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Inverse bremsstrahlung energy absorption rate

Yaakov Shima and Haim Yatom
Soreq Nuclear Research Centre, Yavne, Israel
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The rate of energy absorption by inverse bremsstrahlung of a plasma in an oscillating electric field is calculated. An expression for this rate is obtained that is valid for all ranges of the electron-plasma temperature and density and the electric field flux intensity and polarization. Analytical formulas are obtained for high-flux intensities where the rate is independent of the electron-velocity-distribution function.

I. INTRODUCTION

One of the methods used in heating a plasma to high temperatures is the application of electromagnetic (e.m.) radiation (laser light). In the first stages of this process, when the plasma density is relatively low, $\omega_p < \omega$ (where ω_p is the plasma frequency, $\omega_p = (4\pi n_e e^2/m)^{1/2}$, and ω is the laser frequency), the main mechanism of heating is inverse bremsstrahlung, where electrons absorb energy from the e.m. field when they collide with the ions. The energy thus absorbed per unit volume, per unit time, dW/dt, is a function of the following parameters: e.m. flux intensity and polarization, laser frequency, electron temperature T, and electron plasma density n_e.

The energy absorption rate by inverse bremsstrahlung has already been calculated for low fluxes,[1-3] where the main process is one-photon absorption and the result is given by a simple analytical formula.[4] The calculation at high fluxes, where the main contribution to the absorption is from multiphoton processes, is more difficult and was considered by several authors.[5-10]

We shall see in what follows that at low fluxes the absorption rate does not depend on the polarization of the laser light but may depend on the velocity distribution of the electrons. At high fluxes the situation is reversed. The absorption rate depends strongly on whether the light is linearly or circularly polarized but is quite independent of the electron-velocity distribution.

There are two approaches to the calculation of the absorption rate: classical and quantum-mechanical methods. In the classical approach[1,2,5,6] the Vlasov equation is used to calculate the induced current and the resulting absorbed energy. The electrons are taken to have a Maxwellian distribution and the ions are assumed to be at rest and to be either randomly or nonrandomly distributed. In these calculations there appear divergent integrals which can be avoided by introducing, *ad hoc*, cutting-off limits of the integration.

In the quantum-mechanical approach[3,7-14] one first calculates the transition probability for the elementary process of inverse bremsstrahlung (photon emission or absorption) in an electron-ion collision in an oscillating e.m. field and then takes an appropriate statistical average over all the electron momenta. There is no need for cutting off the integrals and Planck's constant $\hbar$ appears explicitly in the results in the dimensionless parameter q, $q=\hbar\omega/2kT$. Both approaches give the same result when q is extremely small, if $\hbar$ is introduced explicitly in one of the cutting-off limits in the classical calculation, by taking the minimum distance of closest approach in electron-ion collisions to be $\hbar/(mkT)^{1/2}$ instead of the usual Ze^2/kT.[15] When q is not extremely small the classical theory breaks down. Because of interest in laser fusion experiments it is necessary to determine dW/dt for all ranges of electron temperatures and flux intensities in the plasma. To our knowledge this has not been done in a complete, exhaustive manner.

In Sec. II we discuss the quantum-mechanical theory of the elementary process of inverse bremsstrahlung. In Sec. III we calculate the statistical average and obtain an expression for the rate of energy absorption dW/dt valid for all ranges of the parameters, for different distribution functions of the electrons, and also for linearly and circularly polarized light. In Sec. IV we evaluate the rate of energy absorption for three different distribution functions of the electrons, namely, Maxwell-Boltzmann, Fermi-Dirac, and δ-shaped. We give analytical expressions which cover quite accurately almost all ranges of the parameters and show graphically the dependence of dW/dt on flux intensities.

II. TRANSITION PROBABILITY

We shall here calculate, in the first Born approximation, the transition probability per unit time for a specific momentum change in a collision between two free charged particles with masses m_1 and m_2 and charges $Z_1 e$ and $Z_2 e$ in the presence of an electric field oscillating with a given frequency ω. It is assumed that during this process the internal structure of the two particles remains unchanged. This means that if the plasma is not fully ionized we do not take into account possible excitations or ionizations of the ions. We also assume that the particles are nonrelativistic and we neglect the effects due to magnetic fields.

The perturbation $U(r_{12})$ which is responsible for the momentum change is the electrostatic interaction which depends on the distance $r_{12} = |\vec{r}_1 - \vec{r}_2|$ only. The oscillating electric field $\vec{E}$ is treated[7-10] classically and the dipole approximation is used. E is a (periodic) function of time only. The last approximation is valid if the electric field wavelength is large compared with the range of the effective interaction.

An unperturbed state of the two particles with momenta $\vec{p}_1$ and $\vec{p}_2$ is described by a wave function $\psi_{\vec{p}_1,\vec{p}_2}$ which is a product of two one-particle wave functions $\psi_{\vec{p}_1}$ and $\psi_{\vec{p}_2}$:

$$\psi_{\vec{p}_1,\vec{p}_2}(\vec{r}_1,\vec{r}_2,t)=\psi^{(m_1,Z_1)}_{\vec{p}_1}(\vec{r}_1,t)\times\psi^{(m_2,Z_2)}_{\vec{p}_2}(\vec{r}_2,t) \quad . \tag{1}$$

$\psi_{\vec{p}}$ is given by[10]

$$\psi^{(m,Z)}_{\vec{p}}(\vec{r},t) = \exp\left[-i\hbar\vec{p}\cdot\vec{r} - \frac{i}{2m\hbar}\int_{-\infty}^{t}\left(\vec{p}-\frac{Ze}{c}\vec{A}(t')\right)^2 dt'\right] \quad , \tag{2}$$

where the vector potential $\vec{A}(t)$ satisfies

$$\vec{E} + \frac{1}{c}\frac{\partial\vec{A}}{\partial t} = 0 \ , \tag{3}$$

$$\frac{\partial^2\vec{A}}{\partial t^2} + \omega^2\vec{A} = 0 \ . \tag{4}$$

The transition probability per unit time $P(\vec{p}_1, \vec{p}_2 \to \vec{p}_1', \vec{p}_2')$ from an initial state characterized by $\vec{p}_1, \vec{p}_2$ to a final state characterized by $\vec{p}_1', \vec{p}_2'$ under the perturbation $U(r_{12})$ will be evaluated for two kinds of polarization of the electric field $\vec{E}$, namely, for linear polarization and for circular polarization:

(a) linear polarization (l.p.),

$$\vec{E} = E\vec{e}_x \sin\omega t \ ; \tag{5}$$

(b) circular polarization (c.p.),

$$\vec{E} = \frac{1}{2^{1/2}}E(\vec{e}_x \sin\omega t + \vec{e}_y \cos\omega t) \ , \tag{6}$$

where $\vec{e}_x, \vec{e}_y$ are unit vectors perpendicular to the propagation direction (z direction).

