Coherence properties of the N-atom-radiation interaction and the Holstein-Primakoff transformation

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We consider a system of N two-level atoms initially in the ground state and interacting with a small $[\langle n(t=0)\rangle \ll N]$ coherent radiation field. The field is assumed to be monochromatic, and its wavelength is taken to be much larger than the dimensions of the region of the N atoms. The statistical properties of both the atomic system and the radiation field are studied as functions of time. Using the Holstein-Primakoff transformation, it is shown that the initial coherence of the system tends to disappear after the energy of the system has been exchanged between atoms and field a number of times approximately given by $N/\langle n(t=0)\rangle$.

I. INTRODUCTION

In this paper we discuss the following problem. Suppose we have a system of N two-level atoms, all of them in the ground state, and that we apply to this system a small monochromatic and coherent electromagnetic (e.m.) field in resonance with the atoms. The field is assumed to be uniform over the volume occupied by the atoms. As time passes, processes of emission and absorption take place, since the atoms interact with the field. We wish to investigate the coherence properties of the radiation field and the atomic system as a function of time.

We now proceed to state this problem more precisely.

(i) We take the Dicke Hamiltonian

$$\mathcal{H} \equiv \mathcal{H}_0 + V = \omega(S_z + \alpha^\dagger \alpha) + \tfrac{1}{2}\epsilon(S_+\alpha + S_-\alpha^\dagger) \qquad (1.1)$$

to represent the total system, where ω is the frequency of the incident radiation and ϵ is the atom-photon coupling constant. The S operators in (1.1) pertain to the atomic system, and are collective dipole (or spin, as we shall sometime call them in this paper) operators defined by[1]

$$S_i = \sum_{n=1}^{N} S_i^n \quad (i = +, -, z),$$

where S_i^n is a Pauli 2×2 operator acting on the nth atom's state vector. $\alpha^\dagger$ and α are the creation and annihilation operators of the e.m. field, which is assumed to be uniform in the region of the N atoms. Moreover

$$\mathcal{H}_0 = \omega(S_z + \alpha^\dagger \alpha)$$

in (1.1) is the free Hamiltonian, while

$$V = \tfrac{1}{2}\epsilon(S_+\alpha + S_-\alpha^\dagger)$$

is the atom-radiation interaction in the rotating-wave approximation. It is usual to take as a convenient basis for (1.1) the set of eigenstates of $\mathcal{H}_0$ represented by $|n, S, S_z\rangle$, where n is the number of photons and S_z is the z component of the total dipole moment $\vec{S}$ of the atomic system. Since $|\vec{S}|^2$ commutes with $\mathcal{H}$, and since we choose our initial state to have $S = \frac{1}{2}N$, the time evolution of the system shall be confined to the subspace corresponding to this quantum number, and we shall drop the label S from the symbol for the states of our basis. Consequently S_z varies between $-\frac{1}{2}N$ and $\frac{1}{2}N$ in steps of one, so that we may use the index $p = 0, 1, \ldots, N$ instead of S_z and indicate the states by $|n, p\rangle$.

(ii) Following Glauber[2] we define the normalized and coherent states of the field as

$$|z\rangle = e^{-|z|^2/2} e^{z\alpha^\dagger}|0\rangle, \qquad (1.2)$$

where $|0\rangle$ is the vacuum of photons and z is a complex number such that $|z|^2 = \langle n\rangle = \langle z|\alpha^\dagger\alpha|z\rangle$. We observe that states of the form (1.2) select a direction in the complex z plane, given by the argument of z. Coherent kets are right eigenvectors of α, since $\alpha|z\rangle = z|z\rangle$, and their duals are left eigenvectors of $\alpha^\dagger$ since $\langle z|\alpha^\dagger = z^*\langle z|$. From what we have said at the beginning, our initial state is of the form

$$|\psi(t=0)\rangle = e^{-|a|^2/2} e^{a\alpha^\dagger}|n=0, \; p=0\rangle, \qquad (1.3)$$

with $|a|^2 \ll S$. We also introduce coherent spin states[3,4] by the relation

$$|\mu\rangle = [1/(1+|\mu|^2)]^S e^{\mu S_+}|0\rangle, \qquad (1.4)$$

where $|0\rangle$ is the $p=0$ state with all spins down and μ is a complex number such that

$$\langle\mu|S_z|\mu\rangle = -S(1-|\mu|^2)/(1+|\mu|^2). \qquad (1.5)$$

States (1.4) are not exact right eigenvectors of S_-, since

$$S_-|\mu\rangle = 2S\mu|\mu\rangle - \mu^2 S_+|\mu\rangle.$$

Furthermore, coherent spin states have a nonvanishing transverse component in the x-y plane, since

$$\langle \mu | S_- | \mu \rangle = (\langle \mu | S_+ | \mu \rangle)^* = 2S\mu/(1+|\mu|^2), \quad (1.6)$$

and if we put $\mu = e^{-i\varphi}\tan(\theta/2)$ in (1.5) and (1.6), we find

$$\langle \mu | S_x | \mu \rangle = S\sin\theta\cos\varphi, \langle \mu | S_y | \mu \rangle = S\sin\theta\sin\varphi,$$
$$\langle \mu | S_z | \mu \rangle = -S\cos\theta,$$

which shows that the transformation (1.4) changes the orientation of the dipole moment from the vertical (0, 0) to the (θ, φ) direction, and it does not introduce any additional spread in the components, since we have

$$(\langle \mu | S_x | \mu \rangle)^2 + (\langle \mu | S_y | \mu \rangle)^2 + (\langle \mu | S_z | \mu \rangle)^2 = S^2$$

after the transformation. It has in fact been shown[3,4] that the rotation operator

$$R_{\theta,\varphi} = \exp[-i\theta(S_x \sin\varphi - S_y \cos\varphi)]$$

generates coherent spin states in the sense that $R_{\theta,\varphi}|0\rangle = |\mu\rangle$. By defining rotated spin operators

$$S_i' = R_{\theta,\varphi} S_i R_{\theta,\varphi}^{-1} \quad (i = \pm, z),$$

one therefore obtains

$$S_-' | \mu \rangle \equiv R_{\theta,\varphi} S_- R_{\theta,\varphi}^{-1} | \mu \rangle = 0, \quad (1.7)$$

which expresses the fact that coherent spin states are the eigenstates of the rotated spin operator S_-' belonging to the 0 eigenvalue. It is possible to show also that

$$(\Delta S_x')^2 (\Delta S_y')^2 = \tfrac{1}{4}\langle S_z' \rangle^2 \quad (1.8)$$

with the equality sign, where $(\Delta S_i')$ is the variance of S_i'.

