Kinetics of the Compton scattering and the Bose condensation of a photon gas

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We study the kinetics of photons coupled to an electron gas by Compton collisions. We show that, for the limiting cases of linear and nonlinear evolutions, an initial narrow spectrum exhibits the same tendency of shifting towards the red frequencies. Concerning the Bose condensation of the photon gas which is predicted by standard thermodynamic arguments when the total number A of photons exceeds some critical value A_c, we show that the nonlinear kinetic evolution leads to a condensationlike phenomena. This condensation, which always appears when $A > A_c$, can also occur when $A < A_c$ for sufficiently large and sharply peaked initial photon distributions. Such long-lived quasi-equilibrium states can be called "metastably condensed."

I. INTRODUCTION

Many papers have already been devoted to the kinetics of a photon gas interacting with electrons (or with an ionized gas) via Compton scattering, and its physical applications (mainly astrophysical).[1-7] It is certainly a rich subject, including phase-transition processes. However, the approach is made difficult by the complex nonlinear structure of the kinetic equations, and only partial information has been obtained, or conjectured, on the behavior of the solutions.

Our purpose in this paper is to add to previous results some rather qualitative considerations on the various types of evolutions (either linear or nonlinear) which may be expected in this problem, and on the mechanism of Bose condensation.

As is well known, the kinetic equation for photons contains both linear and nonlinear terms, and the resulting evolution can be mainly linear or nonlinear according to the physical situation we consider.

The nonlinearity of the kinetic equation tends to shift an initial photon distribution towards lower frequencies, and to steepen the red side of the distribution curve, ending with a vertical front after a finite time lapse. The subsequent evolution has been studied in some detail by Reinisch[8]: The existence of the diffusion term (however small it is) in the Compton-Fokker-Planck equation describing the photons' kinetics prevents the solution from being multivalued and causes the vertical "shock front" to move towards lower frequencies. We will show that this shock front collides with the zero-frequency wall after a finite time interval, and then the spectrum begins to pile up into a very small frequency domain near $\nu=0$. Such an evolution occurs when considering intense initial spectrums localized around an average frequency ν_0 (laser or maser radiations for instance).

Linear evolutions are associated with small occupation numbers (such as those which usually pertain to a hard-x-ray spectrum). In this case, if the total number of photons and electrons is conserved, we obtain the unexpected result that the linearized kinetic equation is also characterized (in the limit of vanishing diffusion coefficient) by a global red shift of the initial spectrum, accompanied by a narrowing of the photon distribution. In this last case, however, the approach to the low-frequency domain is gently progressive, in contradistinction with the sudden catastrophe which appears in the nonlinear evolution.

This "catastrophe" has been related to the phenomena of Bose condensation of the photon gas. In the case where the temperature T of the electron scatterers is maintained fixed, the condensation is predicted to occur on the basis of standard thermodynamic arguments, when the total number of photon A exceeds the critical value A_c corresponding to the equilibrium Planck distribution at temperature T. (Remember that, in this problem, A is conserved.) In this case the equilibrium state would contain a "condensed phase" of photons of quasizero energy coexisting with a Planckian distribution of uncondensed photons associated with a zero value of the chemical potential. We show that, for appropriate initial conditions, in the course of the nonlinear evolution the system can reach a quasi-equilibrium where a condensed phase (of nearly zero energy) may coexist together with a noncondensed phase whose chemical potential is different from zero. Of particular interest is the case where the condensation occurs for $A<A_c$, that is, for subcritical thermodynamic conditions. The existence of such quasi-stationary states, which may be called "metastably condensed," is peculiar to the photon system because of the zero rest mass of these boson particles.

II. SOME CHARACTERISTIC FEATURES OF VARIOUS TYPES OF EVOLUTION (LINEAR AND NONLINEAR)

We consider the evolution of an initial ensemble of photons and electrons interacting via only

Compton interactions (dilute media). Therefore, the total number of particles of each species is conserved in time, and their distributions functions [$N(\vec{\nu})$ for photons, $f(\vec{p})$ for electrons] obey the following kinetic equations (assuming spatial homogeneity and assuming low electronic densities so that electrons can be considered as bare particles):

$$\partial_t N(\vec{\nu}) = \int_0^\infty \nu'^2\, d\nu' \int d\vec{\Omega}' \int d\vec{p}\, \sigma_{\vec{p}}(\vec{\nu} \to \vec{\nu}')\{N(\vec{\nu}')[1+N(\vec{\nu})]f(\vec{p}') - N(\vec{\nu})[1+N(\vec{\nu}')]f(\vec{p})\}, \tag{1}$$

$$\partial_t f(\vec{p}) = \int_0^\infty \nu^2\, d\nu \int d\vec{\Omega}\, d\vec{p}'\sigma\{N(\vec{\nu}')[1+N(\vec{\nu})]f(p') - N(\vec{\nu})[1+N(\vec{\nu}')]f(p)\}, \tag{2}$$

where $\vec{\nu}$ stands for $\nu\vec{\Omega}$, $\vec{\Omega}$ being the unit vector of the $\vec{\nu}$ photon, and $\sigma(\vec{\nu} \to \vec{\nu}')$ is the Klein-Nishina cross section (in the frame where the electron is at rest).

The limit of low-energy photons and low-energy electrons is defined by the conditions

$$h\nu/mc^2 \ll 1, \tag{3}$$

$$T_e/mc^2 \ll 1, \tag{4}$$

T_e being the average electron energy [condition (3) expresses the fact that $\Delta\nu/\nu \ll 1$ in a Compton collision].