In the first Born approximation we obtain for P

$$P(\vec{p}_1, \vec{p}_2 \to \vec{p}_1', \vec{p}_2') = \frac{2\pi}{\hbar}\delta_{\vec{p}_1+\vec{p}_2,\vec{p}_1'+\vec{p}_2'}|U_{p_1\to p_1'}|^2 \times \sum_{n=-\infty}^{\infty} J_n^2(X_{12})\delta(W_1'+W_2'-W_1-W_2-n\hbar\omega) \ , \tag{7}$$

where

$$X_{12} = \frac{eE}{\hbar\omega^2}\left(\frac{Z_2}{m_2}-\frac{Z_1}{m_1}\right)(\Delta p_1)_E \ ,$$

$$(\Delta p_1)_E = (\vec{p}_1' - \vec{p}_1)\cdot\vec{e}_x \quad \text{for l.p.} \tag{8}$$

$$= \frac{1}{2^{1/2}}\{[(\vec{p}_1'-\vec{p}_1)\cdot\vec{e}_x]^2 + [(\vec{p}_1'-\vec{p}_1)\cdot\vec{e}_y]^2\}^{1/2} \quad \text{for c.p.}$$

J_n is the usual Bessel function of order n and $U_{\vec{p}_1\to\vec{p}_2}$ is the Fourier transform of the interaction $U(r)$,

$$U_{\vec{p}\to\vec{p}'} = \int d^3r\, U(r) e^{-(i/\hbar)[(\vec{p}'-\vec{p})\cdot\vec{r}]} \ . \tag{9}$$

Positive n corresponds to absorption of n photons whereas negative n corresponds to the emission of $|n|$ photons. J_n^2 is a measure of the strength of these processes and includes all orders of perturbation in the electric field intensity. Both absorption and emission of n photons have the same probability (detailed balance). It is evident that P is symmetrical with respect to the interchange of particles 1 and 2, and satisfies the requirement of energy and momentum conservation.

For particles with the same Z/m ratio, $X_{12} = 0$, the field intensity E disappears from the equation and P corresponds to the transition probability for elastic collisions. This means that there is no exchange of energy with the oscillating electric field during collisions between particles that have the same Z/m ratio (electron-electron collisions). Consider now electron-ion collisions where $m_1 = m$, $m_2 = M$, $Z_1 = -1$, $Z_2 = Z$, $1/m \gg Z/M$. X_{12} depends on electron parameters only if we neglect Z/M compared with $1/m$. We can also show that one can neglect the difference between the kinetic energies of the ion compared with the difference between the kinetic energies of the electron in the energy-conservation factor. It then follows that the only place in Eq. (7) where the ion momenta appears is in the $\delta_{\vec{p}_1+\vec{p}_1',\vec{p}_2+\vec{p}_2'}$ factor. Summing Eq. (7) over all the ion momenta we get the usual[7,9-14] transition probability per unit time for one electron from state $\vec{p}$ to state $\vec{p}'$ by collision with the ions in a unit volume,

$$P(\vec{p}\to\vec{p}\,')=\frac{2\pi}{\hbar}n_i|U_{\vec{p}\to\vec{p}\,'}|^2 \times\sum_{n=-\infty}^{\infty}J_n^2(X)\delta(W'-W-n\hbar\omega)\ , \qquad (10)$$

where

$$W=\frac{p^2}{2m},\quad X=\frac{eE}{\hbar m\omega^2}(\Delta p)_E\ ,$$

and n_i is the number of ions per unit volume. We have assumed in Eq. (10) only one kind of ion with a charge distribution that enters through the potential U. For more than one kind of ion, we would have to take an appropriate average.

III. STATISTICAL AVERAGE

The energy absorbed by the electrons per unit time per unit volume, dW/dt, is given by

$$\frac{dW}{dt}=\sum_{\vec{p}}f(\vec{p},t)\sum_{\vec{p}\,'}P(\vec{p}\to\vec{p}\,')(1-f(\vec{p}\,',t))(W'-W)$$
$$=\tfrac{1}{2}\sum_{\vec{p},\vec{p}\,'}[f(\vec{p},t)-f(\vec{p}\,',t)](W'-W)P(\vec{p}\to\vec{p}\,')\ , \qquad (11)$$

where $f(\vec{p},t)$ is the electron-distribution function, satisfying

$$\sum_{\vec{p}}f(\vec{p},t)=n_e\ ,$$

$$\frac{\partial f(\vec{p},t)}{\partial t}=\sum_{\vec{p}\,'}[f(\vec{p}\,',t)-f(\vec{p},t)]P(\vec{p}\,'\to\vec{p})\ . \qquad (12)$$

To evaluate Eq. (11) we approximate $f(\vec{p},t)$ by a time-independent initial isotropic distribution $f(\vec{p})$. This approximation is valid when the energy imparted to the electrons is small compared to the initial energy of the electrons. This is the case for the usual short pulses used in laser plasma experiments and low-flux lasers. For high-flux lasers, which are required for the initiation of thermonuclear reactions, the electron temperature rises by several orders of magnitude and the validity of the approximation might be in doubt. However, as mentioned in the Introduction, we shall see in what follows that for high fluxes the absorption rate is independent of the assumed initial distribution function and depends only on the flux intensity, the mean electron energy, and the electron density. We replace the sums over the momenta in Eq. (11) by integrals according to $\sum_{\vec{p}}\to(1/2\pi\hbar)^3\int d^3p$ and make the transformations

$$\vec{\rho}=(\vec{p}\,'-\vec{p})/2p_0,\quad \vec{\sigma}=(\vec{p}\,'+\vec{p})/2p_0\ , \qquad (13)$$

where p_0 is a characteristic momentum of the plasma, $f=f(p^2/p_0^2)$. In the new coordinates the Fourier transform of the potential $U_{\vec{p}\to\vec{p}\,'}$ can be written

$$U_{\vec{p}\to\vec{p}\,'}=-(\pi Ze^2\hbar^2/p_0^2)\phi(\rho)\ . \qquad (14)$$

Using Eqs. (10)–(14), the expression for dW/dt becomes

$$\frac{dW}{dt}=C\tfrac{1}{2}q^{3/2}\int d^3\rho\,|\phi(\rho)|^2\sum_{n=1}^{\infty}nJ_n^2\left(\frac{s}{q}(\rho)_E\right)\int d^3\sigma[\bar{f}(\rho^2+\sigma^2-2nq)-\bar{f}(\rho^2+\sigma^2+2nq)]\delta(\vec{\rho}\cdot\vec{\sigma}-nq)\ , \qquad (15)$$

where

$$q=\frac{m\hbar\omega}{2p_0^2},\quad s=\frac{eE}{\omega p_0}=\left(\frac{2\pi}{c}\right)^{1/2}\frac{2e}{\omega p_0}I^{1/2}\ ,\quad \bar{f}=\frac{1}{n_e}\left(\frac{p_0}{2\pi\hbar}\right)^3f\left(\frac{p^2}{p_0^2}\right),\quad C=4n_e^2e^4Z\left(\frac{2}{m\hbar\omega}\right)^{1/2}\ , \qquad (16)$$

$$(\rho)_E=\rho_x \qquad \text{for l.p.}$$
$$=[\tfrac{1}{2}(\rho_x^2+\rho_y^2)]^{1/2} \quad \text{for c.p.}\ ,$$

and I is the flux intensity, $I=cE^2/8\pi$, with c the light velocity.

