omega_4^2}\right)=-\frac{ie^2\omega_p^2}{8m^2c^2\omega_1\omega_3\omega_4}\left(3a_1a_3a_4+2b_1a_3b_4-b_1b_3a_4-\frac{(a_1a_3-b_1b_3)(k_1+k_3)^2c^2}{(\omega_1+\omega_3)^2-\omega_p^2}a_3\right.$$
$$\left.-\frac{2(a_1a_4+b_1b_4)(k_2+k_3)^2c^2}{(\omega_2+\omega_3)^2-\omega_p^2}a_3\right), \qquad (2.19)$$

$$\frac{\partial b_2}{\partial t}-\frac{ie^2\omega_p^2 b_2}{8m^2c^2\omega_2}\left(\frac{a_3^2+b_3^2}{\omega_3^2}+\frac{a_4^2+b_4^2}{\omega_4^2}\right)=\frac{ie^2\omega_p^2 b_2}{8m^2c^2\omega_1\omega_3\omega_4}\left(2a_1b_3a_4+3b_1b_3b_4-a_1a_3a_4+\frac{(a_1a_3-b_1b_3)(k_1+k_3)^2c^2}{(\omega_1+\omega_3)^2-\omega_p^2}b_4\right.$$
$$\left.-\frac{2(a_1a_4+b_1b_4)(k_2+k_3)^2c^2}{(\omega_2+\omega_3)^2-\omega_p^2}b_3\right). \qquad (2.20)$$

Assuming the time variation of a_1, b_1, a_2, and b_2 to be proportional to $e^{i\omega t}$, a fourth-degree equation in ω is obtained from (2.17)–(2.20), which yields, in general, four frequency shifts and four accompanying growth rates. This means the development of four new colors from the two colors of the two driver waves. The one with the largest growth rate will be most prominent.

It must be remembered that the nonlinear growth does not become apparent when the linear damping of the excited waves is larger than their nonlinear growth, and that the pump-wave amplitudes must exceed certain threshold values in any real plasma when such decay processes are visible. The linear dissipation is taken into account by including the dissipative force $-\nu\vec{v}$ per unit mass, in addition to other terms on the right-hand side of (2.1). This term describes the average effect of collision between electrons and ions in the motion of the former. For threshold estimation we should replace $\partial/\partial t$ by $\partial/\partial t+\nu\omega_p^2/2\omega_1^2$ in (2.17), and by $\partial/\partial t+\nu\omega_p^2/2\omega_2^2$ in (2.19), and then put $\partial/\partial t=0$. See, for example, Rosenbluth and Sagdeev[19] and Larsson and Stenflo[20] for a discussion on the estimation of the threshold powers.

It is known[18] that a strong elliptically polarized wave undergoes self-precession of its polarization ellipse as it propagates through the plasma. We have not considered this effect in this paper.

When the difference between two of the frequencies is in the neighborhood of ω_p, the parametric interactions become enhanced and some of the nonrelativistic nonlinear sources dominate over the relativistic ones. To study such near-resonance situations, we set

$$\omega_4-\omega_1=\omega_2+\omega_3=\omega_p(1-\delta),\quad 0<\delta\ll 1, \qquad (2.21)$$

Then the following cases arise:

(a) The two pump waves propagate in opposite directions. So $\omega_3<0$ when $\omega_4>0$ and

$$\omega_1+\omega_3=\omega_4-\omega_2=\omega_p(1-\delta)+\omega_1-\omega_2\approx\omega_p(1-\delta)\quad \text{if}\quad \omega_1\approx\omega_2. \qquad (2.22)$$

(b) The two pump waves are in the same direction. Then in order to satisfy (2.12) we must have $\omega_2<0$ whenever $\omega_1>0$, $\omega_3>0$, and $\omega_4>0$; and

$$\omega_1+\omega_3=\omega_4-\omega_2=\omega_p(1-\delta)+\omega_1-\omega_2\approx|\omega_1|+|\omega_2|, \qquad (2.23)$$

and $(k_1+k_3)^2c^2/[(\omega_1+\omega_3)^2-\omega_p^2]\approx 1$.

In the case of very high frequencies in a transparent plasma, the dispersion relation $k_i^2c^2=\omega_i^2-\omega_p^2$ $(i=1,2,3,4)$, can be simplified to

$$k_i^2c^2\approx\omega_i^2. \qquad (2.24)$$

If the pump-wave intensities, averaged over one time period of the respective wave fields, are denoted by P_3 and P_4 (in cgs units) then

$$P_3\approx ca_3^2/8\pi,\quad P_4\approx ca_4^2/8\pi, \qquad (2.25)$$

if the waves are plane polarized.

