Inverse scattering transform for wave-wave scattering*

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An inverse scattering transform method is presented for wave-wave scattering problems such as the Raman effect, distributed parametric amplifiers, and wave-wave interactions in plasmas. This transform method can be viewed as a nonlinear generalization of the Fourier-transform technique; and, in the form presented here, it describes the dynamics of the scattering medium.

I. INTRODUCTION

The inverse scattering transform method for finding a solution of a nonlinear wave equation through a succession of linear computations was introduced in 1967 by Gardner, Greene, Kruskal, and Miura[1] as a special calculation adapted to the initial-value problem for the Korteweg–de-Vries equation. This method was generalized by Lax[2] in the following way: Consider a nonlinear wave equation represented abstractly by

$$\mathcal{Y}_t = N(\mathcal{Y}) \tag{1.1}$$

where N denotes a nonlinear operator on some suitable space of functions. Suppose it is possible to find linear operators L and B which depend upon $\mathcal{Y}$ and which satisfy the operator equation

$$iL_t = BL - LB \tag{1.2}$$

when $\mathcal{Y}$ satisfies the nonlinear wave equation (1.1). Suppose also that the eigenvalues λ of the operator L are found from

$$L\psi = \lambda\psi \tag{1.3}$$

and that the time evolution of ψ is given by

$$i\psi_t = B\psi. \tag{1.4}$$

It is then easily demonstrated that the eigenvalues of L are independent of time even though $\mathcal{Y}(x,t)$ evolves as in (1.1). When it is possible to associate a scattering problem with the operator L, $\mathcal{Y}(x,t)$ can in principle be computed from $\mathcal{Y}(x,0)$ through three linear steps.

(a) *Direct scattering problem.* Scattering parameters are calculated (such as reflection and transmission coefficients of L) for ψ at $|x| = \infty$ and $t = 0$ from $\mathcal{Y}(x,0)$.

(b) *Time evolution of the scattering data.* Equation (1.4) is used together with the asymptotic form of B at $|x| = \infty$ to calculate the time evolution of the scattering data.

(c) *Inverse scattering problem.* $\mathcal{Y}(x,t)$ is constructed from a knowledge of the scattering data of L as a function of time.

Although the conditions expressed in Eqs. (1.2)–(1.4) appear rather special, they are satisfied for a rather large class of nonlinear wave equations which are of independent physical interest.[3] In each of these cases, the (constant) eigenvalues in (1.3) correspond to the fixed velocities of "solitons" or localized traveling-wave solutions which can undergo nondestructive collisions among themselves. As Ablowitz, Kaup, Newell, and Segur (AKNS) have emphasized,[4] the inverse scattering transform method can be viewed as a generalization of the Fourier-transform method for solving a linear wave equation. In the linear case step (a) is a decomposition of the initial condition into spatial harmonic components, step (b) is a calculation of time evolution for these components, and step (c) is their Fourier reconstruction. In the nonlinear calculation, solitons play a role analogous to that of Fourier components in a linear calculation.

During an independent investigation, Lamb[5] developed an inverse scattering transform technique for the self-induced-transparency (SIT) problem of nonlinear optics. In SIT a classical plane wave interacts with a resonant two-level atomic medium, and the appropriate inverse scattering calculation was obtained by Lamb directly from the underlying physical problem. Following a suggestion of Steudel,[6] we consider here a problem which is roughly the "dual" of SIT. As is indicated in Fig. 1, we treat the resonant interaction of two plane waves (an *incident* wave of amplitude A_1 and radian frequency ω_1, a *scattered* wave of amplitude A_2 and frequency ω_2) with a medium which can propagate a classical *scattering* wave (of amplitude $\mathcal{Y}$ and frequency ω_3). We show how an inverse scattering transform method can be developed to calculate the time

evolution for $\mathcal{Y}(x,t)$, and we indicate how this method is related to the physical problem.

Note carefully that the term "scattering" is used in two senses in the above paragraph: first, as the physical scattering which is the key aspect of the phenomena under investigation, and second, as the abstract scattering of ψ in (1.3). These two uses are unrelated.