The main dimensionless parameters of the problem are s and q. s is the ratio of two momenta: eE/ω and p_0. eE/ω is a characteristic momentum which an electron acquires in the oscillating external field. $4q$ is the ratio of two energies, the photon energy $\hbar\omega$ and the characteristic energy W_0 of an electron in the plasma, where $W_0=p_0^2/2m$. $\bar{f}$ is a dimensionless distribution function and depends only on the squares $(\vec{\rho}\pm\vec{\sigma})^2$. The integration $\int d\Omega_\sigma$ over the angles of $\vec{\sigma}$ in Eq. (15) can be performed readily, because the angles appear in $\vec{\rho}\cdot\vec{\sigma}$ only. It can be easily shown that

$$\int\delta(\vec{\rho}\cdot\vec{\sigma}-nq)d\Omega_\sigma=\frac{2\pi}{\rho\sigma}\Theta\left(1-\frac{nq}{\rho\sigma}\right)\ , \qquad (17)$$

where $\Theta(x)$ is the step function. $\Theta(x)=0$ when $x<0$; $\Theta(x)=1$ when $x>0$.

Putting Eq. (17) in Eq. (15) we obtain

$$\frac{dW}{dt}=C\pi q^{3/2}\int\frac{d^3\rho}{\rho}|\phi(\rho)|^2\sum_{n=1}^{\infty}nJ_n^2\left(\frac{s}{q}(\rho)_E\right)$$
$$\times\int_{nq/\rho}^{\infty}d\sigma\,\sigma[\bar f(\rho^2+\sigma^2-2nq)-\bar f(\rho^2+\sigma^2+2nq)]\ . \tag{18}$$

We shall evaluate Eq. (18) for three types of distribution functions: Maxwell-Boltzmann (hot plasma), where the electron temperature T is large compared with the degenerate electron-gas temperature T_g ($T>T_g$); Fermi-Dirac (cold plasma), where $T_g>T$; and δ-shaped functions.

A. Hot plasma

For a hot plasma the distribution function can be taken to be Maxwellian,

$$\bar f=\frac{1}{(2\pi)^{3/2}}e^{-p^2/2p_0^2}\ ,$$
$$p_0^2=mkT,\quad q=\hbar\omega/2kT\ . \tag{19}$$

The expression (18) becomes, after integrating over σ,[9]

$$\frac{dW}{dt}=C\frac{1}{(2\pi)^{1/2}}q^{3/2}\int\frac{d^3\rho}{\rho}|\phi(\rho)|^2$$
$$\times\sum_{n=1}^{\infty}n\sinh(nq)\ e^{-(\rho^2+n^2q^2/\rho^2)/2}J_n^2\left(\frac{s}{q}(\rho)_E\right)\ . \tag{20}$$

The angular dependence of $\vec\rho$ enters only in the argument of the Bessel functions $s(\rho)_E/q$. Therefore, the integration $\int d\Omega_\rho$ will give two different forms depending on whether the light is polarized linearly or circularly. The result is

$$\frac{dW}{dt}=C(2\pi)^{1/2}\ q^{3/2}\int_0^{\infty}d\rho\,\rho|\phi(\rho)|^2$$
$$\times\sum_{n=1}^{\infty}n\sinh(nq)\ e^{-(\rho^2+n^2q^2/\rho^2)/2}D_n\left(\frac{s}{q}\rho\right)\ , \tag{21}$$

where D_n has the following two forms:

$$D_n(t)=\int_0^{\pi}J_n^2(t\cos\theta)\sin\theta d\theta$$
$$=\frac{2}{t}\int_0^{t}J_n^2(y)dy\qquad\text{for l.p.}$$
$$=\int_0^{\pi}J_n^2\left(\frac{1}{2^{1/2}}t\sin\theta\right)\sin\theta d\theta$$
$$=\frac{4}{t}\int_0^{t/2^{1/2}}J_n^2(y)\frac{y\,dy}{(t^2-2y^2)^{1/2}}\quad\text{for c.p.} \tag{22}$$

B. Cold plasma

For a cold plasma we take a Fermi-Dirac distribution function and the corresponding $\bar f$ is

$$\bar f=\frac{3}{4\pi}\Theta\left(1-\left(\frac{p}{p_F}\right)^2\right). \tag{23}$$

The characteristic momentum p_0 is now p_F. If we substitute p_F for p_0 in the parameters s and q and introduce $\bar f$ from Eq. (23) in Eq. (18), the result for the energy absorption rate becomes

$$\frac{dW}{dt}=C(3\pi/4)q^{3/2}\left[\sum_{n=1}^{\infty}n\int_{a_n^-}^{a_n^+}d\rho\,\rho\,|\phi(\rho)|^2[1-(\rho-nq/\rho)^2]D_n\left(\frac{s}{q}\rho\right)\right.$$
$$\left.-\sum_{n=1}^{[1/4q]}n\int_{b_n^-}^{b_n^+}d\rho\ \rho|\phi(\rho)|^2[1-(\rho+nq/\rho)^2]D_n\left(\frac{s}{q}\rho\right)\right]\ , \tag{24}$$

where

$$a_n^{\pm}=\tfrac12[(1+4nq)^{1/2}\pm1],\quad b_n^{\pm}=\tfrac12[1\pm(1-4nq)^{1/2}]\ , \tag{25}$$

and $[1/4q]$ is the largest integer smaller than $1/4q$. The D_n's are defined in Eq. (22).

C. δ-shaped distribution

Here we consider an extreme case of a δ-shaped isotropic distribution function. The corresponding $\bar f$ is

$$\bar f=\frac{1}{2\pi}\delta\left(1-\left(\frac{p}{p_0}\right)^2\right). \tag{26}$$

From Eq. (18) we obtain, with the same notation as in Eqs. (22) and (25),

$$\frac{dW}{dt}=C(\pi/2)q^{3/2}\left[\sum_{n=1}^{\infty} n\int_{a_n^-}^{a_n^+} d\rho\,\rho\,|\phi(\rho)|^2 D_n\left(\frac{s}{q}\rho\right)-\sum_{n=1}^{[1/4q]} n\int_{b_n^-}^{b_n^+} d\rho\,\rho\,|\phi(\rho)|^2 D_n\left(\frac{s}{q}\rho\right)\right]. \quad (27)$$

IV. RESULTS AND DISCUSSION

As the potential ϕ was not specified explicitly the results of Sec. III are quite general and can, therefore, be applied to different cases. In what follows we shall use a pure Coulomb potential, $\phi(\rho)=1/\rho^2$. This is valid at high fluxes (many-photon processes) and also at low fluxes and high electron temperatures. One has to take into account the Debye screening and the possible charge distribution of the ions only when both the flux and electron temperature are not high. This will not be done here because one-photon processes were discussed extensively and correctly in the literature.