In this limit, and making the additional assumption of isotropy in velocity space (we recall that the anisotropies usually relax on shorter times than the isotropic part of the distributions), the kinetic equations reduce to

$$\partial_t N = (\alpha/\nu^2)\partial_\nu[\nu^4(N + N^2 + \mu\, \partial_\nu N)], \tag{5}$$

$$\partial_t T_e = \alpha h \int \nu^2(N + N^2 + \mu\, \partial_\nu N)\, d\nu, \tag{6}$$

where

$$\mu = T_e/h, \tag{7}$$

$$\alpha = h(n_e \sigma c)/mc^2 \tag{8}$$

(where n_e is the electronic density and σ is the square of the classical electron radius). We postpone the study of the stationary solutions of the above equations to Sec. III, and we now focus on the kinetics of a photon gas interacting with electrons in thermal equilibrium at the constant temperature T. The evolution is then only described by the "Compton-Fokker-Planck" equation (5), in which α and μ are given coefficients.

The radiation spectra produced by nonequilibrium sources are characterized by occupation numbers $N(\nu)$ which may vary in an enormous range, when passing from radio to x-ray frequencies. When $N \gg 1$ (which is easily realized in the radio-frequency range) the evolution is frequently of the nonlinear type (which means that the N^2 term in Eq. (5) is very important). However, when $N \ll 1$ (x-ray range), the evolution is usually of the linear type. We shall briefly consider these two cases.

A. Nonlinear type of evolution

As was already shown by Zeldovich and Levich,[2] a striking feature of the nonlinear evolution is the appearance of a condensation-like phenomenon after a finite time interval. Since the behavior of the photon distribution may be pathological near $\nu = 0$, it is advisable to consider an initial spectrum $N^0(\nu)$ which does not extend up to the very low frequencies. We choose $N_0(\nu)$ to be a peaked function around ν_0, with a finite characteristic width $\Delta\nu$. Such an initial condition, which is pictured in Fig. 1, is of course of interest in the case of maser or laser sources.

Let us first rewrite Eq. (5) in terms of the density of occupation numbers in ν space, $u = \nu^2 N$,

$$\partial_\tau u = (u^2 + x^2 u)' + \mu(x^2 u'' - 2u), \tag{9}$$

where x stands for the frequency and $\tau = \alpha t$. Assuming that, for large enough u, the right-hand side of Eq. (9) reduces to its nonlinear term [$\partial_\tau u = (u^2)'$], the solution is known (cf. Ref. 2) and is given by the implicit relation

$$u = u_0(x + 2u\tau). \tag{10}$$

Starting with the above $N^0(\nu)$, this solution is characterized by a progressive steepening of the red side of the initial distribution, and after a time interval τ_1 of the order of $\Delta\nu/u^0_{\max} \sim \Delta\nu/\nu_0^2 N_0$, a

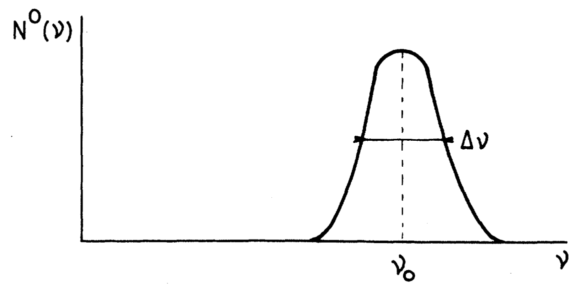

FIG. 1. Inital spectrum of photons.

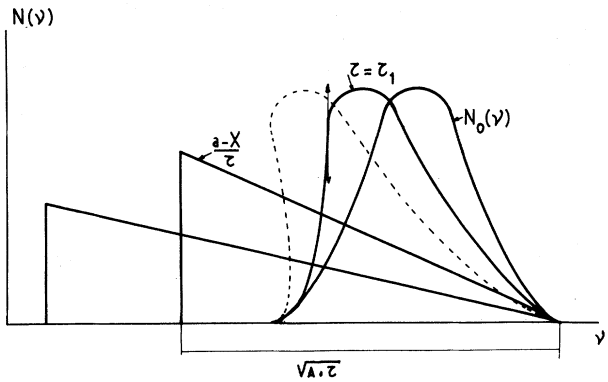

FIG. 2. Burgers' type evolution. Dotted curves correspond to the solution in the absence of diffusion process.

quasi-vertical slope has been built up. Later on, multivalued solutions would be obtained. Such an evolution for $\tau > \tau_1$ is not allowed if one takes into account the linear diffusion term in Eq. (9). It has been shown by Reinisch[8] that in the limit $\mu \to 0$, Eq. (9) may be approximated on convenient time scale by

$$\partial_\tau u = (u^2)' + \tilde{\mu} u'' ,$$

with $\tilde{\mu} = \mu x_c^2$ being the abscissa of the shock front. Such an equation, which is of the Burgers type,[8] may be integrated, and its solution (for $\tau < \tilde{\mu}^{-1}$) has the triangular form represented in Fig. 2. The condition of existence of this triangular solution is known (see Appendix A for an elementary derivation) and reads

$$A/\tilde{\mu}^2 \gg 1 ,$$

where

$$A = 4\pi \int_0^\infty u \, d\nu \approx x_0^2 \Delta\nu N .$$

Putting x of the order of x_0, this condition reads

$$N \gg \mu/\Delta\nu . \qquad (11)$$

The slope of the nonvertical side of the triangle is inversely proportional to time, and the actual "width" of the vertical front is of the order of $(\tilde{\mu}^2 t/A)^{1/2}$ (see Appendix A). A remarkable feature is that the kinetics of the shock front is independent of $\tilde{\mu}$. At time