The results we obtain may be applied to a variety of physical problems including Raman and Brillouin scattering of laser light in solids and liquids,[7] wave-wave interactions in plasma,[8,9] and the distribution parametric amplifier.[10,11] The main assumptions are plane semi-infinite geometry, lossless medium, and equal group velocity for the incident and scattered waves. We expect the latter requirement to be satisfied if $\omega_3 \ll \omega_1$ or ω_2. It should perhaps be emphasized that the inverse scattering transform method outlined here is not simply a degenerate form of that sketched recently by Zakharov and Manakov[12] for three wave interactions when all group velocities differ; it is more closely related to the formalism developed by AKNS.[4]

In Sec. II we indicate how the basic physical equations may be obtained for the case of Raman scattering. Our reasons for this didactic excursion are two-fold: the technique for casting the physical equations in a form suitable for application of the inverse scattering transform method is not obvious, and we intend in Sec. III to suggest a physical interpretation for the inverse scattering transform equation corresponding ot Eq. (1.3). Finally, in Sec. IV we outline the details of the inverse scattering transform method for the operators in Sec. III, and we consider some specific properties of soliton solutions into which the initial conditions must eventually evolve.

II. RAMAN SCATTERING AS A CONCRETE EXAMPLE

In this section we derive the basic equations to be analyzed for the special case of one-dimensional (forward) Raman scattering in an infinite medium. The incident wave is at frequency ω_1, and the scattered-wave frequency ω_2 is assumed to be lower (Stokes) than ω_1 so $\omega_3 = \omega_1 - \omega_2$. In this treatment we have ignored higher-order Raman radiation effects, which are important for intense fields.[13] Following Yariv,[14] the molecules in the Raman medium are represented by classical harmonic oscillators. Since they are uncoupled, the group velocity of the scattering (ω_3) wave is zero in the laboratory frame. If X is the vibrational coordinate of a molecule, the equation of motion for a harmonic oscillator is

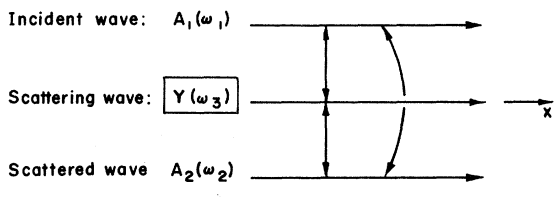

FIG. 1. Diagram of the wave-wave interaction under study.

$$X_{tt} + \omega_R^2 X = aE^2 - qE, \tag{2.1}$$

where a is a constant. In (2.1), ω_R is the resonant vibrational frequency, aE^2 represents the anharmonic character of the molecular bond under the electric field E, and q is the electronic charge of the molecule. The one-dimensional wave equation for the propagation of the electric field is

$$E_{xx} - E_{tt}/u^2 = 2\mu_0 a(XE)_{tt}, \tag{2.2}$$

where u is the group velocity of the electromagnetic wave, μ_0 is the magnetic permeability of the medium, and $2aXE$ is the nonlinear polarization. E and X are expanded as

$$E = \frac{1}{2}\left(\sum_{j=1}^{2} \mathcal{E}_j(x,t) e^{i(\omega_j t - \kappa_j x + \gamma_j)} + \text{c.c.}\right), \quad \gamma_j \text{ constants} \tag{2.3a}$$

$$X = \tfrac{1}{2}\left(\chi e^{i(\omega_3 t - \kappa_3 x + \gamma_3)} + \text{c.c.}\right), \quad \gamma_3 \text{ constant} \tag{2.3b}$$

where the $\mathcal{E}_j$'s and χ are assumed to be slowly varying amplitudes of space and time. In (2.3a), $\mathcal{E}_1$ and $\mathcal{E}_2$ are the slowly varying amplitudes of the incident and scattered electric fields, respectively.