In addition to the flux intensity, the other important physical parameter is q, which is a measure of the ratio of the one-photon energy to the mean electron kinetic energy. For neodymium and CO_2 lasers, at the usual temperatures of interest in laser fusion experiments, this ratio is very small. Nevertheless, it is very instructive to consider, also, the case $q \gg 1$, which applies to x-ray and γ-ray lasers. In what follows we shall consider the two extreme cases separately.

1. $q \gg 1$

In general dW/dt is determined by the excess of absorbed to emitted photons. In the assumed case $q \gg 1$, the photon energy is large compared to the energy of a great majority of the plasma electrons. Thus there is very little induced emission since there are very few high-energy electrons which can emit quanta $n\hbar\omega$, with the result that only absorption takes place. Moreover, the rate is independent of the distribution and temperature of the electrons. It depends only on the electron density and electric flux intensity in the plasma. We can see this mathematically from the formulas of Sec. III. All the expressions for the absorption rate for the three distribution functions, Eqs. (21), (24), and (27), reduce, at $q \gg 1$, to the same expression (Appendix A):

$$\frac{dW}{dt}=C(\pi/2)\sum_{n=1}^{\infty}\frac{1}{n^{1/2}}D_n\left(s\left(\frac{n}{q}\right)^{1/2}\right). \quad (28)$$

The only parameter in the last equation which depends on the flux intensity is s^2/q,

$$\frac{s^2}{q}=\frac{2e^2E^2}{m\hbar\omega^3}=4\frac{\omega_p^2}{\omega^2}\frac{n_{\rm ph}}{n_e}=\frac{I}{I_\omega}=\alpha^2, \quad (29)$$

where $n_{\rm ph}$ is the number of photons per unit volume (photon density), I is the flux intensity, and $I_\omega=m\hbar c\omega^3/16\pi e^2$. α is the main dimensionless parameter of this case. It is a measure of the flux intensity, independent of the plasma temperature. When $\alpha<1$ (low flux), the main contribution in Eq. (28) comes from $n=1$ (one-photon processes). Moreover, we can expand in power series and obtain

$$\frac{dW}{dt}=C(\pi/12)\alpha^2 \quad (30)$$

for both l.p. and c.p.

In the other extreme case, $\alpha \gg 1$, high flux, many-photon processes contribute to the absorption. In fact, all n values from $n=1$ up to approximately $n\approx\alpha^2$ in the l.p. case and $n\approx\alpha^2/2$ in the c.p. case have to be taken into account and the result is given by the approximate asymptotic formulas (Appendix A)

$$\frac{dW}{dt}=C\frac{1}{\alpha}[\ln^2(2\alpha)+\gamma\ln 2\alpha-\tfrac{1}{12}\pi^2] \quad \text{for l.p.} \quad (31a)$$

$$=C\frac{2^{1/2}\pi}{\alpha}\ln\alpha-\tfrac{1}{2}(\ln 2-\gamma)] \quad \text{for c.p.,} \quad (31b)$$

where $\gamma=0.5772\cdots$ is Euler's constant.

We see that at low fluxes the energy absorption rate is independent of the polarization of the incident light. At very high fluxes the energy absorption rate is much stronger for linearly polarized light than for circularly polarized light. In Fig. 1

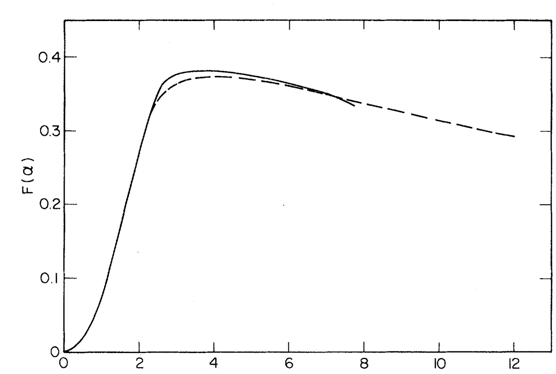

FIG. 1. Inverse bremsstrahlung energy absorption rate dW/dt as a function of α at large q for linearly polarized electric field E, Eq. (28). $dW/dt=C\pi F(\alpha)$, $C=4n_e^2e^4Z(2/m\hbar\omega)^{1/2}$, $\alpha=eE(2/m\hbar\omega^3)^{1/2}$, $q=\hbar\omega/2kT$. Dashed curve, asymptotic approximation, Eq. (31a).

we plot results of computer calculations of Eq. (28) for the l.p. case. For $\alpha > 4$ the asymptotic formula, Eq. (31), fits the numerical calculations within less than 1%. The rate of absorption increases linearly with the flux I at small fluxes and decreases slowly at high fluxes. It has a maximum at about $\alpha \approx 3.2$ or $I/I_\omega \approx 10$.

2. $q \ll 1$

The case $q \ll 1$ applies to infrared lasers and electron temperaturesof more than 1 eV. In this case the parameter s as defined in (16) is a measure of the flux intensity. If $s \ll 1$ the main contribution to the absorption rate is determined by one-photon processes and if $s \gg 1$ many-photon processes contribute to the rate. In this case, however, the absorption rate may also depend on the electron temperature as well as the distribution function.

For $s \ll 1$, the results for the three distribution functions are (Appendix B)

$$\frac{dW}{dt} = C[(2\pi)^{1/2}/6]\,s^2 q^{1/2}[\ln(2/q) - \gamma] \quad \text{(hot plasma)} \tag{32a}$$

$$= C(\pi/2)s^2 q^{1/2}\ln(1/q) \quad \text{(cold plasma)} \tag{32b}$$

$$= C(\pi/3)s^2 q^{1/2} \quad (\delta\text{-shaped distribution}). \tag{32c}$$

For s not small there are no simple analytical approximations. Only when $s \gg 1$, i.e., when the oscillating electron energy in the electric field is large compared to the characteristic electron energy, is the absorption rate approximately independent of the distribution function. However it does depend on the mean electron energy, as discussed in Appendix B. The result is

$$\frac{dW}{dt} = C\frac{q^{1/2}}{s}[\ln^2(2s) + 2\ln 2s\,\ln(1/q)] \quad \text{for l.p.} \tag{33a}$$

$$= C2^{1/2}\pi\frac{q^{1/2}}{s}[\ln s + \ln(1/q)] \quad \text{for c.p.} \tag{33b}$$

The results [Eqs. (33)] for linear and for circular polarization are essentially different from the expressions given in Refs. 9 and 10.