$$\tau_c = x_0^2/A \sim (1/\nu_0)[(\nu_0/\Delta\nu)N]^{-1}$$

the front reaches the origin with a vanishing width. Then the x dependence of $\tilde{\mu}$ can no longer be ignored in Eq. (9), and all the terms of Eq. (1) would be needed in order to describe the subsequent evolution. However, solution (10) still remains valid, if one excludes a very small domain near $\nu = 0$ (of the order of the shock width at $\tau = \tau_c$). As a result the major part of the distribution still remains linear, with a slope which still decreases as $1/\tau$. Since the norm A is conserved in the course of the process, we therefore expect that the whole photon distribution progressively piles up in the vicinity of $\nu = 0$. This is the condensation phenomenon. It is worth noting that the width of the condensed distribution may eventually become so small that the condition of small momentum transfer

$$\left(N(\nu + \delta\nu) \sim N(\nu) + \delta\nu \frac{\partial N}{\partial \nu} \right)$$

on which Eq. (5) is based may be invalidated, and that the ultimate evolution could only be described by Eq. (1). It is also important to note that, without the knowledge of the solution of the kinetic equation, we can neither predict which fraction of the initial spectrum will be condensed, nor answer the question: would an appreciable amount of energy be stored in the very-low-frequency domain in the presence of a constant source of photons? A last remark concerns the characteristic times and light paths which are needed in order to observe the condensation of a nonequilibrium radiation spectrum. Coming back to the original time scale, we have

$$t_c \sim (\alpha\nu_0)^{-1}[(\nu_0/\Delta\nu)N]^{-1} . \qquad (12)$$

It is interesting to note that, in the case of a Planckian initial distribution (cf. Chapline, Cooper, Slutz[7]) (owing to the large width of this distribution), the characteristic time for the onset of the condensation is not given by expression (12). It has been estimated in Ref. 7 as the time at which, according to the nonlinear solution (10), particles appear at $\nu = 0$. It is easily verified that, at this time, all the derivatives of N with respect to ν are infinite. Therefore, we recover the notion that the condensation sets up when a vertical distribution reaches the origin. Here t_c is of the order of $(\alpha\nu_0)^{-1}$, where ν_0 is the average frequency of the initial value. Such a value, expressed in terms of the corresponding initial number of photons, may be much larger than the above value associated with a sharp initial spectrum provided the occupation numbers are large enough. It is also worth noting that the light path L_c corresponding to the above characteristic times must be evaluated by taking into account the Brownian motion of the photons with the nonlinear Compton mean free path $l_{NL} = 1/N(n_e\sigma)$. Therefore, L_c is of the order of

$$L_c \sim l_{NL}(ct_c/l_{NL})^{1/2} = (l_{NL}ct_c)^{1/2} \sim (ct_c/n_e\sigma N)^{1/2}. \qquad (13)$$

B. Linear evolution

The term associated with spontaneous emission in the kinetic equation [$\nu^4 N$ term in Eq. (5) or $(x^2u)'$ term in Eq. (9)] has been neglected in the above nonlinear model. It is easily seen that the above triangular profile is compatible with this approximation if $N \gg \nu_0/\Delta\nu$. In the opposite case the evolution is governed (on convenient time scale) by the linear terms. Roughly speaking it may be said that linear evolutions are of interest in the case of nonequilibrium radiation where the occupation numbers are very small (as in the case of x-ray spectra). Let us make the additional assumption that the diffusion term in Eq. (9) is negligible, which amounts to considering situations where $\mu/\Delta\nu \ll 1$ ($\Delta\nu$ being the width of the photon spectrum at any time). The kinetic equation then reduces to

$$\partial_\tau u = (x^2 u)' . \tag{14}$$

Its solution is easily obtained by the method of characteristics, and reads (see Appendix B)

$$u(x, \tau) = [1/(1 - x\tau)^2] u^0 x/(1 - x\tau) . \tag{15}$$

Here also the evolution is characterized by a global red shift of the initial spectrum.

Starting from a rectangular profile around $x = x_0$, the evolution is pictured on Fig. 3. When $\tau \gtrsim 1/x_0$, an asymptotic behavior is reached in which u is growing like τ^2, while the spectrum width narrows like $\Delta X/x^2\tau^2$. Since the coherence time of the electromagnetic field inside the wave packet increase like τ^{-1}, such an evolution produces a more and more organized structure. We also remark that the displacement towards lower frequencies is here quite regular, in contradistinction with the sudden condensation which appears in the nonlinear process.

We can also ask the question: What could be the energy stored in the electromagnetic field in the low-frequency domain in the presence of a time-independent photon source $S(x)$? Solving the equation

$$\partial_\tau u = (x^2 u)' + S(x)$$

is easily done in terms of the Green's function of Eq. (14) (see Appendix A):

$$G(x, x', \tau) = \frac{1}{(1 - x\tau)^2}\, \delta\left(\frac{x}{1 - x\tau} - x'\right) .$$

Dropping the initial condition term, we obtain

$$u(x, t) = \int_0^t d\tau \frac{1}{(1 - x\tau)^2} S\left(\frac{x}{1 - x\tau}\right) .$$

In the case of a localized source function $S(x) = \delta(x - x_0)$, we obtain

$$u(x, t) = x^{-2} Y[t - (x_0 - x)/x_0 x],$$

where $Y(x)$ is the Heaviside step function. The time evolution of $u(x, t)$ is pictured on Fig. 4.