On the basis of what precedes, our intention is to investigate the coherence properties of the state

$$|\psi(t)\rangle = e^{-i\mathcal{H}t} e^{-|a|^2/2} e^{a\alpha^\dagger} |0, 0\rangle. \quad (1.9)$$

In principle, the solution of this problem can be obtained by diagonalization of the Dicke Hamiltonian, since the time development of the initial state can be expressed by a suitable linear combination of the harmonic time developments of the eigenstates of this Hamiltonian. In practice, however, the diagonalization of the Dicke Hamiltonian has been accomplished only approximately; as a consequence in some cases analytic approximations to the eigenstates belonging to certain energy ranges are not known,[5] or even the approximate expressions obtained for eigenstates and eigenvalues are too involved and do not lend themselves to an easy reconstruction of the state at time $t \neq 0$,[6] or the time development obtained is valid only for short times.[7]

We must now explain why we are interested in this problem. If we substitute in our system a second harmonic oscillator for the atomic part, Hamiltonian (1.1) becomes that of two linearly coupled harmonic oscillators, and the problem reduces to the investigation of the statistical properties of such a system, one oscillator being in a coherent state and the other in the ground state at $t = 0$. This problem has actually been solved,[8] since it has been shown that the state of the system remains coherent at all times, the two harmonic oscillators exchanging periodically the energy without any additional spread in the wave functions. This happends in any system of n harmonically coupled oscillators for which the Heisenberg equations of motion can be written

$$\alpha_j(t) = F_j[\{\alpha_k(t)\}, t] \quad (j = 1, 2, \ldots, n), \quad (1.10)$$

F_j being any function of its arguments. It is easy to see that condition (1.10) is not satisfied in the case of Hamiltonian (1.1); consequently we should expect that the initial coherence of our system, described by (1.3), somehow gets lost in the course of time. On the other hand this conclusion may seem surprising, since one should also expect that the information contained at $t = 0$ in the phase of the harmonic oscillator in the a complex plane could not get lost in the absence of external forces. We therefore want to discuss what happens to this information during the time development of the system. We have chosen $\langle n(t=0)\rangle = |a|^2 \ll S$, so that our spin system should behave very much, but not quite, like a harmonic oscillator: therefore we shall be interested in the deviations from harmonic behavior of the atomic part, since these deviations should depend intrinsically on the two-level nature of each of the N atoms. The Holstein-Primakoff transformation, coupled to this condition, shall allow us to follow the behavior of the deviation from coherent harmonic motion at long times, which is of course desirable, but shall exclude from the scope of this paper avalanche and superradiant emission processes.

While several papers[9] exist dealing with the time evolution of N atoms in a Dicke state and the radiation in an incoherent state at time $t = 0$, to the best of our knowledge, apart from Senitzky's approach,[10] which however is valid for short times, the problem with an initially coherent radiation field has been treated only quite recently by Lu and Smithers.[7] In their work however they seem to treat the system only in the limit $S \to \infty$ or alternatively for finite S but short times, which should correspond to describing the N atoms by a single harmonic oscillator, as discussed above. Here we try to go beyond this approximation, as we shall expose in Secs. II–VI of this paper.

II. GENERAL THEORY

Since $[\mathcal{H}_0, V]=0$ in (1.1), we may write $|\psi(t)\rangle$ in (1.9) as

$$|\psi(t)\rangle = e^{-|a|^2/2}e^{-iVt}e^{-i\mathcal{H}_0 t}e^{a\alpha^\dagger}e^{i\mathcal{H}_0 t}e^{-i\mathcal{H}_0 t}|0,0\rangle$$
$$= e^{i\varphi_0}e^{-|a|^2/2}e^{-iVt}\exp(ae^{-i\omega t}\alpha^\dagger)e^{iVt}|0,0\rangle$$
$$= e^{i\varphi_0}e^{-|a|^2/2}\exp[e^{-iVt}a(t)\alpha^\dagger e^{iVt}]|0,0\rangle , \qquad (2.1)$$

where φ_0 includes the initial phase and $a(t)=ae^{-i\omega t}$. In (2.1) we have used the equality

$$e^{-iVt}e^{a(t)\alpha^\dagger}e^{iVt} = \sum_{m=0}^{\infty}\frac{1}{m!}[e^{-iVt}a(t)\alpha^\dagger e^{iVt}]^m$$
$$= \exp[e^{-iVt}a(t)\alpha^\dagger e^{iVt}] .$$

We are therefore led to study the operator

$$e^{-iVt}\alpha^\dagger e^{iVt} = \alpha^\dagger + [\alpha^\dagger, iVt] + (1/2!)([\alpha^\dagger, iVt], iVt) + \cdots . \qquad (2.2)$$

The trouble with (2.2) is that the commutators do not have a simple recursive formula, because of the commutation properties of the total spin operators. The origin of this trouble is ultimately related to the finiteness of the number of eigenvalues of S_z, so that one should expect in the limit of large N an approximation to exist for the commutation relations of the spin operators that eliminates the core of this difficulty. This approximation is related to the Holstein-Primakoff (HP) transformation.[11] In our notation

$$S_+|p\rangle = (p+1)^{1/2}(2S-p)^{1/2}|p+1\rangle ,$$

$$S_-|p\rangle = p^{1/2}(2S-p+1)^{1/2}|p-1\rangle ,$$

and expanding in power series of $p/2S$ and of $(p-1)/2S$, respectively, we have

$$\frac{S_+}{\sqrt{2S}}|p\rangle = (p+1)^{1/2}\left[1-\frac{1}{2}\frac{p}{2S}-\frac{1}{8}\left(\frac{p}{2S}\right)^2-\cdots\right]|p+1\rangle ,$$

$$\frac{S_-}{\sqrt{2S}}|p\rangle = p^{1/2}\left[1-\frac{1}{2}\frac{p-1}{2S}-\frac{1}{8}\left(\frac{p-1}{2S}\right)^2-\cdots\right]|p-1\rangle . \qquad (2.3)$$

We obtain the first HP approximation for $p \ll S$ by neglecting terms $O(p/2S)$ in (2.3), so that

$$(S_+/\sqrt{2S})|p\rangle \simeq (p+1)^{1/2}|p+1\rangle \equiv \sigma_+|p\rangle ,$$
$$(S_-/\sqrt{2S})|p\rangle \simeq p^{1/2}|p-1\rangle \equiv \sigma_-|p\rangle . \qquad (2.4)$$