The typical values of the parameters to be used in the numerical estimation of our formulas are

$$P_3\approx P_4\approx 10^{23}\ \text{erg/cm}^2\,\text{sec},\quad \omega_p\approx 10^{11}\ \text{sec}^{-1}, \qquad (2.26)$$
$$|\omega_1|\approx|\omega_2|\approx|\omega_3|\approx|\omega_4|\approx 10^{15}\ \text{sec}^{-1},\quad \delta=0.1.$$

III. RELATIVISTIC FREQUENCY SHIFTS

It is noted that in (2.17)–(2.20) the relativistic effects have yielded the second term on the left-hand sides and the first three terms inside the square brackets on the right-hand sides. An interesting interference effect occurs if

$$\frac{(k_2+k_3)^2c^2}{(\omega_2+\omega_3)^2-\omega_p^2}\approx 1,\quad \frac{(k_1+k_3)^2c^2}{(\omega_1+\omega_3)^2-\omega_p^2}\approx 1. \qquad (3.1)$$

These conditions are possible for very high frequencies for which (2.24) holds and none of the near-resonance conditions expressed through (2.21) to (2.23) are valid. Then the right-hand sides of (2.17)–(2.20) vanish because a part of the relativistic contributions cancels the sum of all the other nonlinear effects; thus we are left with

$$\frac{\partial}{\partial t}(a_i,b_i)\approx\frac{ie^2\omega_p^2}{8m^2c^2}\left(\frac{a_3^2+b_3^2}{\omega_3^2}+\frac{a_4^2+b_4^2}{\omega_4^2}\right)\frac{(a_i,b_i)}{\omega_i}, \qquad (3.2)$$

for $i=1,2$. Hence (2.17)–(2.20) lose the character of simultaneous equations and so are not mutually connected. Solving (3.2) we get the relative frequency shift

$$\frac{\delta\omega_i}{\omega_i}=-\frac{e^2\omega_p^2}{8m^2c^2\omega_i^2}\left(\frac{a_3^2+b_3^2}{\omega_3^2}+\frac{a_4^2+b_4^2}{\omega_4^2}\right), \qquad (3.3)$$

which is towards the red side. Hence some of the high-frequency weak oscillations can induce red shifts by nonlinear mixing with some powerful high-frequency vibrations when subjected to the linear phase-matching condition (2.14) in a plasma. This interesting effect should be thoroughly investigated with the help of some experiments for their detection.

Putting $b_i=0$, $i=1,2,3,4$, and using the values of (2.25) and (2.26), we get the numerical estimation

$$\frac{\delta\omega_i}{\omega_i}\approx 3.6\times 10^{-11}. \qquad (3.4)$$

At this time it is not clear that such a relative frequency shift of this order of magnitude is detectable.[21]

IV. NEAR-RESONANT PARAMETRIC INTERACTIONS

A. Plane-polarized pump fields

In (2.17)–(2.20), assuming the solution $e^{i\omega t}$, and considering plane-polarized waves only, we get for the increment to the frequencies ω_1 and ω_2 the quadratic equation

$$\left[\omega-\frac{e^2\omega_p^2}{8m^2c^2\omega_1}\left(\frac{a_3^2}{\omega_3^2}+\frac{a_4^2}{\omega_4^2}\right)\right]\left[\omega-\frac{e^2\omega_p^2}{8m^2c^2\omega_2}\left(\frac{a_3^2}{\omega_3^2}+\frac{a_4^2}{\omega_4^2}\right)\right]$$
$$=\left(\frac{e^2\omega_p^2a_3a_4}{8m^2c^2\omega_3\omega_4}\right)^2\frac{1}{\omega_1\omega_2}\left[3-\frac{(k_2+k_3)^2c^2}{(\omega_2+\omega_3)^2-\omega_p^2}-\frac{2(k_2+k_3)^2c^2}{(\omega_1+\omega_3)^2-\omega_p^2}\right]\left[3-\frac{(k_1+k_3)^2c^2}{(\omega_1+\omega_3)^2-\omega_p^2}-\frac{2(k_2+k_3)^2c^2}{(\omega_2+\omega_3)^2-\omega_p^2}\right]. \qquad (4.1)$$

The two factors in the square brackets on the right-hand side contain terms whose denominators are $(\omega_2+\omega_3)^2-\omega_p^2$ and $(\omega_1+\omega_3)^2-\omega_p^2$. When one of the near-resonant conditions (2.22) and (2.23) is satisfied, these terms dominate over the others. Since these are the only terms which come exclusively from nonrelativistic nonlinear effects, the relativistic contributions can be totally ignored in such interactions.