At time t (and for $t \gg x_0^{-1}$) the spectrum grows like x^{-2} up to

$$x_m = x_0/(1 + x_0 t) \sim 1/t .$$

The total photon energy E grows like $\log t$ and concentrates near the lower bound x_m of the spectrum.

In the initial value problem an appreciable narrowing and relative red shift of the initial distribution is achieved for times of the order of $(\alpha\nu_0)^{-1}$ (in the original time scale).

In actual physical situations the above type of evolution ends when the neglected diffusion term starts acting. This happens when the width of the spectrum becomes of the same order as the characteristic diffusion frequency μ, then the diffusion process enlarges the spectrum towards higher fre-

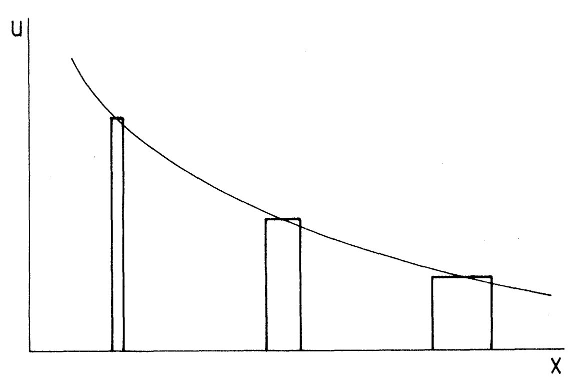

FIG. 3. Linear evolution of an initial condition.

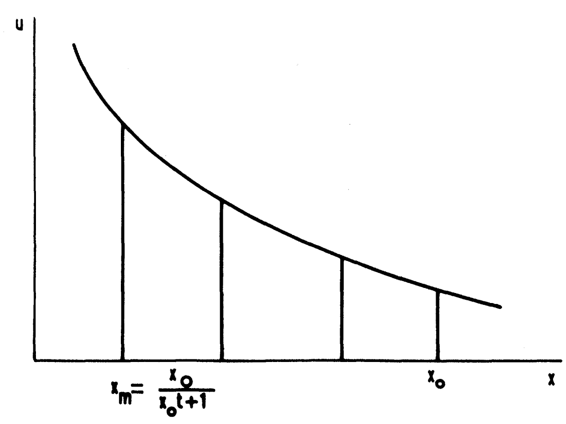

FIG. 4. Linear evolution in the presence of a source term at x_0.

quencies. [This can be seen with the help of the Green function of the diffusion operator of Eq. (9), already obtained by Zeldovich and Syunyaev.[9]] In astrophysical situations the linear evolution could be eventually responsible of a large red shift of a hard x-ray radiations. The interacting length must be here expressed in terms of the *linear* Compton mean free path, and reads $L\sim 10^{-38}(N_e^2\nu)^{-1/2}$. As a numerical application of the astrophysical type, if the photon energies are of the order of 10 keV and the electronic density $N_e\sim 10\ \mathrm{cm}^{-3}$, we find $L\sim 10^{22}$ m. In the case of higher photon energy, we may reasonably expect shorter interaction length, but the kinetics is then governed by the integral Eq. (1).

III. STATIONARY SOLUTIONS: CONDENSED AND METASTABLY CONDENSED STATES

Up to now we restrict ourselves to study some particular types of evolutions, only relevant on limited time scales. Therefore, the problem of stationary solutions of the kinetic equations is still open. It is known that coupled equations (5, 6) for photons and electrons are satisfied by a Maxwellian distribution at temperature T for electrons and for the photons, by the Planckian-like distribution

$$N(\nu)=(e^{h(\nu+\nu_0)/T}-1)^{-1}\,, \tag{16}$$

where $h\nu_0$ plays the role of a chemical potential and is determined by the initial conditions of the problem. Equation (16) is the general solution of

$$N+N^2+\mu\,\partial_\nu N=0\,,$$

which ensures the vanishing of the collision term of the photon kinetic equation. The coupled kinetics of electrons and photons conserve three constants of the motion, namely: the total energy E_0 (photon + electron), the number $\mathfrak{N}$ of electrons, and the number of photons A. These invariants, together with the Bose-Einstein statistics of the photons, impose strong constraints on the possible stationary states. We see that, in order to ensure that the general stationary solution (16) be positive definite, we must satisfy the condition $\nu_0\geq 0$. This condition is easily formulated in terms of initial conditions in the case of the non-self-consistent problem ($T_e=C^{te}$). It amounts to saying that the total number of photons associated with distribution (16) is necessarily smaller than the corresponding number pertaining to the Planck distribution (that is for $\nu_0=0$):

$$A=A(\nu_0,T_e)=4\pi\int\frac{\nu^2\,d\nu}{e^{h(\nu+\nu_0)/T}-1}<A(0,T_e)\,.$$

Since

$$A(0T_e)=(V/c^3)\,\tfrac{4}{15}\pi^5(T/h)^3=\tfrac{4}{15}\pi^5\mu^3V/c^3\,,$$

the above condition reads

$$A<\gamma\mu^3,\quad \gamma=\tfrac{4}{15}\pi^5V/c^3\,,$$

which reads approximately for a sharply peaked $N(\nu)$ with width $\Delta\nu$

$$N<\gamma(\mu/\Delta\nu)(\mu/\nu)^2 \tag{17}$$

(in the case of a localized distribution of the type pictured on Fig. 1). Starting from an initial Planckian distribution with temperature T_0, the above condition would imply that $T_0<T_e$. In the case of the general self-consistent problem [where N and T_c evolve simultaneously according to Eqs. (5) and (6)], the existence criterion of a stationary solution may be written in the form (cf. Appendix C)

$$E_0\geq(15/4\pi^5)^{1/3}h(c/v)^{1/3}\times[3A^{1/3}\mathfrak{N}_0^e+2.404(15/4\pi^4)A^{4/3}]\,. \tag{18}$$