Under the assumptions that (a) the amplitudes of the waves are large only at the frequencies ω_1, ω_2, and ω_3, (b) ω_1/κ_1 and ω_2/κ_2, the phase velocities of the incident and Stokes wave are equal to u, the group velocity of the electric field, (c) $\kappa_1 - \kappa_2 - \kappa_3 = \Delta\kappa$, $\gamma_1 - \gamma_2 - \gamma_3 = 0$, and (d) a is a small quantity, the coefficients of $e^{i\omega_j t}$, $j = 1,2,3$, can be balanced in Eqs. (2.1) and (2.2). Keeping only first-order terms, Eq. (2.1) becomes

$$\chi_t - i\delta\chi = -iq_3\mathcal{E}_1\mathcal{E}_2^* e^{-i\Delta\kappa x} \tag{2.4a}$$

and Eq. (2.2) becomes

$$(1/u)\mathcal{E}_{1t} + \mathcal{E}_{1x} = -iq_1\mathcal{E}_2\chi e^{i\Delta\kappa x} \tag{2.4b}$$

$$(1/u)\mathcal{E}_{2t} + \mathcal{E}_{2x} = -iq_2\mathcal{E}_1\chi^* e^{-i\Delta\kappa x}. \tag{2.4c}$$

In Eqs. (2.4a–c),

$$\delta \equiv [(\Delta\omega)^2 - 2(\Delta\omega)\omega_3]/2\omega_3,$$

$$q_3 \equiv a/2\omega_1,$$

$$q_j = (a\mu_0/2\kappa_j)[\omega_j^2 - 2\omega_3(\Delta\omega) + (\Delta\omega)^2] \quad j = 1, 2$$

where[7] $\Delta\omega \equiv \omega_3 - \omega_R$ defines the difference between the frequency of vibration of the oscillators ω_3 and the resonant frequency of the oscillators ω_R.

Under an independent variable transformation defined by

$$x \to \zeta = x, \quad t \to \tau = t - x/u,$$

and the normalization of the dependent variables as

$$A_1 = (q_2 q_3)^{\frac{1}{2}} \mathcal{E}_1, \quad A_2 = (q_1 q_3)^{\frac{1}{2}} \mathcal{E}_2, \quad Y = (q_2 q_1)^{\frac{1}{2}} \chi,$$

Eqs. (2.4a) – (2.4c) become

$$Y_\tau = i\delta Y - iA_1 A_2^* e^{-i\Delta\kappa\zeta} \tag{2.5a}$$

$$A_{1\zeta} = -iA_2 Y e^{i\Delta\kappa\zeta},$$

$$A_{2\zeta} = -iA_1 Y^* e^{-i\Delta\kappa\zeta}. \tag{2.5b}$$

Then writing

$$\mathfrak{A}_1 = A_1 e^{-i\Delta\kappa\zeta}, \quad \mathfrak{A}_2 = A_2 e^{i\Delta\kappa\zeta}, \quad \mathcal{Y} = Y e^{-i\Delta\kappa\zeta},$$

Eqs. (2.5a) and (2.5b) become

$$\mathcal{Y}_\tau = i\delta\mathcal{Y} - i\mathfrak{A}_1\mathfrak{A}_2^* \tag{2.6a}$$

$$\mathfrak{A}_{1\zeta} = -i\Delta\kappa\mathfrak{A}_1 - i\mathcal{Y}\mathfrak{A}_2,$$

$$\mathfrak{A}_{2\zeta} = i\Delta\kappa\mathfrak{A}_2 - i\mathcal{Y}^*\mathfrak{A}_1. \tag{2.6b}$$

Following Feynman *et al.*,[15] we define the quantities

$$\mathfrak{U} = i\mathfrak{A}_1\mathfrak{A}_2^*, \quad \mathfrak{W} = \mathfrak{A}_1\mathfrak{A}_1^* - \mathfrak{A}_2\mathfrak{A}_2^*, \tag{2.7}$$

where $\mathfrak{W}$ represents the difference in intensity between the incident and Stokes waves. In terms of these new variables, Eqs. (2.6a) and (2.6b) are

$$\mathcal{Y}_\tau = i\delta\mathcal{Y} - \mathfrak{U}, \tag{2.8a}$$

$$\mathfrak{U}_\zeta = -\mathcal{Y}\mathfrak{W} - 2i\Delta\kappa\mathfrak{U}, \tag{2.8b}$$

$$\mathfrak{W}_\zeta = 2(\mathfrak{U}\mathcal{Y}^* + \mathfrak{U}^*\mathcal{Y}). \tag{2.8c}$$

These are the fundamental equations to be studied in this paper. A corresponding development is easily carried through when the scattering oscillators are coupled so their group velocity is not zero.[15] When $\delta = 0$, Eqs. (2.8a)–(2.8c) are similar to those discussed by Lamb[16] as a representation of SIT.