Using (2.22) we obtain

$$\omega\approx\frac{3e^2\omega_p^2a_3a_4}{8m^2c^2\omega_3\omega_4(\omega_1\omega_2)^{1/2}}\left(1+\frac{1}{2\delta}\right). \qquad (4.2)$$

The data of (2.26) yield

$$\omega\approx 6.5\times 10^5\ \text{sec}^{-1}. \qquad (4.3)$$

The condition (2.23) gives the growth rate

$$\gamma\approx\frac{e^2\omega_p^2a_3a_4(1+1/2\delta)^{1/2}(1+1/\delta)^{1/2}}{8m^2c^2\omega_3\omega_4|\omega_1\omega_2|^{1/2}}. \qquad (4.4)$$

Putting the values of (2.26) into (4.4), we get

$$\gamma\approx 9.25\times 10^5\ \text{sec}^{-1}. \qquad (4.5)$$

In this case, the threshold values of P_3 and P_4 should be such that

$$\frac{\nu}{2\omega_1\omega_2}=\frac{\pi e^2(P_3P_4)^{1/2}(1+1/2\delta)^{1/2}(2+1/\delta)^{1/2}}{m^2c^3\omega_3\omega_4|\omega_1\omega_2|^{1/2}}, \qquad (4.6)$$

where ν is defined in Sec. II.

B. Circularly polarized pump fields

Putting $b_3=a_3$ and $b_4=a_4$ in (2.17)–(2.20), we find that the waves of $a_1\pm b_1$ are parametrically coupled, respectively, to those of $a_2\pm b_2$. There will then be four frequency shifts which are the solutions of the following two quadratic equations:

$$A/B=(\tfrac{1}{4},1), \qquad (4.7)$$

where

$$A=\left[\omega-\frac{e^2\omega_p^2}{4m^2c^2\omega_1}\left(\frac{a_3^2}{\omega_3^2}+\frac{a_4^2}{\omega_4^2}\right)\right]\left[\omega-\frac{e^2\omega_p^2}{4m^2c^2\omega_2}\left(\frac{a_3^2}{\omega_3^2}+\frac{a_4^2}{\omega_4^2}\right)\right],$$

$$B=\left(\frac{e^2\omega_p^2a_3a_4}{2m^2c^2\omega_3\omega_4}\right)^2\frac{1}{\omega_1\omega_2}\left[1-\frac{(k_1+k_3)^2c^2}{(\omega_1+\omega_3)^2-\omega_p^2}\right]\times\left[1-\frac{(k_2+k_3)^2c^2}{(\omega_2+\omega_3)^2-\omega_p^2}\right].$$

As in Sec. IV A, when any one of the near-resonant conditions (2.22) and (2.23) is satisfied, the relativistic effects become insignificant. For (2.22) we get the two frequency shifts

$$\omega\approx\frac{e^2\omega_p^2a_3a_4(1+1/2\delta)}{2m^2c^2\omega_3\omega_4(\omega_1\omega_2)^{1/2}}(\tfrac{1}{2},1). \qquad (4.8)$$

If (2.26) is used, their numerical values are

$$\omega\approx 8.6\times 10^5(\tfrac{1}{2},1)\ \text{sec}^{-1}. \qquad (4.9)$$

When (2.23) holds, (4.7) yields two growth rates γ, which are numerically identical to the ω of (4.8) and (4.9). The corresponding threshold value of the geometric mean $(P_3P_4)^{1/2}$ of the pump-field intensities is given by

$$(P_3P_4)^{1/2}=\frac{\nu m^2c^3\omega_3\omega_4(2,1)}{8\pi e^2|\omega_1\omega_2|^{1/2}(1+1/2\delta)}. \quad (4.10)$$