There is no general formula for the different distributions when s is of the order of unity. In Appendix B we see that for a Boltzmann distribution we obtain the form (B10) for dW/dt; the functions $A(s)$ and $B(s)$ are plotted in Figs. 2 and 3. We see that for $s > 4$ there is, at most, a difference of only a few percent between the numerically evaluated and the asymptotic values.

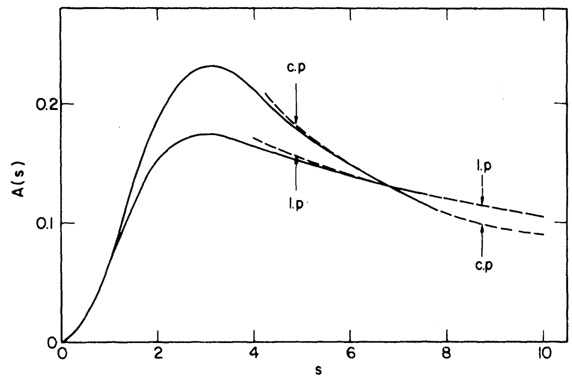

FIG. 2. Function $A(s)$ for a Maxwell-Boltzmann distribution function, Eq. (B13a), for linearly and circularly polarized light. $s = eE/\omega(mkT)^{1/2}$. Dashed curves, asymptotic approximations, Eqs. (B11a) and (B12a).

In Figs. 4 and 5 we plot the absorption rate in the l.p. case for a Maxwell-Boltzmann distribution as a function of s for two values of the parameter q, 0.07 and 0.0007, which correspond, in the case of a neodymium laser, to electron temperatures of 10^5 and 10^7 K. The absorption rate is initially proportional to the flux intensity I. As s increases, the slope decreases and eventually the rate decreases slowly. At very large s, which implies very high flux intensity, the rate is proportional to $\ln^2 I/I^{1/2}$ (in the c.p. case the rate is proportional to $\ln I/I^{1/2}$). The absorption rate attains a maximum at a value of $s \approx 3.2$ and the value at the maximum can be approximated by dW/dt

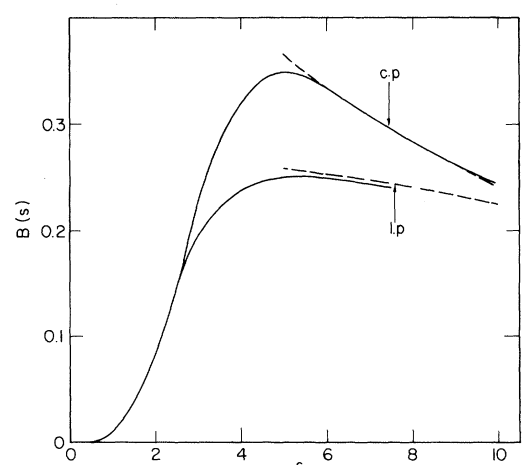

FIG. 3. Function $B(s)$ for a Maxwell-Boltzmann distribution function, Eq. (B13b), for linearly and circularly polarized light. $s = eE/\omega(mkT)^{1/2}$. Dashed curves, asymptotic approximations, Eqs. (B11b) and (B12b).

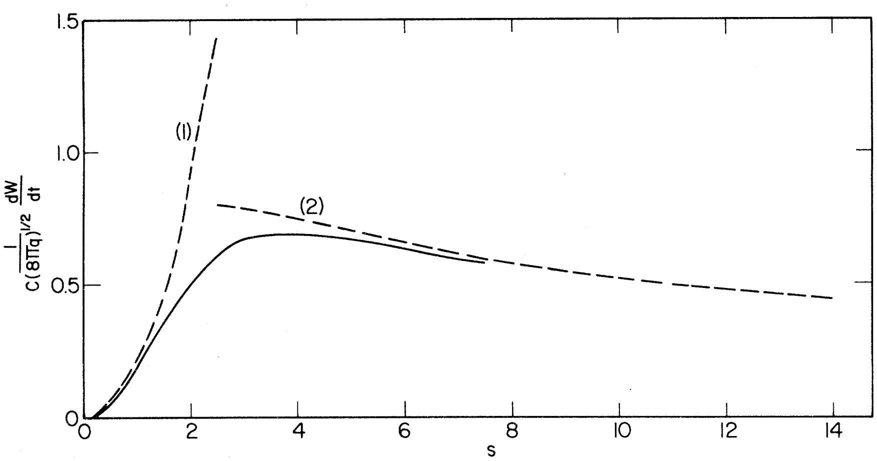

FIG. 4. Energy absorption rate dW/dt for a Maxwell-Boltzmann distribution function and a linearly polarized electric field as a function of s, Eq. (B10). $q = 0.07$ ($T = 10^5$ K for a neodymium laser) and $s = eE/\omega(mkT)^{1/2} = 2\times 10^{-7} I^{1/2}$ (for a neodymium laser), where I is the flux in W/cm^2. Dashed curve (1), low-flux limit, Eq. (32a); dashed curve (2), asymptotic limit at high fluxes, Eq. (B11).

$\approx Cq^{1/2}\ln(1/q)$. As q is inversely proportional to the temperature we obtain the result that the maximum absorption rate is proportional to $\ln(aT)/T^{1/2}$, where $a = 2k/\hbar\omega$.

As in the case where q is large, the asymptotic formulas (33a) and (33b) give good approximations to the numerically evaluated values for $s > 4$. We also see that the low-flux approximation, Eq. (32), which is used extensively in computer codes[16,17] is valid only for $s < 0.7$. At large s the absorption rate depends only logarithmically on the temperature.

In Fig. 6 we show for the neodymium-laser case the regions, in flux and temperature, where thé different approximation formulas are valid. In the "low" region the low-flux (one-photon processes) approximation, Eq. (32), is good. In the "high" region Eq. (33) gives an agreement within a few percent with numerically calculated values. In the "intermediate" region the rate has to be evaluated numerically.

We note again that in the high region dW/dt does not depend on the assumed velocity-distribution function and also in this region the result is essentially different from that of Ref. 10. The effective collision rate ν, defined as

$$\nu = \frac{2m\omega^2}{n_e e^2 E^2}\frac{dW}{dt},$$

becomes

$$\nu = \frac{8Zn_e em\omega^3}{E^3}\ln 2s\ln(2s/q^2) \quad \text{for l.p.}$$

$$= \frac{8Z2^{1/2}\pi n_e em\omega^3}{E^3}\ln(s/q) \quad \text{for c.p.} \qquad (34)$$

APPENDIX A

We consider here the formulas for the absorption rate dW/dt at $q \gg 1$ for the distribution func-