Consequently,

$$\sigma_+\sigma_-|p\rangle = p|p\rangle, \quad \sigma_-\sigma_+|p\rangle = (p+1)|p\rangle ,$$
$$[\sigma_-, \sigma_+] = 1 , \qquad (2.5)$$

and we see that σ_+ and σ_- behave as creation and annihilation operators of a harmonic oscillator. In the second HP approximation we neglect terms $O((p/2S)^2)$, and using (2.4) and (2.5) we get

$$(S_+/\sqrt{2S})|p\rangle \simeq \sigma_+(1-\sigma_+\sigma_-/4S)|p\rangle ,$$
$$(S_-/\sqrt{2S})|p\rangle \simeq (1-\sigma_+\sigma_-/4S)\sigma_-|p\rangle . \qquad (2.6)$$

These approximations, as it is fairly obvious, are valid only if the unphysical states with $p > 2S$ that we have introduced by (2.5) are never appreciably populated. This is in line with our original assumption that the initial number of photons in the field is small with respect to the number of atoms. In fact, if at $t=0$ the atomic system is in the ground state, since the total number of excitations $S_z + \alpha^\dagger\alpha$ commutes with $\mathcal{H}$, the average number of atoms ever excited is at most equal to the number of initial photons. The possibility exists of eliminating the unphysical states by a Dyson-Maleev transformation,[12] but we shall not consider here this problem. Our technique, which is, in principle, very simple, consists in making approximations (2.4) or (2.6) in operator (2.2) and in using commutation relations (2.5). In the first HP approximation the problem is equivalent to that of two coupled harmonic oscillators for which, as we have discussed in Sec. I, the solution is known: we discuss it in Sec. III. in order to text our technique and to allow detailed comparison with the results of the second HP approximation, which is treated in Sec. IV.

III. FIRST HP APPROXIMATION

We use (2.4) and (2.5) and we approximate

$$\mathcal{H}_0 = \omega(\sigma_+\sigma_- - S + \alpha^\dagger\alpha) ,$$
$$V = \tfrac{1}{2}\epsilon\sqrt{2S}(\sigma_+\alpha + \sigma_-\alpha^\dagger) . \qquad (3.1)$$

Consequently,

$$[\alpha^\dagger, V] = -\tfrac{1}{2}\epsilon\sqrt{2S}\sigma_+ , \quad \big[[\alpha^\dagger, V], V\big] = (\tfrac{1}{2}\epsilon\sqrt{2S})^2\alpha^\dagger ,$$
$$\big[\big[[\alpha^\dagger, V], V\big], V\big] = -(\tfrac{1}{2}\epsilon\sqrt{2S})^3\sigma_+ , \ \ldots ,$$

and from (2.2)

$$e^{-iVt}\alpha^\dagger e^{iVt} = \sum_{m=0}^{\infty}(i\tfrac{1}{2}\epsilon\sqrt{2S}t)^{2m}\frac{1}{(2m)!}\alpha^\dagger - \sum_{m=0}^{\infty}(i\tfrac{1}{2}\epsilon\sqrt{2S}t)^{2m+1}\frac{1}{(2m+1)!}\sigma_+$$
$$= \alpha^\dagger\cos\tfrac{1}{2}\epsilon\sqrt{2S}t - i\sigma_+\sin\tfrac{1}{2}\epsilon\sqrt{2S}t .$$

We introduce a dimensionless time $\tau = \frac{1}{2}\epsilon\sqrt{2S}t$, and substituting in (2.1) we find

$$|\psi(t)\rangle = e^{i\varphi_0} e^{-|a|^2/2} e^{a(t)\alpha^\dagger \cos\tau} e^{-i a(t)\sigma_+ \sin\tau} |0,0\rangle . \tag{3.2}$$

Within our approximation state (3.2) factorizes into a product of a coherent and normalized field state and of a coherent and normalized spin state. To show this we replace σ_+ by $S_+/\sqrt{2S}$ and write (3.2) neglecting the phase factor $e^{i\varphi_0}$ as

$$|\psi(t)\rangle = \frac{e^{-(1/2)|a|^2\sin^2\tau}}{[1+(|a|^2/2S)\sin^2\tau]^{-S}} \frac{e^{-(1/2)|a|^2\cos^2\tau}}{[1+(|a|^2/2S)\sin^2\tau]} \times e^{a(t)\alpha^\dagger\cos\tau} e^{-ia(t)(1/\sqrt{2S})S_+ \sin\tau} |0,0\rangle .$$

For large S

$$[1+|a|^2/2S\sin^2\tau]^{-S} = e^{-(1/2)|a|^2\sin^2\tau}[1+O(1/S)],$$

so that neglecting terms $O(1/S)$ consistently with the first HP approximation we have

$$|\psi(t)\rangle = (e^{-|z|^2/2} e^{z\alpha^\dagger})[1/(1+|\mu|^2)]^S e^{\mu S_+}|0,0\rangle , \tag{3.3}$$

where

$$z = ae^{-i\omega t}\cos\tau, \quad \mu = -i(a/\sqrt{2s})e^{-i\omega t}\sin\tau . \tag{3.4}$$

From (3.4) we have

$$z^2 - S(1-2|\mu|^2) = |a|^2 - S, \tag{3.5}$$

which expresses conservation of the total number of excitations since

$$|z|^2 = \langle n(t)\rangle , \quad \langle\mu|S_z|\mu\rangle \sim -S(1-2|\mu|^2). \tag{3.6}$$

The length of the transverse component of the dipole moment in the x, y plane varies like

$$|\langle\mu|S_-|\mu\rangle| \simeq 2S|\mu| = \sqrt{2S}\,|a|\sin\tau ,$$

while the component itself rotates at frequency ω, since

$$\langle\mu|S_x|\mu\rangle = \mathrm{Re}\langle\mu|S_-|\mu\rangle \simeq \sqrt{2S}\,|a|\sin\omega t\sin\tau ,$$
$$\langle\mu|S_y|\mu\rangle = -\mathrm{Im}\langle\mu|S_-|\mu\rangle \simeq \sqrt{2S}\,|a|\cos\omega t\sin\tau . \tag{3.7}$$

Therefore the motion of the total dipole can be likened to that of a conical pendulum of period ω in which the angle of the aperture of the cone varies periodically like $\sin\tau$. Since the spin part of (3.3) is coherent at all times, there is no further spreading of the wave packet, and the motion is almost classical. At the same time the amplitude of the radiation field varies harmonically with the same frequency $\frac{1}{2}\epsilon\sqrt{2S}$ but 90° out of phase. The average number of photons is given by

$$\langle n(t)\rangle = |a|^2\cos^2\tau . \tag{3.8}$$

This result coincides with that obtained by Lu and Smithers,[8] as well as that of the variance of n, for which we find $(\Delta n)^2 = \langle n(t)\rangle$. This is fairly obvious, since in the first HP approximation the e.m. field is coherent at all times, as shown above. By the same argument we also have $(\Delta\alpha)^2 = 0$ for the variance of α.