III. INVERSE SCATTERING TRANSFORM EQUATIONS

Since the equations describing wave-wave scattering [Eqs. (2.8)] are similar to those for SIT, linear operators L and B which satisfy (1.2)–(1.4) can be found as a modification of results obtained by AKNS[4] and Lamb.[17] Assuming that Eqs. (2.8a)–(2.8c) correspond to Eq. (1.1), we can write Eq. (1.3) (the "L equation") as

$$\psi_{1\zeta} + i\lambda\psi_1 = \mathcal{Y}\psi_2, \quad \psi_{2\zeta} - i\lambda\psi_2 = -\mathcal{Y}^*\psi_1, \tag{3.1}$$

and Eq. (1.4) (the "B equation") as

$$\psi_{1\tau} = [\tfrac{1}{4}i\mathfrak{W}/(\lambda - \Delta\kappa) + \tfrac{1}{2}i\delta]\psi_1 + \tfrac{1}{2}i\mathfrak{U}/(\lambda - \Delta\kappa)\psi_2,$$

$$\psi_{2\tau} = \tfrac{1}{2}i\mathfrak{U}^*/(\lambda - \Delta\kappa)\psi_1 - [\tfrac{1}{4}i\mathfrak{W}/(\lambda - \Delta\kappa) + \tfrac{1}{2}i\delta]\psi_2. \tag{3.2}$$

Since most L and B equations have hitherto been found by the "classical method" (i.e., guessing), Lambs's approach to the SIT problem is particularly interesting. His inverse scattering transform calculation was directly related to the physical problem under study. Motivated by this, McLaughlin and Corones[18] studied the propagation of magnetic flux along a Josephson transmission line and again indicated how the L equation can be related to the underlying physical problem. This relation may also be established in the case of Raman scattering. To see this, consider Eq. (2.6b), which describes the electromagnetic aspect of Raman scattering. It can be rewritten as

$$\mathfrak{A}_{1\zeta} + i\Delta\kappa\mathfrak{A}_1 = \mathcal{Y}(-i\mathfrak{A}_2),$$

$$(-i\mathfrak{A}_2)_\zeta - i\Delta\kappa(-i\mathfrak{A}_2) = -\mathcal{Y}^*(A_1). \tag{3.3}$$

Comparing Eq. (3.3) with Eq. (3.1), it is seen that these two sets of equations are identical if

$$\mathfrak{A}_1 \equiv \psi_1, \quad -i\mathfrak{A}_2 \equiv \psi_2, \quad \text{and } \Delta\kappa = \lambda.$$

Thus the L equation for the inverse scattering transform method can be identified with the electromagnetic equations describing slow spatial variation of incident and Stokes wave amplitudes under the special condition $\Delta\kappa = \lambda$, which makes the denominators of the B operator [Eq. (3.2)] go to zero. With this condition the determinant of the B matrix reduces to

$$|B| = (\mathfrak{A}_1\mathfrak{A}_1^* + \mathfrak{A}_2\mathfrak{A}_2^*)^2/16(\lambda - \Delta\kappa)^2, \tag{3.4}$$

which is indeterminate if the incident and Stokes wave energies remain of order $\lambda - \Delta\kappa$ as $\lambda \to \Delta\kappa$. Such a decoupling seems necessary because Eq. (3.2) does not give the variation of the electromagnetic wave amplitudes with τ.