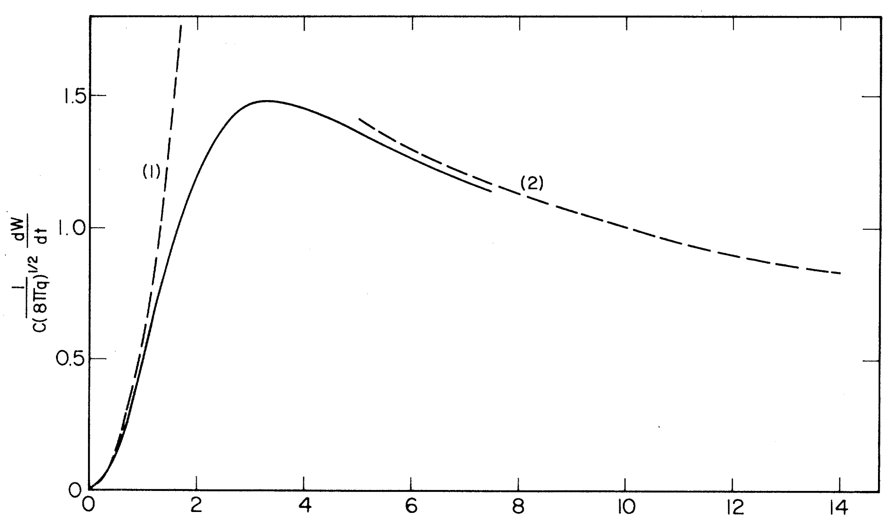

FIG. 5. Energy absorption rate dW/dt for a Maxwell-Boltzmann distribution function and a linearly polarized electric field as a function of s, Eq. (B10). $q = 0.0007$ ($T = 10^7$ K for a neodymium laser) $s = eE/\omega(mkT)^{1/2} = 2\times 10^{-8} \times I^{1/2}$ (for a neodymium laser), where I is the flux in W/cm^2. Dashed curve (1), low-flux limit, Eq. (32a); dashed curve (2), asymptotic limit at high fluxes, Eq. (B11).

tions considered in Sec. III.

1. Hot plasma

We start with Eq. (21) and use a pure Coulomb potential $\phi(\rho)=1/\rho^2$. We change the variable from ρ to z, $z=\rho/(nq)^{1/2}$, and obtain for the absorption rate

$$\frac{dW}{dt}=C(2\pi q)^{1/2}\sum_{n=1}^{\infty}\sinh(nq)\times\int_0^{\infty}\frac{dz}{z^3}e^{-nq(z^2+1/z^2)/2}D_n\left(s\left(\frac{n}{q}\right)^{1/2}z\right). \tag{A1}$$

Since $\sinh(nq)\approx\frac{1}{2}e^{nq}$ we obtain

$$\frac{dW}{dt}=C\left(\frac{\pi q}{2}\right)^{1/2}\sum_{n=1}^{\infty}\int_0^{\infty}\frac{dz}{z^3}e^{-nq(z-1/z)^2/2}\times D_n\left(s\left(\frac{n}{q}\right)^{1/2}z\right). \tag{A2}$$

The exponential term in the integral varies more rapidly than all other factors and the method of deepest descent can be applied to evaluate the z integration. The main contribution comes from $z=1$ and we obtain

$$\frac{dW}{dt}=C(\pi/2)\sum_{n=1}^{\infty}\frac{1}{n^{1/2}}D_n\left(s\left(\frac{n}{q}\right)^{1/2}\right)=C(\pi/2)\sum_{n=1}^{\infty}\frac{1}{n^{1/2}}D_n(\alpha n^{1/2}), \tag{A3}$$

where $\alpha=s/q^{1/2}$. α is the only parameter which appears in the last equation.

Another form is obtained by expanding in a power series the squares of Bessel functions[18] of order n that appear in the definition of D_n [Eq. (22)],

$$J_n^2(y)=\sum_{M=n}^{\infty}(-)^{M-n}\binom{2M}{M}\frac{1}{(M+n)!(M-n)!}\left(\frac{y^2}{4}\right)^M, \tag{A4}$$

and integrating term by term. The result is

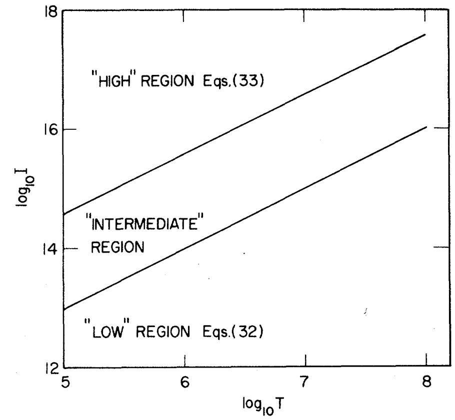

FIG. 6. Regions in the $\log_{10}T$-$\log_{10}I$ plane where the different analytic expressions for the energy absorption rate for a neodymium laser are valid. T is the electron temperature in K and I is the flux in W/cm^2. In the "intermediate" region there are no simple analytic expressions and the absorption rate has to be evaluated numerically. For a CO_2 laser the $\log_{10}I$ scale has to be lowered by two units.

$$\frac{dW}{dt}=C\pi\sum_{M=1}^{\infty}\frac{(-)^M}{2M+1}C_M\left(\frac{\alpha^2}{2}\right)^M\times\sum_{n=1}^{M}(-)^n\frac{n^{M-1/2}}{(M+n)!(M-n)!}, \tag{A5}$$

where

$$C_M=\frac{1}{2^M}\binom{2M}{M}\quad\text{for l.p.}$$
$$=1\quad\text{for c.p.} \tag{A6}$$

For small α, $dW/dt=C\pi\alpha^2/12$. For large α we can obtain an asymptotic expansion from Eq. (A3). $J_n^2(y)$ is very small if $y<n$. It behaves like $1/\pi y$ for $y\gg n$. Thus, for $n>\alpha^2$ in the l.p. case and for $n>\alpha^2/2$ in the c.p. case, the nth-term contribution to the sum (A3) is extremely small. For $n<\alpha^2$ in the l.p. case and for $n<\alpha^2/2$ in the c.p. case, a good approximation for integration purposes is $J_n^2(y)=1/\pi(y^2-n^2)^{1/2}$ and we obtain for the D_n functions

$$D_n(\alpha n^{1/2})=\frac{2}{\pi\alpha n^{1/2}}\int_n^{\alpha n^{1/2}}\frac{dy}{(y^2-n^2)^{1/2}}=\frac{2}{\pi\alpha n^{1/2}}\ln\left[\frac{\alpha}{n^{1/2}}+\left(\frac{\alpha^2}{n}-1\right)^{1/2}\right]\quad\text{for l.p.}$$

$$=\frac{4}{\pi\alpha n^{1/2}}\int_n^{\alpha(n/2)^{1/2}}\frac{y\,dy}{[(y^2-n^2)(\alpha^2n-2y^2)]^{1/2}}=\frac{1}{\alpha}\left(\frac{2}{n}\right)^{1/2}\quad\text{for c.p.} \tag{A7}$$

Putting this in (A3) we obtain

$$\frac{dW}{dt}=C\frac{1}{\alpha}\sum_{n=1}^{[\alpha^2]}\frac{1}{n}\ln\left[\frac{\alpha}{n^{1/2}}+\left(\frac{\alpha^2}{n}-1\right)^{1/2}\right]\text{ for l.p.}$$

$$=C\frac{\pi}{2^{1/2}\alpha}\sum_{n=1}^{[\alpha^2/2]}\frac{1}{n}\qquad\text{for c.p.}\tag{A8}$$

The last expressions can be approximated by an integral and correction terms, according to the formula

$$\sum_{n=n_1}^{n=n_2}\Phi(n)=\int_{n_1}^{n_2}\Phi(x)\,dx+\tfrac{1}{2}[\Phi(n_1)+\Phi(n_2)]+\sum_{r=1}^{\infty}\frac{B_{2r}}{(2r)!}[\Phi^{(2r-1)}(n_1)-\Phi^{(2r-1)}(n_2)]\,,\tag{A9}$$

where B_{2r} are the Bernoulli numbers and $\Phi^{(r)}$ is the rth derivative of Φ. This leads to the asymptotic formulas Eq. (31).