IV. SECOND HP APPROXIMATION

Using (2.6) we now approximate Hamiltonian (1.1) as

$$\mathcal{H}_0 = \omega(\sigma_+\sigma_- - S + \alpha^\dagger\alpha),$$
$$V = \tfrac{1}{2}\epsilon\sqrt{2S}[\sigma_+(1-\sigma_+\sigma_-/4S)\alpha + (1-\sigma_+\sigma_-/4S)\sigma_-\alpha^\dagger]. \tag{4.1}$$

We also introduce the symbol $[A,B]_n$ to indicate the n-fold comutator of A with B, so that, e.g.,

$$[A,B]_3 = \Big[\big[[A,B],B\big],B\Big].$$

Then commutation relations (2.5) give, up to terms $O(1/S)$,

$$[\alpha^\dagger, V] = -\tfrac{1}{2}\epsilon\sqrt{2S}\sigma_+(1-\sigma_+\sigma_-/4S), \quad [\alpha^\dagger, V]_2 = (\tfrac{1}{2}\epsilon\sqrt{2S})^2\alpha^\dagger(1-\sigma_+\sigma_-/S),$$
$$[\alpha^\dagger, V]_3 = (\tfrac{1}{2}\epsilon\sqrt{2S})^3\{-\sigma_+(1-\sigma_+\sigma_-/4S) - (1/S)[\sigma_+\sigma_-\alpha^\dagger, (\alpha\sigma_+ + \alpha^\dagger\sigma_-)]\},$$
$$[\alpha^\dagger, V]_4 = (\tfrac{1}{2}\epsilon\sqrt{2S})^4\{\alpha^\dagger(1-\sigma_+\sigma_-/S) - (1/S)[\sigma_+\sigma_-\alpha^\dagger, (\alpha\sigma_+ + \alpha^\dagger\sigma_-)]_2\},$$
$$[\alpha^\dagger, V]_5 = (\tfrac{1}{2}\epsilon\sqrt{2S})^5\{-\sigma_+(1-\sigma_+\sigma_-/4S) - (1/S)[\sigma_+\sigma_-\alpha^\dagger, (\alpha\sigma_+ + \alpha^\dagger\sigma_-)] - (1/S)[\sigma_+\sigma_-\alpha^\dagger, (\alpha\sigma_+ + \alpha^\dagger\sigma_-)]_3\}, \ldots . \tag{4.2}$$

Results (4.2) lead to the study of the following commutators:

$$[\sigma_+\sigma_-\alpha^\dagger, (\alpha\sigma_+ + \alpha^\dagger\sigma_-)] = \sigma_+\alpha^\dagger\alpha - \sigma_+\sigma_+\sigma_- - \sigma_-\alpha^\dagger\alpha^\dagger,$$
$$[\sigma_+\sigma_-\alpha^\dagger, (\alpha\sigma_+ + \alpha^\dagger\sigma_-)]_2 = -2\sigma_+\sigma_+\alpha + 5\sigma_+\sigma_-\alpha^\dagger - 2\alpha^\dagger\alpha^\dagger\alpha,$$
$$[\sigma_+\sigma_-\alpha^\dagger, (\alpha\sigma_+ + \alpha^\dagger\sigma_-)]_3 = 13\sigma_+\alpha^\dagger\alpha - 7\sigma_+\sigma_+\sigma_- - 7\sigma_-\alpha^\dagger\alpha^\dagger, \ldots . \tag{4.3}$$

It is easy to convince oneself that only the terms $\sigma_+\alpha^\dagger\alpha$, $\sigma_+\sigma_+\sigma_-$, and $\sigma_-\alpha^\dagger\alpha^\dagger$ appear in the odd commutators (4.3), while only $\sigma_+\sigma_+\alpha$, $\sigma_+\sigma_-\alpha^\dagger$, and $\alpha^\dagger\alpha^\dagger\alpha$ appear in the even commutators. Consequently, when results (4.3) are substituted into (4.2) one has

$$[\alpha^\dagger, V]_{2n} = (\tfrac{1}{2}\epsilon\sqrt{2S})^{2n}[\alpha^\dagger - (1/S)a_{2n}^{(1)}\sigma_+\sigma_-\alpha^\dagger + (1/S)a_{2n}^{(2)}(\sigma_+\sigma_+\alpha + \alpha^\dagger\alpha^\dagger\alpha)],$$
$$[\alpha^\dagger, V]_{2n+1} = (\tfrac{1}{2}\epsilon\sqrt{2S})^{2n+1}[-\sigma_+(1-\sigma_+\sigma_-/4S) - (1/S)a_{2n+1}^{(1)}\sigma_+\alpha^\dagger\alpha + (1/S)a_{2n+1}^{(2)}(\sigma_+\sigma_+\sigma_- + \sigma_-\alpha^\dagger\alpha^\dagger)]. \tag{4.4}$$

Coefficients $a_1^{(i)}$ appearing in (4.4) can be shown to be given by the recursion formulas

$$a_{2n+1}^{(1)} = a_{2n}^{(1)} + 4a_{2n}^{(2)}, \quad a_{2n+1}^{(2)} = a_{2n}^{(1)} + a_{2n}^{(2)},$$
$$a_{2n+2}^{(1)} = 1 + a_{2n+1}^{(1)} + 4a_{2n+1}^{(2)}, \quad a_{2n+2}^{(2)} = a_{2n+1}^{(1)} + a_{2n+1}^{(2)}, \tag{4.5}$$

with boundary conditions $a_0^{(1)} = a_0^{(2)} = 0$. A solution to equations (4.5) can be easily checked to be

$$a_{2n}^{(1)} = \tfrac{1}{2}[\tfrac{1}{8}(9^n-1)+n], \quad a_{2n}^{(2)} = \tfrac{1}{4}[\tfrac{1}{8}(9^n-1)-n],$$
$$a_{2n+1}^{(1)} = \tfrac{1}{2}[\tfrac{3}{8}(9^n-1)-n], \quad a_{2n+1}^{(2)} = \tfrac{1}{4}[\tfrac{3}{8}(9^n-1)+n]. \tag{4.6}$$