The nonexistence of stationary solutions of type (16) has been related[2] to the condensation process of a Bose gas. Considering for simplicity the non-self-consistent problem, the thermodynamics indeed predicts that the equilibrium state must contain a condensed phase when the thermal wavelength λ_{DB} of the bosons becomes of the order or larger than their interparticle distance $p^{-1/3}$. In the case of photons $\lambda_{DB}\sim hc/T$, and we see that condition (17) coincides with the thermodynamic condition of noncondensation. It is important to study the physics of the condensation from the point of view of kinetic theory. First of all, we remark that the very notion of a photon loses its meaning when $\nu\to 0$. Indeed, to analyze the electromagnetic field as a photon ensemble is an appropriate picture only if the characteristic time $\delta\tau$ of the kinetic evolution is much larger than $\bar{\nu}^{-1}$ ($\bar{\nu}$ being the average frequency of the condensed photons). Assuming the nonlinear term of the kinetic equation to be dominant, $\delta\tau\sim(N\alpha\bar{\nu})^{-1}$. The photon description is appropriate as long as $N\alpha\bar{\nu}<\bar{\nu}$ or

$$N<N_{\max}=mc^2/h\,(n\sigma c)=Z(nr_0^3)^{-1},$$

where $Z=\frac{1}{137}$ is the fine structure constant. Since the total number of photons is conserved in the course of the evolution, the above condition gives a lower bound of the accessible frequencies: indeed,

$$(\nu_{\min}^3/c^3)N_{\max}\sim(\nu_0^2/c^3)\Delta\nu_0N^0$$

(assuming that $\nu\sim\delta\nu$ in the condensed phase). Therefore,

$$\nu_{\min}\sim(N^0/N_{\max})\nu_0^3(\Delta\nu_0/\nu_0)\,,$$

from which we could also deduce an upper bound of the "condensable" electromagnetic energy, starting from a given initial photon spectrum. It is worth remarking that, in realistic physical problems, other phenomena like the existence of a cutoff frequency (plasma frequency) for the photons, the inverse bremsstrahlung, the multiphoton scattering, or the multiple scattering of a photon (when the Compton mean free path and the interparticle lengths become comparable) may limit the condensation process.

Let us now look for the stationary solution of the original integral equation (1). A distribution containing a condensed part must be defined with some care in a system with continuous energy levels, owing to the fact that the negative frequency domain is strictly forbidden. A convenient form is the following:

$$N(\nu)=\lim_{\epsilon\to 0}\{f\delta(\nu-\epsilon)+[e^{h(\nu+\nu_0)/T}-1]^{-1}\}, \qquad (19)$$

where f is a coefficient having the dimension of a frequency, and representing the amount of condensed particles. Replacing $N(\nu)$ by the above expression in the collision term of Eq. (1) we obtain

$$\int d\nu'\,\bar{\sigma}(\nu,\nu')\{f^2(e^{\nu'-\nu}-1)\delta(\nu'-\epsilon)\delta(\nu-\epsilon)+f(e^{\nu_0+\epsilon}-1)^2e^{\nu'-\epsilon}[\delta(\nu'-\epsilon)-\delta(\nu-\epsilon)]/(e^{\nu+\nu_0}-1)(e^{\nu'+\nu^0}-1)\},$$

where we put $h/T=1$ and $\bar{\sigma}(\nu,\nu')$ is the Compton cross section averaged over the angle variable, whose expression is (see Appendix D)

$$\bar{\sigma}(\nu,\nu')=\left(\frac{n_c\sigma}{\nu}\right)\left(\frac{2mc^2}{\pi T}\right)\int_0^1 du[1-u^2(1-u^2)]\exp\left[-\frac{m}{T}\frac{c^2}{u^2}\left(\frac{\nu-\nu'}{\nu}\right)^2\right]. \qquad (20)$$

Clearly the f^2 term gives no contribution. The second term may be written

$$f(e^{\nu_0+\epsilon}-1)\left(\int d\nu'\frac{e^{\nu_0+\epsilon}-1}{(e^{\nu+\nu_0}-1)(e^{\nu'+\nu_0}-1)}e^{\nu'-\epsilon}[\delta(\nu'-\epsilon)-\delta(\nu-\epsilon)]\bar{\sigma}(\nu,\nu')\right).$$

The integral in the bracketed term is always finite. We therefore conclude that, whatever be the particular form of $\bar{\sigma}(\nu,\nu')$, distribution (19) with zero chemical potential ($\nu_0=0$) is a stationary solution of the kinetic equation: this distribution is nothing but the thermodynamic one, which maximizes the entropy. However, in the present photon problem, we find that distribution (18) is a stationary solution for any value of ν_0. Indeed, it is easily seen in expression (20) that $\bar{\sigma}(\nu,\epsilon)$ behaves like $\delta(\nu-\epsilon)$ in the limit $T/mc^2\to 0$, and also $\bar{\sigma}(\epsilon,\nu')\sim\delta(\nu'-\epsilon)$. As a result the bracketed integral in Eq. (20) vanishes. The fact that we find stationary solutions which do not maximize the entropy, is simply due to the fact that the evolution of our photon system only proceeds via Compton interactions, and the peculiar behavior of the cross sections at low energy which comes from the zero mass of the photons.