2. Cold plasma

We start from Eq. (24) and as before use a pure Coulomb potential. Making the same change of variables from ρ to z, $z=\rho/(nq)^{1/2}$, we obtain

$$\frac{dW}{dt}=C(3\pi/4)q^{3/2}\left(\sum_{n=1}^{\infty}n\int_{d_n^-}^{d_n^+}\frac{dz}{z^5}[(d_n^+)^2-z^2][z^2-(d_n^-)^2]D_n(\alpha n^{1/2}z)-\sum_{n=1}^{[1/4q]}n\int_{e_n^-}^{e_n^+}\frac{dz}{z^5}[(e_n^+)^2-z^2][z^2-(e_n^-)^2]D_n(\alpha n^{1/2}z)\right)\,,\tag{A10}$$

where

$$d_n^{\pm}=\frac{(1+4nq)^{1/2}\pm1}{2(nq)^{1/2}},\quad e_n^{\pm}=\frac{1\pm(1-4nq)^{1/2}}{2(nq)^{1/2}}\,.$$

As q is large the second sum does not contribute, $d_n^+\approx1$, $d_n^+-d_n^-=1/(nq)^{1/2}$, and the z integration leads to Eq. (A3).

3. δ-shaped distribution

We start from Eq. (27) and obtain after the same change of variables as before

$$\frac{dW}{dt}=C(\pi/2)q^{1/2}\left(\sum_{n=1}^{\infty}\int_{d_n^-}^{d_n^+}\frac{dz}{z^3}D_n(\alpha n^{1/2}z)-\sum_{n=1}^{[1/4q]}\int_{e_n^-}^{e_n^+}\frac{dz}{z^3}D_n(\alpha n^{1/2}z)\right)\,.\tag{A11}$$

The z integration leads again to Eq. (A3).

APPENDIX B

We consider here the formulas for the absorption rate dW/dt for the distribution functions considered in Sec. III and pay particular attention to the case where q is small. Here it is more convenient to use the parameters s and q.

1. Hot plasma

Using the expansion formula (A4) for $J_n^2(y)$ in Eq. (A1) and integrating term by term we obtain

$$\frac{dW}{dt}=C(8\pi q)^{1/2}\sum_{M=1}^{\infty}\frac{(-)^{M+1}}{2M+1}C_M\left(\frac{s^2}{2}\right)^M\times\sum_{n=1}^{M}\frac{(-)^{n+1}}{(M+n)!\,(M-n)!}\left(\frac{n}{q}\right)^M\sinh(nq)K_{M-1}(nq)\,,\tag{B1}$$

where C_M is given in Eq. (A6) and $K_r(a)$ is the modified Bessel function of order r. In obtaining Eq. (B1) we used the integral representation of the modified Bessel functions,[18]

$$K_r(a)=\int_0^{\infty}z^{2r-1}e^{-a(z^2+1/z^2)/2}\,dz\,.\tag{B2}$$

Expression (B1) gives dW/dt in a power series in s^2. s is a measure of the flux intensity defined in Eq. (16). For small fluxes the first term in the expression gives the usual formula[4]

$$\frac{dW}{dt}=C[(2\pi)^{1/2}/6]q^{1/2}s^2K_0(q)\,,\tag{B3}$$

which, at small q, leads to Eq. (32a).

At high flux intensities, i.e., when s is not small, we have to take many terms in the expansion (B1). Again, as was done in Appendix A, we can approximate the D_n's by

$$D_n\left(s\left(\frac{n}{q}\right)^{1/2}z\right)=\frac{2}{\pi s z}\left(\frac{q}{n}\right)^{1/2}\ln\left[\frac{sz}{(nq)^{1/2}}+\left(\frac{s^2z^2}{nq}-1\right)^{1/2}\right]\Theta\left(\frac{sz}{(nq)^{1/2}}-1\right)\quad \text{for l.p.}$$

$$=\frac{2^{1/2}}{sz}\left(\frac{q}{n}\right)^{1/2}\Theta\left(\frac{sz}{(2nq)^{1/2}}-1\right)\quad \text{for c.p.}\tag{B4}$$

Inserting (B4) in (A1) we obtain for the l.p. case

$$\frac{dW}{dt}=C\frac{4}{(2\pi)^{1/2}}\frac{q^{1/2}}{s}\sum_{n=1}^{\infty}\frac{\sinh(nq)}{n^2q}\int_0^s x^2e^{-(x^2+n^2q^2/x^2)/2}\psi\left(\frac{s}{x}\right)dx\,,\tag{B5}$$

where $\psi(y)=\ln[y+(y^2-1)^{1/2}]$.

To obtain (B5) we changed variables from z to x, $x=(nq)^{1/2}/z$. The sum over n can be replaced by an integral according to formula (A9),

$$\frac{dW}{dt}=C\frac{4}{(2\pi)^{1/2}}\frac{q^{1/2}}{s}\int_0^s x^2e^{-x^2/2}\;\psi\left(\frac{s}{x}\right)\left(\int_q^\infty\frac{\sinh y}{y^2}e^{-y^2/2x^2}dy+\gamma\right)dx\,.\tag{B6}$$

The Euler's constant γ is the correction term of (A9) for small q. The y integral leads to a logarithmic singularity in q and in order to extract it we integrate by parts,

$$\int_q^\infty\frac{\sinh y}{y^2}e^{-y^2/2x^2}dy\approx 1+\ln(1/q)+\frac{1}{x^2}\int_0^\infty e^{-y^2/2x^2}(y\ln y\cosh y-\sinh y-x^2\ln y\sinh y)\,dy\,.\tag{B7}$$