Using (4.6) we find

$$A \equiv -\sum_{n=0}^{\infty}\frac{1}{(2n)!}(i\tau)^{2n}a_{2n}^{(1)}$$
$$= -\tfrac{1}{2}(\tfrac{1}{8}\cos 3\tau - \tfrac{1}{8}\cos\tau - \tfrac{1}{2}\tau\sin\tau),$$
$$B \equiv -\sum_{n=0}^{\infty}\frac{1}{(2n)!}(i\tau)^{2n}a_{2n}^{(2)}$$
$$= -\tfrac{1}{4}(\tfrac{1}{8}\cos 3\tau - \tfrac{1}{8}\cos\tau + \tfrac{1}{2}\tau\sin\tau),$$
$$C \equiv -\sum_{n=0}^{\infty}\frac{i^{-1}}{(2n+1)!}(i\tau)^{2n+1}a_{2n+1}^{(2)}$$
$$= -\tfrac{1}{2}(\tfrac{1}{8}\sin 3\tau + \tfrac{1}{8}\sin\tau - \tfrac{1}{2}\tau\cos\tau),$$
$$D \equiv -\sum_{n=0}^{\infty}\frac{i^{-1}}{(2n+1)!}(i\tau)^{2n+1}a_{2n+1}^{(2)}$$
$$= -\tfrac{1}{4}(\tfrac{1}{8}\sin 3\tau - \tfrac{7}{8}\sin\tau + \tfrac{1}{2}\tau\cos\tau), \tag{4.7}$$

and substitution of (4.7) into (2.2) yields

$$e^{-iVt}\alpha^\dagger e^{iVt} = \alpha^\dagger\cos\tau - i\sigma_+(1-\sigma_+\sigma_-/4S)\sin\tau + (1/S)[A\sigma_+\sigma_-\alpha^\dagger - B(\sigma_+\sigma_+\alpha + \alpha^\dagger\alpha^\dagger\alpha) + iC\sigma_+\alpha^\dagger\alpha - iD(\sigma_+\sigma_+\sigma_- + \sigma_-\alpha^\dagger\alpha^\dagger)]. \tag{4.8}$$

We now put

$$x = \alpha^\dagger\cos\tau - i\sigma_+(1-\sigma_+\sigma_-/4S)\sin\tau,$$
$$y = (1/S)[A\sigma_+\sigma_-\alpha^\dagger - B(\sigma_+\sigma_+\alpha + \alpha^\dagger\alpha^\dagger\alpha) + iC\sigma_+\alpha^\dagger\alpha - iD(\sigma_+\sigma_+\sigma_- + \sigma_-\alpha^\dagger\alpha^\dagger)],$$

and find, neglecting terms $O(1/S^2)$,

$$[x, y] = (1/S)[F(\sigma_+\sigma_+ + \alpha^\dagger\alpha^\dagger) + iG\alpha^\dagger\sigma_+],$$
$$[x,[x,y]] = [y,[x,y]] = 0,$$

where

$$F \equiv B\cos\tau + D\sin\tau = \tfrac{1}{8}(1-\cos 2\tau - \tau\sin 2\tau),$$
$$G \equiv A\sin\tau - C\cos\tau = \tfrac{1}{8}(\sin 2\tau - 2\tau\cos 2\tau). \tag{4.9}$$

Therefore within our approximation

$$\exp[e^{-iVt}a(t)\alpha^\dagger e^{iVt}] = \exp\{-\tfrac{1}{2}a^2(t)[x,y]\}e^{a(t)x}e^{a(t)y}.$$

Furthermore,

$$e^{a(t)y}|0,0\rangle \equiv |0,0\rangle$$

since each term in y contains an annihilation operator acting to the right on the vacuum; consequently from (2.1)

$$|\psi(t)\rangle = e^{-|a|^2/2}\exp\{-[a^2(t)/2S][F(\sigma_+\sigma_+ + \alpha^\dagger\alpha^\dagger) + iG\alpha^\dagger\sigma_+]\}\exp\{a(t)[\alpha^\dagger\cos\tau - i(2S)^{-1/2}S_+\sin\tau]\}|0,0\rangle$$
$$= e^{-|a|^2/2}\{1-[a^2(t)/2S][F(\sigma_+\sigma_+ + \alpha^\dagger\alpha^\dagger) + iG\alpha^\dagger\sigma_+]\}\exp\{a(t)[\alpha^\dagger\cos\tau - i(1/\sqrt{2S})S_+\sin\tau]\}|0,0\rangle, \tag{4.10}$$

where we have reintroduced the operator S_+ and neglected the phase factor $e^{i\varphi_0}$.

V. STATISTICAL PROPERTIES OF SYSTEM

We now proceed to calculate the average values in state (4.10) of the operators corresponding to physical quantities relevant to the statistical properties of the system. We remark that within the limits of the second HP approximation

$$\alpha e^{z\alpha^\dagger}|0\rangle = ze^{z\alpha^\dagger}|0\rangle,$$
$$\sigma_- e^{\mu(1/\sqrt{2S})S_+}|0\rangle = \sigma_- e^{-\mu^2(1/8S)\sigma_+^2}e^{\mu\sigma_+}|0\rangle = \mu(1-\sigma_+\sigma_-/4S)e^{\mu(1/\sqrt{2S})S_+}|0\rangle, \tag{5.1}$$

and

$$\alpha^\dagger e^{z\alpha^\dagger}|0\rangle = z^* e^{z\alpha^\dagger}|0\rangle + (z^*/|z|^2)X(|z|)e^{z\alpha^\dagger}|0\rangle,$$

$$\sigma_+ e^{\mu(1/\sqrt{2S})S_+}|0\rangle = \mu^* e^{\mu(1/\sqrt{2S})S_+}|0\rangle + (\mu^*/|\mu|^2)\,Y(|\mu|)e^{\mu\sigma_+}|0\rangle + O(1/S)|0\rangle\,, \tag{5.2}$$

where

$$X(|z|) = \alpha^\dagger\alpha - |z|^2\,, \quad Y(|\mu|) = \sigma_+\sigma_- - |\mu|^2\,.$$