Using (2.26) we get

$$(P_3P_4)^{1/2}\approx 5\times 10^8\times\nu \text{ erg/cm}^2\text{ sec}.$$

C. Pump fields oscillating in two mutually perpendicular planes

When the two driver waves are in two mutually perpendicular planes of polarization passing through the common direction of propagation, we can put $a_4=0$, and $b_3=0$ into (2.17)–(2.20) and get two equations connecting a_1 to b_2, and a_2 to b_1. These yield for ω the two quadratic equations

$$\frac{C}{D}=-\frac{1}{2\omega_1\omega_2}\left[\left(1-\frac{(k_1+k_3)^2c^2}{(\omega_1+\omega_3)^2-\omega_p^2}\right)^2,\left(1-\frac{(k_2+k_3)^2c^2}{(\omega_2+\omega_3)^2-\omega_p^2}\right)^2\right], \quad (4.11)$$

where

$$C=\left[\omega-\frac{e^2\omega_p^2}{8m^2c^2\omega_1}\left(\frac{a_3^2}{\omega_3^2}+\frac{b_4^2}{\omega_4^2}\right)\right]\left[\omega-\frac{e^2\omega_p^2}{8m^2c^2\omega_2}\left(\frac{a_3^2}{\omega_3^2}+\frac{b_4^2}{\omega_4^2}\right)\right],\qquad D=\left(\frac{e^2\omega_p^2a_3b_4}{4m^2c^2\omega_3\omega_4}\right)^2.$$

The near-resonant condition (2.22) yields the growth rate

$$\gamma\approx\frac{e^2\omega_p^2a_3b_4(1+1/2\delta)}{4m^2c^2\omega_3\omega_4(2\omega_1\omega_2)^{1/2}}; \quad (4.12)$$

the condition (2.23) gives the frequency shift

$$\omega\approx\frac{e^2\omega_p^2a_3b_4(1+1/2\delta)}{4m^2c^2\omega_3\omega_4(2|\omega_1\omega_2|)^{1/2}} \quad (4.13)$$

and another, numerically very small, shift.

V. RESONANT SECULAR INTERACTIONS

Resonant interaction leading to secular temporal growth of the nonlinear sources is effected when the difference between the magnitudes of the frequencies of two of the four waves exactly equals ω_p. For example, such types of interactions occur when

$$(\omega_4-\omega_2)^2=(\omega_3+\omega_1)^2=\omega_p^2. \quad (5.1)$$

Using this relation and retaining only the secularly growing terms, we get

$$v_2=-\frac{te^2(k_1+k_3)}{4m^2}\left(\frac{a_1a_3-b_1b_3}{\omega_1\omega_3}-\frac{a_2a_4+b_2b_4}{\omega_2\omega_4}\right)\sin(\theta_1+\theta_2). \quad (5.2)$$

With the help of the third and fourth relations of (2.8) and (5.2), the secular terms of E_2 and n_2 are determined. Then such terms of the nonlinear third-order sources $(\vec{v}_2\cdot\vec{\nabla})\vec{v}_1$, $-(e/mc)(\vec{v}_2\times\vec{H}_1)$, and $-1/n_0(\partial/\partial t)(n_2\vec{v}_1)$ are evaluated. Using these results in (2.11) and retaining only the secularly growing parts, we have

$$\ddot{\vec{E}}_3-c^2\nabla^2\vec{E}_3+\omega_p^2\vec{E}_3=-\frac{te^2\omega_p^2(k_1+k_3)^2}{8m^2}\left(\frac{a_1a_3-b_1b_3}{\omega_1\omega_3}-\frac{a_2a_4+b_2b_4}{\omega_2\omega_4}\right)\left[\frac{\omega_1}{\omega_3}(\vec{e}_xa_3\sin\theta_1+\vec{e}_yb_3\cos\theta_1)-\frac{\omega_2}{\omega_4}(\vec{e}_xa_4\sin\theta_2+\vec{e}_yb_4\cos\theta_2)\right]. \quad (5.3)$$

Using on the left-hand side the solution (2.5) and considering only the plane-polarized waves, the equations connecting a_1 and a_2 are

$$2\omega_1\frac{\partial a_1}{\partial t}+i\frac{\partial^2a_2}{\partial t^2}+\frac{te^2\omega_p(k_1+k_3)^2a_3^2}{8m^2\omega_3^2}a_1=\frac{te^2\omega_p\omega_1(k_1+k_3)^2a_3a_4}{8m^2\omega_2\omega_3\omega_4}a_2, \quad (5.4)$$

$$2\omega_2\frac{\partial a_2}{\partial t}+i\frac{\partial^2a_2}{\partial t^2}+\frac{te^2\omega_p(k_1+k_3)^2a_4^2}{8m^2\omega_4^2}a_2=\frac{te^2\omega_p\omega_1(k_1+k_3)^2a_3a_4}{8m^2\omega_1\omega_3\omega_4}a_1. \quad (5.5)$$