Putting (B7) in (B6), the expression for the absorption rate can be expressed as an integral over only one variable,

$$\frac{dW}{dt}=C\frac{4}{(2\pi)^{1/2}}\frac{q^{1/2}}{s}\int_0^s x^2e^{-x^2/2}\{[1+\ln(1/q)-\tfrac{1}{2}(\ln 2-\gamma)+\ln x]\psi(s/x)+\tfrac{1}{2}[\psi(s/x)]^2-\tfrac{1}{24}\pi^2-\tfrac{1}{4}R(u^2)\}dx\,,\tag{B8}$$

where $u=s/x-(s^2/x^2-1)^{1/2}$ and $R(y)$ is the dilogarithm function[19] (Spence function) defined by

$$R(y)=\int_0^y\frac{dz}{z}\ln(1-z),\quad R(1)=-\pi^2/6\,.$$

For the c.p. case we obtain, after similar manipulations, the following expression for dW/dt:

$$\frac{dW}{dt}=C2\pi^{1/2}\frac{q^{1/2}}{s}\int_0^{s/2^{1/2}}x^2e^{-x^2/2}[1+\ln(1/q)-\tfrac{1}{2}(\ln 2-\gamma)+\tfrac{1}{2}\ln(s^2/2-x^2)]dx\,.\tag{B9}$$

Both expressions, (B8) and (B9), are of the form

$$\frac{dW}{dt}=C(8\pi q)^{1/2}\{A(s)[\ln(2/q)-\gamma]+B(s)\}\,,\tag{B10}$$

where, at large s, the functions $A(s)$ and $B(s)$ have the asymptotic forms

$$A(s)=\frac{1}{(2\pi)^{1/2}s}(\ln 2s-0.365)\,,\tag{B11a}$$

$$B(s)=\frac{1}{(2\pi)^{1/2}s}[\tfrac{1}{2}\ln^2(2s)+0.826\ln 2s-1.307]\tag{B11b}$$

for the l.p. case and

$$A(s)=\pi^{1/2}/2s\,,\tag{B12a}$$

$$B(s)=\frac{\pi^{1/2}}{2s}(\ln s+0.48)\tag{B12b}$$

for the c.p. case.

Expressions for $A(s)$ and $B(s)$ which are valid for all values of s can be obtained from the sum (B1) evaluated at small q. At first glance it seems that (B1) is very singular when q is small because, expanding at small argument the modified Bessel function, $K_M(x)\approx\frac{1}{2}(M-1)!/(\frac{1}{2}x)^M$, and $\sinh x\approx x$, each term becomes of the order $1/(nq)^{2(M-1)}$. However, it can be proved that

$$\sum_{n=1}^{M}\binom{2M}{M-n}(-)^{n+1}n^{2k}=0$$

if $1\leq k<M$. The first significant term in the expansion of $(n/q)^M\sinh(nq)K_{M-1}(nq)$ which gives a nonzero contribution to the sum in n is of the form

$$\ln(2/q)-\gamma+a(M,n)\,,$$

where $a(M,n)$ is some function of M and of n, and therefore, for small q, the expression (B10) for the absorption rate is valid for all values of s. The functions $A(s)$ and $B(s)$ have the form

$$A(s)=\sum_{M=1}^{\infty}\frac{(-)^{M+1}}{(2M+1)(M-1)!}C_M\left(\frac{s^2}{4}\right)^M\,,\tag{B13a}$$

$$B(s)=\sum_{M=1}^{\infty}\frac{(-)^{M+1}}{(2M+1)(M-1)!}C_M E_M\left(\frac{s^2}{4}\right)^M, \quad \text{(B13b)}$$

where C_M is given in Eq. (A6), and $E_1=0$ and

$$E_M=\sum_{n=2}^{M}\left[\frac{1}{2(n-1)}-2(-)^{n+M}\frac{n^{2M}}{(M+n)!\,(M-n)!}\ln n - (-)^n\frac{(n-2)!\,(M-1)!}{(2n-1)!\,(M-n)!}2^{2n-3}\right] \quad \text{for } M>1\,.$$

We have computed numerically $A(s)$ and $B(s)$ and the results are shown in Figs. 2 and 3. For $s>4$ the asymptotic formulas (A11) and (B12) give excellent agreement with numerically obtained values.

At very small q and very large s the leading terms of the asymptotic expansions (B11) and (B12) give Eq. (33).

2. Cold plasma

Starting with Eq. (A10) we see that, at small s (one-photon processes), the main contribution comes from the $n=1$ terms, and, because $D_1(t)\approx\frac{1}{6}t^2$ at small t, we obtain for the absorption rate

$$\frac{dW}{dt}\approx C(\pi/8)s^2q^{1/2}\left(\int_{d_1^-}^{d_1^+}\frac{dz}{z^3}[(d_1^+)^2-z^2][z^2-(d_1^-)^2]-\int_{e_1^-}^{e_1^+}\frac{dz}{z^3}[(e_1^+)^2-z^2][z^2-(e_1^-)^2]\right)$$

$$=C\frac{\pi}{8}\frac{s^2}{q^{1/2}}[(1+2q)\ln(d_1^+/d_1^-)-(1-2q)\ln(e_1^+/e_1^-)-(1+4q)^{1/2}+(1-4q)^{1/2}]\,, \quad \text{(B14)}$$

which, at small q, leads to Eq. (32b).

At large s (many-photon processes) we have to take many terms in the expression (A10). By considerations similar to those used for hot plasmas, we obtain for the absorption rate at small q an expression like (B10), where the functions $A(s)$ and $B(s)$ have the following forms:

$$A(s)=\frac{1}{(2\pi)^{1/2}s}(\ln 2s+\tfrac{1}{3})\,,$$
$$B(s)=\frac{1}{(2\pi)^{1/2}s}[\tfrac{1}{2}\ln^2(2s)+0.13\ln 2s-0.77] \quad \text{(B15)}$$

for the l.p. case, and

$$A(s)=\pi^{1/2}/2s\,,$$
$$B(s)=\frac{\pi^{1/2}}{2s}(\ln s-0.22) \quad \text{(B16)}$$

for the c.p. case.

The leading terms of (B15) and (B16) again give Eq. (33).

3. δ-shaped distribution

We start with Eq. (A11) and obtain, for small s, from the $n=1$ terms

$$\frac{dW}{dt}=C\frac{\pi}{12}\frac{s^2}{q^{1/2}}\ln\left(\frac{d_1^+e_1^-}{d_1^-e_1^+}\right), \quad \text{(B17)}$$

which, at small q, leads to Eq. (32c).

As in the hot- and cold-plasma cases, we again get, in the limit of large s and small q, an expression similar to (B10). The functions $A(s)$ and $B(s)$ now have the forms

$$A(s)=\frac{1}{(2\pi)^{1/2}s}\ln 2s\,, \quad \text{(B18)}$$

$$B(s)=\frac{1}{(2\pi)^{1/2}s}(\tfrac{1}{2}\ln^2 2s+0.46\ln 2s-0.59)$$

for the l.p. case and

$$A(s)=\pi^{1/2}/2s\,,$$
$$B(s)=\frac{\pi^{1/2}}{2s}(\ln s+0.11) \quad \text{(B19)}$$

for the c.p. case.

Again, the leading terms give Eq. (33).

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