Using (5.1) we have from (4.10), within terms $O(1/S)$ as usual

$$\begin{aligned}\alpha|\psi(t)\rangle &= \alpha(t)\cos\tau|\psi(t)\rangle - e^{-|a|^2/2}[a^2(t)/2S]\,(2F\alpha^\dagger + iG\sigma_+)\exp\{a(t)[\alpha^\dagger\cos\tau - i(1/\sqrt{2S})S_+\sin\tau]\}\,|0,0\rangle\\ &= \{a(t)\cos\tau - [a^2(t)/2S]\,(2F\alpha^\dagger + iG\sigma_+)\}\,|\psi(t)\rangle\,.\end{aligned} \tag{5.3}$$

From (5.3) and (5.1) we immediately obtain

$$\langle\alpha\rangle \equiv \langle\psi(t)|\alpha|\psi(t)\rangle = a(t)\cos\tau - (|a|^2/2S)a(t)(2F\cos\tau - G\sin\tau)\,, \tag{5.4}$$

where we have used (4.10) and the fact that $|\psi(t)\rangle$ is normalized within $O(1/S)$. We also have

$$\langle\alpha^\dagger\alpha\rangle = \langle n(t)\rangle \equiv \langle\psi(t)|\alpha^\dagger\alpha|\psi(t)\rangle = |a|^2\cos^2\tau - (|a|^4/S)(2F\cos\tau - G\sin\tau)\cos\tau\,, \tag{5.5}$$

and the variance $(\Delta\alpha)^2 = \langle\alpha^\dagger\alpha\rangle - |\langle\alpha\rangle|^2$ is seen to vanish up to terms $O(1/S)$. In spite of the fact that this is the accuracy throughout this paper, we can in this particular case push our error up to $O(1/S^2)$, as can be easily seen from (5.3), which we write in the form

$$(\alpha - \langle\alpha\rangle)|\psi(t)\rangle = -\frac{a^2(t)}{2S}\left(2F\frac{a^*(t)\cos\tau}{|a|^2\cos^2\tau}X(|a|\cos\tau) - G\frac{a^*(t)\sin\tau}{|a|^2\sin^2\tau}Y(|a|\sin\tau)\right)|\psi(t)\rangle\,, \tag{5.6}$$

where we have used (5.2) to the lowest order in $1/S$. The squared modulus of vector (5.6) is of $O(1/S^2)$ and yields directly the variance correct up to $O(1/S^2)$. Therefore, we have

$$\begin{aligned}(\Delta\alpha)^2 = (|a|^4/4S^2)[&(4F^2/|a|^2\cos^2\tau)\langle\psi(t)|X^2(|a|\cos\tau)|\psi(t)\rangle + (G^2/|a|^2\sin^2\tau)\langle\psi(t)|Y^2(|a|\sin\tau)|\psi(t)\rangle\\ &- (4FG/|a|^2\sin\tau\cos\tau)\langle\psi(t)|X(|a|\cos\tau)Y(|a|\sin\tau)|\psi(t)\rangle]\,,\end{aligned} \tag{5.7}$$

since $X(|z|) = X^\dagger(|z|)$ and $Y(|\mu|) = Y^\dagger(|\mu|)$ are Hermitian operators. Using (5.1), to the lowest possible order in S we have

$$\alpha|\psi(t)\rangle = a(t)\cos\tau|\psi(t)\rangle\,, \quad \sigma_-|\psi(t)\rangle = -ia(t)\sin\tau|\psi(t)\rangle\,,$$

and consequently

$$\begin{aligned}\langle\psi(t)|X^2|\psi(t)\rangle &= \langle\psi(t)|(\alpha^\dagger\alpha\alpha^\dagger\alpha - 2\alpha^\dagger\alpha|a|^2\cos^2\tau + |a|^4\cos^4\tau)|\psi(t)\rangle\\ &= \langle\psi(t)|[\alpha^\dagger(1+\alpha^\dagger\alpha)\alpha - 2\alpha^\dagger\alpha|a|^2\cos^2\tau + |a|^4\cos^4\tau]|\psi(t)\rangle\\ &= \langle\psi(t)|\alpha^\dagger\alpha|\psi(t)\rangle = |a|^2\cos^2\tau\,.\end{aligned}$$

Analogously we find

$$\langle\psi(t)|Y^2|\psi(t)\rangle = |a|^2\sin^2\tau\,, \quad \langle\psi(t)|XY|\psi(t)\rangle = 0\,.$$

Substituting in (5.7) we obtain

$$(\Delta\alpha)^2 = (|a|^4/4S^2)(4F^2 + G^2)\,.$$

The variance of n can be calculated from (5.3), to which we apply operator α and obtain

$$\begin{aligned}\alpha^2|\psi(t)\rangle = \{&a^2(t)\cos^2\tau\\ &- [a^2(t)/S]a(t)\cos\tau(2F\alpha^\dagger + iG\sigma_+)\}|\psi(t)\rangle\\ &- [a^2(t)/S]F|\psi(t)\rangle\,.\end{aligned}$$

From this we readily find

$$\begin{aligned}\langle\alpha^{\dagger 2}\alpha^2\rangle = |a|^4\cos^4\tau &- 2(|a|^4/S)F\cos^2\tau\\ &- 2(|a|^6/S)(2F\cos\tau - G\sin\tau)\cos^3\tau\,.\end{aligned}$$

Therefore,

$$\begin{aligned}(\Delta n)^2 &= \langle\alpha^\dagger\alpha\alpha^\dagger\alpha\rangle - \langle\alpha^\dagger\alpha\rangle^2 = \langle\alpha^{\dagger 2}\alpha^2\rangle + \langle\alpha^\dagger\alpha\rangle - \langle\alpha^\dagger\alpha\rangle^2\\ &= |a|^2\cos^2\tau - (|a|^4/S)(4F\cos\tau - G\sin\tau)\cos\tau\,.\end{aligned} \tag{5.9}$$

For the spin variables we have from (5.1)

$$\begin{aligned}(S_-/\sqrt{2S})e^{\mu(1/\sqrt{2S})S_+}|0\rangle &\equiv (1 - \sigma_+\sigma_-/4S)\sigma_- e^{\mu(1/\sqrt{2S})S_+}|0\rangle\\ &= \mu(1 - \sigma_+\sigma_-/2S)e^{\mu(1/\sqrt{2S})S_+}|0\rangle\,.\end{aligned} \tag{5.10}$$