IV. SOLITON SOLUTIONS

Having found the L and B equations we can compute $\mathcal{Y}(\zeta, \tau)$ from $\mathcal{Y}(\zeta, 0)$ using the three steps outlined in the Sec. I. From AKNS,[4] it is given by

$$\mathcal{Y}(\zeta, \tau) = -2K(\zeta, \zeta, \tau), \tag{4.1}$$

where $K(\zeta, y, \tau)$ obeys the Marchenko equation

$$K(\zeta, y, \tau) = F(\zeta + y, \tau) - \int_{-\infty}^{\zeta}\int_{-\infty}^{\zeta} F(y + s_1, \tau)F^*(s_1 + s_2) \times K(\zeta, s_2, \tau)\, ds_1\, ds_2, \tag{4.2}$$

with

$$F(y,\tau)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{b^*(\lambda,\tau)}{a(\lambda,\tau)}e^{-i\lambda y}\,d\lambda - i\sum_{j=1}^{N}\bar{c}_j(\lambda_j,\tau)e^{-i\lambda_j y},$$

where N is the number of discrete eigenvalues of Eq (3.1). If f and g are Jost functions which satisfy Eq. (3.1) with boundary conditions

$$\lim_{x\to-\infty} f\to\begin{pmatrix}1\\0\end{pmatrix}e^{-i\lambda x},\quad \lim_{x\to+\infty} g\to\begin{pmatrix}0\\1\end{pmatrix}e^{i\lambda x},\quad \lambda \text{ real}$$

and for λ complex

$$\bar{g}\equiv\begin{pmatrix}g_2^*(\zeta,\lambda)\\-g_1(\zeta,\lambda^*)\end{pmatrix},$$

then a and b are defined by

$$f=bg+a\bar{g}$$

when λ is real. $a(\lambda,\tau)$ and $b(\lambda,\tau)$ can be analytically continued to the upper half of the λ plane, and $\bar{c}$ is defined as

$$\bar{c}\equiv+b^*(\lambda,\tau)/a_\lambda(\lambda,\tau).$$

The time dependence of a, b, and $\bar{c}$ are calculated to be

$$b(\lambda,\tau)=b(\lambda,0)\exp\{-\tfrac{1}{2}i[1/(\lambda-\Delta\kappa)+2\delta]\},\quad (4.3a)$$

$$a(\lambda,\tau)=a(\lambda,0),\quad (4.3b)$$

and

$$\bar{c}(\lambda_j,\tau)=\bar{c}(\lambda_j,0)\exp\{+\tfrac{1}{2}i[1/(\lambda-\Delta\kappa)+2\delta]\,\tau\}=\bar{c}(\lambda_j,0)\exp(\omega_{rj}\tau+i\,\omega_{ij}\tau),\quad (4.3c)$$

where

$$\omega_{rj}=+\lambda_{ij}/(2[(\lambda_{rj}-\Delta\kappa)^2+\lambda_{ij}^2])$$

$$\omega_{ij}=(\lambda_{rj}-\Delta\kappa)/(2[(\lambda_{rj}-\Delta\kappa)^2+\lambda_{ij}^2])+\delta,$$

$$\lambda_j\equiv\lambda_{rj}+\lambda_{ij}.\quad (4.3d)$$

The values $b(\lambda,0)$, $a(\lambda,0)$ and $\bar{c}(\lambda_j,0)$ are determined by the initial conditions.

If $b(\lambda,0)=0$, and $N=1$ in Eqs. (4.1) and (4.2), the solution $\mathcal{Y}$ is of the form

$$\mathcal{Y}(\zeta,\tau)=[2i\bar{c}_1(\lambda_1,0)/|\bar{c}_1(\lambda_1,0)|]\,\lambda_{i1}\exp[i(\omega_{i1}\tau-2\lambda_{r1}\zeta)]\,\mathrm{sech}[2\lambda_{i1}\zeta+\omega_{r1}\tau+\ln(|\bar{c}_1(\lambda_1,0)|/2\lambda_{i1})].\quad (4.4)$$