A solution of the mutually connected equations is sought in the form

$$(a_1, a_2) = (A_1, A_2)\exp[i(\alpha_1 t + \alpha_2 t^2 + \alpha_3 t^3 + \cdots)] \tag{5.6}$$

where A_1, A_2 are independent of time. Eliminating a_1 and a_2 between (5.4) and (5.5), a polynomial in t, containing as coefficients of different powers of t some functions of the parameters α_1, α_2, ... are obtained,

$$\Big[2\omega_1\alpha_1 i - i\alpha_1^2 - 2\alpha_2 + \Big(2\omega_1\times 2\alpha_2 i - 3\times 2\alpha_3 - 2\alpha_1\times 2\alpha_2 i + \frac{e^2\omega_p(k_1+k_3)^2 a_3^2}{8m^2\omega_3^2}\Big) + (2\omega_1\times 3\alpha_3 i - 4\times 3\times\alpha_4 - 2\alpha_1\times 3\alpha_3 i)t^2 + \ldots\Big]$$
$$\times\Big[2\omega_2\alpha_1 i - i\alpha_1^2 - 2\alpha_2 + \Big(2\omega_2\times 2\alpha_2 i - 3\times 2\times\alpha_3 - 2\alpha_1\times 2\alpha_2 i + \frac{e^2\omega_p(k_1+k_3)^2 a_4^2}{8m^2\omega_4^2}\Big) + (2\omega_2\times 3\alpha_3 i - 4\times 3\times\alpha_4 - 2\alpha_1\times 3\alpha_3 i)t^2 + \ldots\Big] = t^2\Big(\frac{e^2\omega_p(k_1+k_3)^2 a_3 a_4}{8m^2\omega_3\omega_4}\Big)^2. \tag{5.7}$$

The coefficients of t^0 and t' in the left-hand side of (5.7) vanish if

$$\alpha_2 = i\omega_1\alpha_1 - \tfrac{1}{2}i\alpha_1^2 = i\omega_2\alpha_1 - \tfrac{1}{2}i\alpha_1^2. \tag{5.8}$$

Since $\omega_1 \neq \omega_2$, the solutions are

$$\alpha_1 = 0, \quad \alpha_2 = 0. \tag{5.9}$$

Equating coefficients of t^2 from both sides of (5.7) and using in them (5.8) and (5.9), we get

$$6\alpha_3 = 0, \frac{e^2\omega_p(k_1+k_3)^2}{8m^2}\Big(\frac{a_3^2}{\omega_3^2} + \frac{a_4^2}{\omega_4^2}\Big). \tag{5.10}$$

Similarly, coefficients of t^3 yield

$$2\alpha_4 = i\alpha_3\omega_1. \tag{5.11}$$

Since α_3 of (5.10) is real, it is an increment to periodicity in time. For the same reason, α_4 of (5.11) being an imaginary quantity, gives a growth rate in time.

Using (5.9) the relation (5.6) can be written as

$$(a_1, a_2) = (A_1, A_2)\exp[i(\omega t)^3 - (\gamma t)^4 + \cdots]. \tag{5.12}$$

As seen below, the quantities ω and γ are the closest equivalents of a frequency shift and a growth rate, respectively, for an equation like (5.3).

$$\omega = \Big[\frac{e^2\omega_p(k_1+k_3)^2}{8m^2}\Big(\frac{a_3^2}{\omega_3^2} + \frac{a_4^2}{\omega_4^2}\Big)\Big]^{1/3}, \tag{5.13}$$

$$\gamma = \Big[\frac{e^2\omega_p(k_1+k_3)^2\omega_1}{16m^2}\Big(\frac{a_3^2}{\omega_3^2} + \frac{a_4^2}{\omega_4^2}\Big)\Big]^{1/4}. \tag{5.14}$$

Using (2.26) their numerical estimates are

$$\omega \approx 1.9\times 10^{10}\,\text{sec}^{-1}, \quad \gamma \approx 1.6\times 10^{11}\,\text{sec}^{-1}. \tag{5.15}$$

Hence the growth due to the factor $\exp(\gamma^4 t^4)$ is about $\exp(10^4)$ times in a time period $2\pi/\omega_1$, and more than $\exp(10^{44})$ times in 1 sec. So the system should break in a time much shorter than $2\pi/\omega_1$, and the theory of resonant secular interaction will fail even much before this time is attained.