From (5.10) and using the same technique which led us to expressions (5.3)–(5.5), we find after some algebra

$$(S_-/\sqrt{2S})\,|\psi(t)\rangle = \{-ia(t)\sin\tau - [a^2(t)/2S][(2F-\sin^2\tau)\sigma_+ + iG\alpha^\dagger]\}\,|\psi(t)\rangle\,, \tag{5.11}$$

$$\langle S_x\rangle = \mathrm{Re}\langle S_-\rangle = -|a|\sqrt{2S}\,\sin\omega t\,\sin\tau - (|a|^3/\sqrt{2S})\sin\omega t[(2F-\sin^2\tau)\sin\tau + G\cos\tau]\,, \tag{5.12}$$

$$\langle S_z\rangle = -S + \langle\sigma_+\sigma_-\rangle = -S + |a|^2\sin^2\tau + (|a|^4/S)(2F\cos\tau - G\sin\tau)\cos\tau\,. \tag{5.13}$$

The above expressions are correct up to terms $O(1/S)$. Another quantity of interest for the spin system is the variance of S_z

$$(\Delta S_z)^2 \equiv \langle S_z^2\rangle - \langle S_z\rangle^2 = \langle(-S+\sigma_+\sigma_-)^2\rangle - \langle S_z\rangle^2 = \langle(S^2 + \sigma_+^2\sigma_-^2 - (2S-1)\sigma_+\sigma_-)\rangle - \langle S_z\rangle^2\,. \tag{5.14}$$

Since

$$\langle\sigma_+^2\sigma_-^2\rangle = |a|^4\sin^4\tau + (|a|^4/S)\sin^2\tau(2F - \tfrac{1}{2}\sin^2\tau) + 2(|a|^6/S)\sin^2\tau\cos\tau(2F\cos\tau - G\sin\tau)\,,$$

using (5.13) we get from (5.14)

$$(\Delta S_z)^2 = |a|^2\sin^2\tau + (|a|^4/S)[(4F-\sin^2\tau)\sin\tau + G\cos\tau]\sin\tau\,. \tag{5.15}$$

Also, using the rotated spin operators introduced in Sec. I it is possible to show that up to terms $O(1/S)$

$$\langle\psi(t)|\,S_-'\,|\psi(t)\rangle = 0\,,$$

provided we choose

$$\varphi = \omega t\,,$$

$$tg\tfrac{1}{2}\theta = (|a|/\sqrt{2S})[\sin\tau + (|a|^2/S)(F\sin\tau + \tfrac{1}{2}G\cos\tau)]\,. \tag{5.16}$$

Analogously to (5.6) we can then calculate

$$(S_-' - \langle S_-'\rangle)\,|\psi(t)\rangle = -i[a(t)/\sqrt{2S}]\,[(2F/\sin\tau)Y(|a|\sin\tau) + (G/\cos\tau)X(|a|\cos\tau)]\,|\psi(t)\rangle\,. \tag{5.17}$$

The squared modulus of vector (5.17) yields the variance of the spin lowering operator in the rotated system, correct up to $O(1/S)$ terms as

$$(\Delta S_-')^2 = (|a|^4/2S)(4F^2 + G^2) \equiv 2S(\Delta\alpha)^2\,. \tag{5.18}$$

VI. DISCUSSIONS AND CONCLUSIONS

We now wish to comment on the results we have obtained. First we calculate from (4.9)

$$2F\cos\tau - G\sin\tau = \tfrac{1}{4}(\sin^2\tau\cos\tau - \tau\sin\tau)\,,$$
$$(2F\cos\tau - G\sin\tau)\cos\tau = \tfrac{1}{16}(\sin^2 2\tau - 2\tau\sin 2\tau)\,,$$
$$(4F^2 + G^2) = \tfrac{1}{64}(\tfrac{3}{2}\cos 4\tau - 8\cos 2\tau + 2\tau\sin 4\tau - 8\tau\sin 2\tau + 4\tau^2 + \tfrac{13}{2})\,,$$
$$(4F\cos\tau - G\sin\tau)\cos\tau = \tfrac{1}{16}[3\sin^2 2\tau - 2\tau\sin 2\tau(1 + 2\cos^2\tau)]\,,$$
$$(2F - \sin^2\tau)\sin\tau + G\cos\tau = \tfrac{1}{4}(-3\sin^3\tau + \sin\tau - \tau\cos\tau)\,,$$
$$[(4F - \sin^2\tau)\sin\tau + G\cos\tau]\sin\tau = \tfrac{1}{16}(\sin^2 2\tau - 2\tau\sin 2\tau - 8\tau\sin^3\tau\cos\tau)\,. \tag{6.1}$$

Expressions (6.1) appear in the $O(1/S)$ terms of the expressions (5.4) and (5.8) for the average value of the field amplitude and its variance, of the field energy (5.5) and its variance (5.9), of the transverse dipole moment (5.12), and of the dipole energy (5.13) and its variance (5.15). They share the property of having terms whose magnitude increase at least linearly with time. On the other hand, the $O(1/S)$ terms are related to the departure from full coherence, as can be seen by comparing the results for the variances of the field variables in the first HP approximation in Sec. III to those of the second approximation in Sec. V. Therefore we may conclude that in the course of time the evolution of the system is characterized by a progressive loss of coherence. At large times, a point is reached when even the $O(1/S)$ terms become so important that our approximation breaks down and the motion of the system is not even approximately coherent. In all the physical quantities we have discussed, this happens when $|a|^2\tau/S$ is of the order of unity, or

$$t \simeq (2S/|a|^2)/\epsilon\sqrt{2S} \equiv t^*\,. \tag{6.2}$$

Now $\epsilon\sqrt{2S}$ is the frequency of exchange of energy between atoms and field in the first HP approximation, and $|a|^2$ is the average number of photons initially present in the field. As an example, in a cavity at microwave frequency we may take $2S = 10^{18}$ and 10^{10} photons at $t=0$; then the system should show evident signs of incoherence after the energy has been exchanged 10^8 times between atoms and field. Taking $\epsilon\sqrt{2S} \sim 10^6$ Hz yields a coherence time of the order of minutes. After this time the system is likely to lose the initial coherence properties, as it is particularly evident for example from expression (5.8) for the variance of the field amplitude. In other words, the interaction between field and atoms can be studied approximately in classical terms up to time t^*. For times larger than t^* the effects of the nonharmonic nature of the spin system spoils the initial coherence, and the behavior of the system cannot be described in classical terms. The system also behaves very differently at large times from the two coupled harmonic oscillators which we have discussed in the Introduction. This difference can be attributed to the difference in the commutation rules of the dynamic variables of a spin and of a harmonic oscillator. Of course, in this paper we have represented both by harmonic oscillators, but in the second HP approximation we have tried to keep some of the angular momentum properties by using the non harmonic interaction Hamiltonian (4.1).