Now what kind of evolution can we predict towards the stationary state? As was shown in Sec. II, the condensation begins suddenly at some critical time, building up a sharp peak near $\nu=0$ of "condensed photons." This peak will be very weakly coupled to the uncondensed distribution, owing to the very small values of the Compton collision term in the low-energy domain. Physically the energy transfer in a Compton collision is very small and the redistribution of condensed particles towards higher energies is a very slow process as soon as the uncondensed particles have reached a quasi-stationary distribution of the form (16).

A first possible situation is the one where the thermodynamic criterium of condensation is satisfied [$N>(\mu/\Delta\nu)(\mu/\nu_0)^2$ in the case of an initial peaked distribution around frequency ν_0]. Then, too many photons are present at the initial time, and the spectrum cannot reach its stationary state without condensation. If $\mu/\nu_0>1$, the former criterion (11) of nonlinear evolution is also satisfied and we know that, after a finite time interval, a quasi-vertical distribution collides with the zero-frequency wall, starting on the condensation phenomenon. One may expect that, at least, all the photons in excess with respect to A_p (A_p being the number of photons corresponding to the Planck distribution) will condense into the low-frequency peak. Later on, the uncondensed photons will redistribute themselves according to the Planck-like distribution. This second stage will proceed along a larger time scale (at least the inverse diffusion time μ^{-1}). In this second stage, the chemical potential of the uncondensed distribution may or may not have a zero value. In the latter case the total distribution is not the equilibrium one, and, since the condensed phase have a finite energy spread in the course of the evolution, a slow evolution may still go on until the final equilibrium state (with $\nu_0=0$) is reached.

More unusual is the case where

$$\mu/\Delta\nu < N < (\mu/\Delta\nu)(\mu/\nu)^2 .$$

According to the results of Sec. II, the evolution is of the nonlinear type and the condensation starts at $t \sim t_e$, while the thermodynamics does not predict the condensation of the equilibrium state. Here we predict that in the first stage, a substantial part of the initial spectrum get "condensed" into a narrow peak in the low-frequency domain. Then the remaining photons redistribute themselves, along time scale t_c, into a Planck-like distribution with chemical potential ν_0' (associated with the number of uncondensed particle at the end of the first stage). Later on the narrow (but finite width) peak still interacts with the uncondensed phase, and we may reasonably expect that this process goes on (along a much larger time interval) until all particles escape from the peak, the final distribution being the Planckian with chemical potential ν_0 corresponding to the over-all initial distribution. Such transitory (but long lived) condensed states which appear for subcritical value of A may be called "metastably condensed."

The above evolution takes place only if the absorptive processes such as bremsstrahlung are negligible, that is for very low density of matter. As a consequence, the time evolution of the matter's temperature would have to be taken into account. However, as said in Sec. II A, the kinetics of the condensation does not depend on $\tilde{\mu}$ (which is a reduced electron temperature). Moreover, it is easily shown[5] that the asymptotic value of the electronic temperature associated with our metastably condensed state is the chemical potential ν_0' of the Planck-like distribution of the uncondensed photons. This chemical potential has no reason to be large and it can be shown there exists a class of initial conditions where ν_0' is equal to the initial electronic temperature. In other words, the condensation mechanism is essentially a redistribution of the photon energy inside the photon's spectrum via the Compton collisions.

APPENDIX A

Let us consider an initial condition $u_0(x)$ of the form

$$u_0(x) = A\,\delta(x - x_0) . \tag{A1}$$

(Actually $u_0(x)$ must be looked at as a peaked function around $x = x_0$ with the small width σ.)

Using the Hopf's change of function,[8] we put

$$u = \tilde{\mu}\,\partial_x \ln\varphi = \tilde{\mu}\,\frac{\partial_x \varphi}{\varphi} ,$$

or

$$\varphi = \exp\left(\frac{1}{\tilde{\mu}} \int u\,\partial_{x'} \right). \tag{A2}$$

$\varphi_0(x)$ is pictured in Fig. 5.

If $A/\tilde{\mu} \gg 1$, $\partial_x \varphi_0(x)$ behaves like $e^{A/\tilde{\mu}}\delta(x - x_0)$. Now $\varphi(x,t)$ obeys the diffusion equation

$$\partial_t \varphi = \tilde{\mu}\varphi'' ,$$

whose solution is of the form

$$\varphi \sim \frac{1}{\sqrt{\tilde{\mu} t}} \int dy\, e^{-y^2/4\tilde{\mu}t}\, \varphi_0(x - y) .$$

In the space derivative

$$\partial_x \varphi = \frac{1}{\sqrt{\tilde{\mu} t}} \int dy\, e^{-y^2/4\tilde{\mu}t}\, \partial_x \varphi_0(x - y) ,$$

approximating $\partial_x \varphi_0(x-y)$ by $e^{A/\tilde{\mu}}\delta(x - x_0)$ is justified as soon as $t \gg \sigma^2/\tilde{\mu}$. Then we have

$$\tilde{\mu}\partial_x \varphi \sim (\tilde{\mu} e^{A/\tilde{\mu}}/\sqrt{\tilde{\mu} t}\,) e^{-(x-x_0)^2/4\tilde{\mu}t}$$