VI. CONCLUDING REMARKS

The parametric interactions involving four electromagnetic waves in a cold plasma are quite powerful effects when two of these waves are available as powerful pump waves and a linear phase-matching condition is satisfied. When any one of the beat frequencies does not either equal the electron plasma frequency ω_p or lie in its neighborhood, the relativistic corrections give a large contribution to the frequency shift.

Ignoring the relativistic effects, the parametric frequency shift ω for plane-polarized fields is given by a relation of the type

$$\omega^2 = \alpha, \tag{6.1}$$

where α is a function of a_1, a_2, a_3, a_4, ω_1, ω_2, ω_3, and ω_4, and the frequencies are restricted by the relation (2.12). But when relativistic effects are also included, the equation for ω can be expressed as

$$(\omega+\beta_1)(\omega+\beta_2) = \alpha^1 \tag{6.2}$$

where β_1 and β_2 depend purely on the relativistic effect and $\alpha^1 \neq \alpha$. If $\omega_i^2 \gg \omega_p^2$, $\omega_i > 0$, $i = 1, 2, 3, 4$, that is, when the conditions (2.24) and (3.1) are fulfilled

$$\omega \approx -\beta_1, \; -\beta_2, \quad \alpha^1 \approx 0, \quad \alpha \neq 0. \tag{6.3}$$

So two different shifts are obtained.

Using the more general set of equation (2.12)–(2.20) the equation for ω can be expressed as

$$\begin{vmatrix} \omega+\alpha & 0 & \gamma_1 & \delta_1 \\ 0 & \omega+\beta & \gamma_2 & \delta_2 \\ \alpha_3 & \beta_3 & \omega+\gamma & 0 \\ \alpha_4 & \beta_4 & 0 & \omega+\delta \end{vmatrix} = 0 . \qquad (6.4)$$

If the relativistic terms are ignored, $\alpha=0$, $\beta=0$, $\gamma=0$, and $\delta=0$, and some parts of the other elements of (6.4) also disappear. Then the coefficients of ω^3 and ω become zero and the biquadratic (6.4) reduces to an equation of the type

$$(\omega^2+p\omega+q)(\omega^2-p\omega+v)=0 . \qquad (6.5)$$

But, on the other hand, if the conditions (2.24) and (3.1) are fulfilled, except for α, β, γ, and δ, all the coefficients of (6.4) vanish and so Eqs. (2.17)–(2.20) are not mutually connected.

The physical implication of the appearance of several frequency shifts and growth rates of signal waves is that a combination of several different colors is developed by the interaction of other types of colors with a plasma medium. The question of the choice of any one of them to the exclusion of the others does not arise, because, naturally, that color which has the largest growth rate will be most prominent.

When the difference between two of the four frequencies is close to the characteristic plasma frequency ω_p, the relativistic contribution to the frequency shift is negligible. When the frequency difference exactly equals ω_p, moreover, the equations for the parametric effects contain secularly growing sources.

The physical processes responsible for the appearance of frequency shifts in some cases and temporal growths in the other cases are not yet clear to us. We also do not know the physical reason why the "resonant" case behaves so differently from the "nonresonant" cases. Side-scattering effects are important in the artificial glow when created in the atmosphere by powerful signals sent from man-made instruments. So, if an investigation is made of the two-dimensional propagation, in which the direction of propagation of all the four waves are different but coplanar, then, the side-scattering effects can be studied; moreover, then other peculiarities of the four-waves interactions, some of which we have noted but could not explain, may become clear.

All waves in plasmas in the absence of a static magnetic field must have frequencies greater than the characteristic plasma frequency ω_p. When the frequencies of two of the waves differ by ω_p, a resonance caused by sources containing secular terms occurs. In the near-resonance cases in which this frequency difference is close to ω_p, some of the sources producing the parametric interaction become very powerful and others negligible. A satisfactory study of the physical reason for the behavior of the medium subject to the resonant and near-resonant parametric interactions has not yet been attempted. For these studies it seems advisable to start with model equations of the Klein-Gordon type having simple cubic nonlinearities; for example, the model of Chandra[22] may be used for that purpose.

Wang[23,24] has studied the energy-transfer problems for three-wave interactions involving ordinary nonlinear differential equations and the question of bounds of solutions of these equations. Witham[25] has investigated the energy-transfer process and its slow variation for a single wave in nonlinear dispersive media. These references should be useful in future works on the analysis of energy transfer from the pump fields to the signal fields in parametric four-wave interactions of the type studied here.

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