This is a single-soliton solution. If $R(\lambda,0)=0$, and there are N discrete eigenvalues, the solution $\mathcal{Y}$ will consist of N such solitons and these solitons will interact nonlinearly, but asymptotically $\mathcal{Y}$ will consist of the superposition of N solitons in the form of Eq. (4.4). The N-soliton formula for $\mathcal{Y}$ is

$$\mathcal{Y}=-2\sum_k\{e^{+i\lambda_k\zeta}[(1+ZZ^*)^{-1}G]_k\},$$

where Z is an $N\times N$ matrix with elements defined by

$$Z_{jl}=\bar{c}_j e^{-i(\lambda_j-\lambda_l^*)\zeta}/(\lambda_j-\lambda_l^*)$$

and G is a $n\times 1$ matrix with elements defined by

$$G_{l1}=-i\bar{c}_l e^{-2i\lambda_l\zeta}.$$

One interesting type of solution is the "breather." These are real solutions of Eq. (3.1) formed by two solitons whose associated eigenvalues λ_1 and λ_2 are related by

$$\lambda_1=-\lambda_2^*\equiv\lambda\equiv\lambda_r+i\lambda_i.$$

Using Eqs. (4.1) and (4.2a), a breather solution is

$$\mathcal{Y}=\frac{-2\lambda_i}{|c(\lambda,0)|\,\lambda_r}\left(\frac{\lambda_r\cosh\theta_1\sin\theta_2+\lambda_2\sinh\theta_1\cos\theta_2}{\cosh^2\theta_1+(\lambda_i^2/\lambda_r^2)\cos^2\theta_2}\right)\quad (4.5)$$

where

$$\theta_1=2\lambda_i\zeta+\omega_r\tau+\ln\left(\frac{|c(\lambda,0)|}{2\lambda_i}\frac{\lambda_r}{|\lambda|}\right),$$

$$\theta_2=-2\lambda_r\zeta+\omega_i\tau-i\ln\left(\frac{c(\lambda,0)}{|c(\lambda,0)|}\frac{|\lambda|}{\lambda}\right),$$

with $\bar{c}_1=-\bar{c}_2^*\equiv c$, and $\omega_r=\omega_{r1}$ and $\omega_i=\omega_{i1}$ where ω_{r1} and ω_{i1} are defined by Eq. (4.3d). A sketch of $|\mathcal{Y}|$ when $\Delta\kappa=0.1$, $\delta=-0.1$, $\lambda_r=\lambda_i=1$, and $c=\lambda\sqrt{2}$ is shown in Fig. 2.

In its present state of development, this inverse scattering transform method has two unsatisfactory features:

(a) When $b(\lambda,0)\neq 0$, the Marchenko equation (4.2a) cannot be analytically solved. However, the contribution to $\mathcal{Y}(\zeta,\tau)$ from the continuous eigenvalues decays away, so asymptotically, at least, the solution will consist only of solitons.

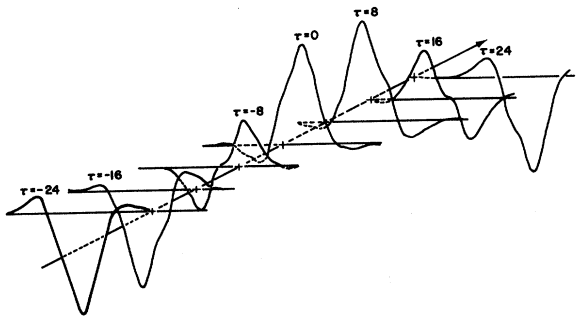

FIG. 2. Evolution of a "breather" soliton indicated in Eq. (4.5).

(b) The inverse scattering operators have been found only in the (ζ,τ) coordinates. An initial value calculation implies that

$$\mathcal{Y}(\zeta,\tau=0)=\mathcal{Y}(x,\ t=x/u)$$

has to be specified. A more realistic initial value problem with initial condition $\mathcal{Y}(x,\ t=0)=\mathcal{Y}(\zeta,\tau=-\zeta/u)$ is yet to be solved.

V. CONCLUSIONS

We have sketched the elements of an inverse scattering transform method for calculating dynamic evolution of the scatterer in a wave-wave scattering problem. (Note the double use of the term "scattering".) The main simplifying assumptions are a lossless scattering meduim, one-dimensional semi-infinite geometry, and equal group velocities for the incident and scattered waves. If the scattering medium is not infinite but long compared with the width of a soliton, some soliton structure should be expected to emerge. Our calculations should then be useful for times less than one transit of the initial pulse.

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