Evaluating now $\varphi(x,t)$, we have for $t \gg \sigma^2/\tilde{\mu}$

$$\varphi \sim e^{A/\tilde{\mu}} \int_{-\infty}^{(x-x_0)/2\sqrt{\tilde{\mu}t}} e^{-y^2} dy + \int_{(x-x_0)/2\sqrt{\tilde{\mu}t}}^{+\infty} e^{-y^2} dy . \tag{A3}$$

For simplicity we consider time evolution smaller than the diffusion time μ^{-1}. It is easily seen that φ takes relatively large values only for $x - x_0 < 0$ and $|x - x_0|/\sqrt{\tilde{\mu}t} \gg 1$. Putting $\lambda = (x_0 - x)/2\sqrt{\tilde{\mu}t}$, we have

$$\int_{-\infty}^{-\lambda} e^{-y^2} dy = 1 - \phi(\lambda) \sim e^{-\lambda^2/2} \quad (\text{for } \lambda \gg 1) ,$$

$\phi(\lambda)$ being the error function. It is easily seen that $\varphi(x,t)$ have quite different values according to the value of parameter $e^{A/\tilde{\mu}} e^{-\lambda^2}/\lambda$: If

$$e^{A/\tilde{\mu}} e^{-\lambda^2}/\lambda \gg 1, \quad \varphi \sim (e^{-\lambda^2}/\lambda) e^{A/\tilde{\mu}} .$$

If

$$x < x_0\sqrt{A t} ,$$

$$u \sim (\tilde{\mu}/\sqrt{\tilde{\mu}t}\,) e^{A/\tilde{\mu}}\, e^{-(x-x_0)^2/4\tilde{\mu}t} \ll (x_0 - x)/t .$$

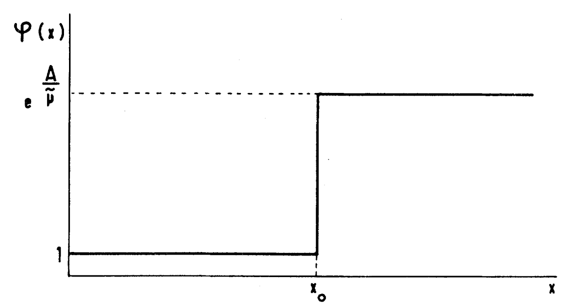

FIG. 5. Sketch of the function $\varphi(x)$.

The limiting value of λ separating the two above regimes is given by $e^{A/\tilde{\mu}} e^{-\lambda^2}/\lambda \sim 1$ or $\lambda \approx \sqrt{A/\tilde{\mu}}$, which implies $x_0 - x = \sqrt{At}$. Expressing $u(x,t)$ in terms of φ by Eq. (A1), we obtain: if

$$x > x_0 - \sqrt{At} \; , \quad u = (x_0 - x)/t \; ;$$

if

$$x < x_0 - \sqrt{At} \; ,$$

$$u = (\tilde{\mu} / \sqrt{\tilde{\mu} t}) e^{A/\tilde{\mu}} e^{-(x-x_0)^2/4\tilde{\mu} t} \ll (x_0 - x)/t \; .$$

We therefore find, if $A/(\tilde{\mu})^2 \gg 1$, the quasi-triangular profile which is pictured on Fig. 2 of the main text. The effective width of the vertical side is the width of $\varphi(x)$ given by (A3) around $x = x_0$. It is easily found that it is of the order of $(\tilde{\mu}^2 t/A)^{1/2}$.

APPENDIX B

Let us put

$$x^2 u(x,\tau) = \nu(x,\tau) \; . \tag{B1}$$

Equation (14) reads, in terms of $\nu(x,\tau)$,

$$(\partial_\tau - x^2 \partial_x)\nu = 0 \; . \tag{B2}$$

The solution of Eq. (B1) is immediately obtained by the method of characteristics and reads

$$\nu(x,\tau) = \nu^0 x/(1 - x\tau) \; ,$$

where $\nu^0(x)$ is the initial condition for $\nu(x)[\nu^0(x) = x^2 u^0(x)]$. Expressing now $\nu(x,\tau)$ in terms of $u(x,\tau)$ through Eq. (B1), we obtain for $u(x,\tau)$ expression (14) of the main text.

The Green's function of Eq. (14) $G(x,x',\tau)$ obeys the equations

$$\partial_\tau G(x,x',\tau) - \partial_x[x^2 G(x,x',\tau)] = 0 \; ,$$

$$G(x,x',\tau=0) = \delta(x - x') \; .$$

It is easily verified that the solution of the above equations is

$$G(x,x',\tau) = [1/(1-x\tau)^2]\,\delta[x/(1-x\tau) - x'] \; .$$

Considering now the equation

$$\partial_\tau u = (x^2 u)' + S(x) \; ,$$

of the main text, the solution of this equation in terms of $G(x,x',\tau)$ reads

$$u(x,t) = \int dx'\, G(x,x',\tau) u^0(x) + \int_0^t d\tau \int dx'\, G(x,x',t-\tau) S(x') \; .$$

Dropping the initial condition term and changing τ for $t - \tau$, we are left with

$$u(x,t) = \int_0^t d\tau \int dx'\, G(x,x',\tau) S(x') = \int_0^t d\tau \frac{1}{(1-x\tau)^2} S\left(\frac{x}{1-x\tau}\right) \; .$$