We now make the following remarks:

(i) In spite of the presence of the $O(1/S)$ terms, the average number of photons $\langle\alpha^\dagger\alpha\rangle$ and the atomic energy $\langle S_z\rangle$ move between the same extrema as in the first HP approximation. This is so because the $O(1/S)$ terms vanish when $\langle\alpha^\dagger\alpha\rangle$ and $\langle S_z\rangle$ reach their respective extremum values. This means that periodically we reach a state devoid of photons and with all the initial energy stored in the atomic system, and that periodically the energy is entirely stored in the field with all the atoms in their lower state.

(ii) At times less than t^*, but such that $\tau \gg 1$, we obtain by the use of (5.11), (5.12), and (6.1) the following asymptotic expressions for the average values of the spin components:

$$\langle S_x\rangle = -|a|\sqrt{2S}\,\sin\omega t(\sin\tau - |a|^2\tau\cos\tau/8S),$$
$$\langle S_y\rangle = |a|\sqrt{2S}\,\cos\omega t(\sin\tau - |a|^2\tau\cos\tau/8S), \quad (6.3)$$
$$\langle S_z\rangle = -S + |a|^2\sin\tau(\sin\tau - |a|^2\tau\cos\tau/4S).$$

It is useful to study in a frame rotating at frequency ω about the z axis the evolution of the transverse component perturbed by the $O(1/S)$ terms. In this frame the transverse component of the dipole moment becomes zero at times given by the equation

$$\tan\tau = |a|^2\tau/8S, \quad (6.4)$$

while it assumes the maximum value of $|a|\sqrt{2S}$ (neglecting as usual $O(1/S^2)$ terms with respect to one) at times given by the equation

$$\cot\tau = -|a|^2/8S/(1-|a|^2/8S). \quad (6.5)$$

Therefore the perturbation tends to retard the motion of the dipoles with respect to the original harmonic motion in the first HP approximation, introducing at $O(1/S)$ an aperiodicity in the system. The retardation tends asymptotically to 90° as it can be easily seen by a graphic solution of (6.3), while the extrema of the transverse component in the rotating frame are, within our approximation, the same as in the first HP approximation. We also find the following relation among components (6.3):

$$\langle S_x\rangle^2 + \langle S_y\rangle^2 = 2S(S+\langle S_z\rangle).$$

(iii) The asymptotic displacement of the harmonic oscillator corresponding to the radiation field is proportional to

$$\mathrm{Re}\langle\alpha\rangle \simeq |a|\cos\omega t(\cos\tau + |a|^2\tau\sin\tau/8S), \quad (6.6)$$

where we have used expressions (5.4) and (6.1). Therefore the displacement vanishes at times which are solutions of

$$\cot\tau = -|a|^2\tau/8S,$$

while it is maximum at times given by

$$\tan\tau = \frac{|a|^2/8S}{1-|a|^2/8S}\,\tau.$$

Both these times coincide within $O(1/S)$ with (6.5) and (6.4), respectively, as they should, and since from (5.5) and (6.1)

$$\langle\alpha^\dagger\alpha\rangle \simeq |a|^2\cos\tau(\cos\tau + |a|^2\tau\sin\tau/4S), \quad (6.7)$$

we find, as for the spin variables

$$(\mathrm{Re}\langle\alpha\rangle)^2 + (\mathrm{Im}\langle\alpha\rangle)^2 = \langle\alpha^\dagger\alpha\rangle.$$

Moreover the extrema of the displacement in the rotating frame are approximately 0 and $|a|$ as in the first HP approximation. The variance of the field amplitude and of the rotated S'_- operator in the asymptotic limit discussed above are given by

$$(\Delta\alpha)^2 = (1/2S)(\Delta S'_-)^2 \simeq |a|^4\tau^2/S^2 \quad (6.8)$$

and increase quadratically with time. Consequently the spread of the field wave packet and of the transverse spin component in the rotated system increase steadily without oscillations, at least as long as $\tau < S/|a|^2$.

(iv) We find that expressions (5.9) and (5.15) for the variance of n and S_z at large enough times can be approximated as

$$(\Delta n)^2 \simeq \langle n(t)\rangle + |a|^4\tau\sin 2\tau\cos^2\tau/4S,$$
$$(\Delta S_z)^2 \simeq \langle S_z\rangle + S - |a|^4\tau\sin 2\tau\sin^2\tau/4S, \quad (6.9)$$

where $n(t)$ and $\langle S_z\rangle$ are given by (5.5) and (5.13), respectively. Expressions (6.9) also indicate a progressive loss of coherence both in the atomic system and in the radiation field. The apparent non vanishing of the variances for $\langle n(t)\rangle = 0$ and for $\langle S_z\rangle = -S$ is not a real difficulty and it should be an effect of our neglecting the $O(1/S^2)$ terms. In fact, within the limits of the second HP approximation, we could have written (6.9)

$$(\Delta n)^2 \simeq \langle n(t)\rangle (1 + |a|^2 \tau \sin 2\tau/4S),$$

$$(\Delta S_z)^2 \simeq (\langle S_z\rangle + S)(1 - |a|^2 \tau \sin 2\tau/4S),$$

which vanish at the desired times. We remark that for a coherent spin state (1.4) the variance of S_z can be shown to be exactly given by

$$(\Delta S_z)^2 = \langle S_z\rangle + S - 2S|\mu|^4/(1+|\mu|^2)^2, \qquad (6.10)$$

Up to $O(1/S)$ we may put in (6.10)

$$|\mu|^2 \simeq (|a|^2 \sin^2\tau)/2S,$$

so that if the atomic system had remained in a coherent state we should have obtained

$$(\Delta S_z)^2 \simeq \langle S_z\rangle + S - |a|^4 \sin^4\tau/2S,$$

which is clearly different from (6.9).

We wish to conclude by answering explicitly the question we posed in the Introduction. The progressive loss of coherence that we have found does not imply loss of the initial information, since we see from (6.3) and (6.6) that the motion of the average values of dipole components and of field coordinates is predictable from the information at $t=0$, and it does not tend to be damped out in the span of times for which our calculations are valid. On the other hand, consideration of the $O(1/S)$ terms has shown that the loss of coherence, which manifests itself in expressions (6.8) and (6.10) for the variances, tends to become important after time t^* given by (6.2), when the energy between atoms and radiation field has been exchanged $S/|a|^2$ times. After this time the non classical features of the system should become more evident.

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