APPENDIX C

Let us consider, in the self-consistent problem, the stationary state

$$N(\nu) = (e^{h(\nu+\nu_0)/T} - 1)^{-1} \; ,$$

which is eventually reached when $t \to \infty$ (T_∞ being the asymptotic temperature). The conservation of the total energy reads

$$E_0 = 3\mathfrak{N}_0^e T_\infty + A\epsilon \; , \tag{C1}$$

where ϵ is the average energy per photon, defined as

$$\epsilon = \int_0^\infty \frac{\nu^3 d\nu}{e^{h(\nu+\nu_0)/T} - 1} \Big/ \int_0^\infty \frac{\nu^2 d\nu}{e^{h(\nu+\nu_0)/T} - 1} \; .$$

Condition $\nu_0 \geqslant 0$ ensures that ϵ is a monotonic growing function of ν_0. Indeed we may write $\partial_{\nu_0}\epsilon$ in the form

$$\partial_{\nu_0}\epsilon = \int d\nu\, d\nu' [(e^{\nu+\nu_0} - 1)(e^{\nu'+\nu_0} - 1)^2 (e^{\nu'+\nu_0} - 1)(e^{\nu+\nu_0} - 1)^2]^{-1} \times \{ \nu^3 \nu'^2 (e^{\nu+\nu_0} - 1)(e^{\nu'+\nu_0} - 1)[e^{\nu'+\nu_0}(e^{\nu+\nu_0} - 1) - e^{\nu+\nu_0}(e^{\nu'+\nu_0} - 1)]\} \, .$$

The numerator of the above integrand may be put after symmetrization ($\nu \longleftrightarrow \nu'$), in the form

$$\tfrac{1}{2}\nu^2 \nu'^2 (e^{\nu+\nu_0} - 1)(e^{\nu'+\nu_0} - 1) e^{\nu+\nu_0} \{(\nu - \nu')[1 - e^{\nu'-\nu}]\} \, ,$$

which is a positive definite function of ν and ν'.

We therefore conclude that $\epsilon > \epsilon_P(T_\infty)$, where $\epsilon_P(T_\infty)$ is the Planck distribution at temperature $T_\infty (\epsilon_P = \epsilon_{\nu_0 = 0})$. Then Eq. (C1) yields the following inequality:

$$E_0 \geqslant 3\mathfrak{N}_0^e T_\infty + A\epsilon_P(T_\infty) .$$

Now we have

$$A < A(T_\infty) = \gamma_1 T_\infty^3 \, ,$$

where

$$\gamma_1 = (\nu/c^3 h^3) \tfrac{4}{15} \pi^5 \; .$$

Since $\epsilon_P(T_\infty) = \gamma_2 T_\infty$ with $\gamma_2 = 2.404(15/\pi^4)$ we have the inequalities

$$E_0 \geqslant 3\mathfrak{N}_0^e T_\infty + \gamma_2 A T_\infty \geqslant \gamma^{1/3}(3\mathfrak{N}^e A^{1/3} + \gamma_2 A^{4/3}) \; .$$

Therefore

$$E_0 \geqslant \gamma_1^{1/3}(3\mathcal{K}_0^e A^{1/3} + \gamma_2 A^{4/3});$$

the equal sign being associated with the exact Planckian distribution, the above condition is necessary and sufficient for obtaining no condensation.

APPENDIX D

In the limit of small energies the conservation of energy and momentum in a Compton collision (Fig. 6) reads

$$\frac{p^2}{2m} + h\nu = \frac{p'^2}{2m} + h\nu', \quad \delta\vec{p} = \frac{h}{c}(\nu'\vec{\Omega}' - \nu\vec{\Omega})$$

from which we obtain (in the limit $|p/mc| \ll 1$)

$$\delta\nu/\nu = (\vec{p}/mc)\cdot(\vec{\Omega} - \vec{\Omega}')\,.$$

Now the Klein Nishina cross section (for unpolarized radiation) is

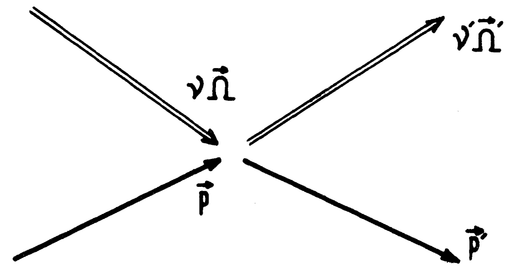

FIG. 6. Compton collision.

$$d\sigma = \tfrac{1}{2} r_0^2 (1 + \cos\theta)\, d\Omega\,,$$

where r_0 is the electron classical radices. Therefore the transition probability $P(\vec{\nu} \to \vec{\nu}')$ by unit of time is

$$P(\vec{\nu} \to \vec{\nu}')\, d\Omega' = \int d\vec{p}\, f(\vec{p}) \tfrac{1}{2} c r_0^2 (1 + \cos^2\theta)\, \delta\left[\nu' - \nu - (\nu\vec{p}/mc)\cdot(\vec{\Omega} - \vec{\Omega}')\right] d\Omega'\,.$$

Let us consider a Maxwellian distribution for $f(\vec{p})$ and integrate over the velocities perpendicular to $(\vec{\Omega} - \vec{\Omega}')$. We obtain

$$P\, d\Omega' = n(m/2\pi T)^{1/2} d\Omega' \int dv_x\, e^{-mv_x^2/2T} \tfrac{1}{2} c r_0^2 (1 + \cos^2\theta)\, \delta\left[\nu' - \nu - \nu(v_x/c)(2\sin\tfrac{1}{2}\theta)\right]\,.$$

Integrating over the parallel velocity v_x, and over Ω', we finally obtain expression (20) of $\bar{\sigma}(\nu, \nu')$ given in the main text.

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