Electrohydrodynamics of a charge-separated plasma

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We derive and discuss the dispersion relation and polarization of waves in a one-component, cold, homogeneous plasma moving at arbitrary speed. The relevance to wave propagation (both low frequency and radio) through the pulsar magnetosphere is briefly discussed.

I. INTRODUCTION

Whereas most astrophysical and laboratory plasmas are, on the average, charge neutral (due to the long range and strength of electric forces), the plasma in a pulsar magnetosphere is probably *charge separated* to a very high degree[1] (due to the strong homopolar induction field). An understanding of the origin of pulsar radiation therefore seems to necessitate an understanding of wave propagation in a charge-separated plasma. We will provide such an understanding within the framework of electrohydrodynamics (EHD).

For a neutral, cold plasma, Ohm's law[2] $j_a = \sigma_{ab} E_b$ leads to a Hermitian, wave-vector-independent *dielectric tensor*

$$\epsilon_{ab} = \delta_{ab} + (4\pi i/\omega)\sigma_{ab}$$

with eigenvalues

$$\epsilon_c = \left(1 - \sum_\alpha \tilde{\omega}_\alpha^{-2},\ 1 - \sum_\alpha (\tilde{\omega}_\alpha^2 \mp \hat{\omega}_\alpha)^{-1}\right)$$

for waves of angular frequency ω in a local rest system, where α numbers plasma components, $\tilde{\omega}_\alpha \equiv \omega(4\pi e_\alpha \rho_\alpha / m_\alpha)^{-1/2}$, and $\hat{\omega}_\alpha \equiv \omega(4\pi c \rho_\alpha / B)^{-1}$. This same relation can be shown to hold for a one-component plasma, with just one term occurring under the summation sign. However, a one-component plasma cannot be (globally) at rest in any inertial system, and in view of the applications one would hesitate to restrict the analysis to local rest frames. We thus work in a general inertial system, and start directly from the linearized equation of motion plus Maxwell's equations.

Our main result is contained in Fig. 1 which shows the dispersion relation of a cold one-component plasma. Unfamiliar are three descending branches ending in three zero-frequency (i.e., static) modes: They correspond to the fact that we work in a global inertial system in which matter must necessarily move (in order to be in stationary equilibrium), and that there are three phase velocities which can compensate the velocity of the medium. More importantly, the diagram shows no resonances, but shows three "cutoffs" (breakoffs) at the Larmor frequency ω_L, the plasma frequency ω_p, and at the low-frequency $\omega_r \equiv \omega_p^2/\omega_L$ which roughly equals the rotation frequency Ω of the homopolar inductor ($\omega_r = 2\gamma^2 \vec{\Omega}\cdot\vec{B}/B$ for corotation where γ is the Lorentz factor of the plasma). We prove the completeness of our modes, and evaluate their polarization and energy-flow velocities.

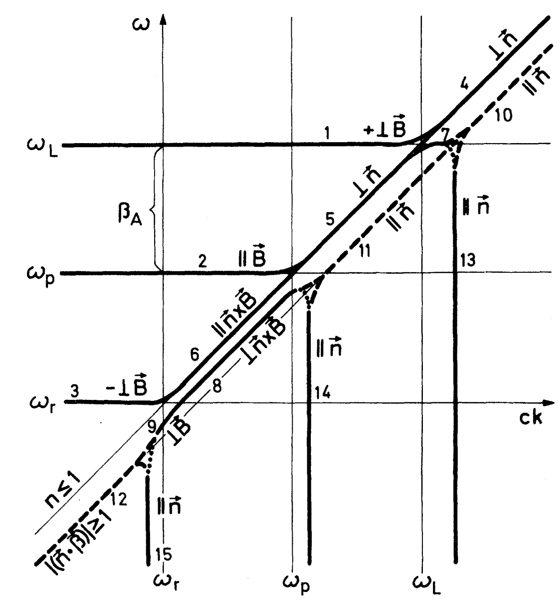

FIG. 1. Dispersion diagram $\omega_j(k)$ for fixed angles, in double-logarithmic representation, for $\beta \ll 1$, $\beta_A = |\cos(\vec{n}, \vec{B})|^{-8}$, $\beta_A = |\beta \cos(\vec{n}, \vec{\beta})|^{-4}$. The dashed and dotted branches correspond to $\cos(\vec{n}, \vec{B}) \{\gtrless\} 0$ respectively. $n \equiv ck/\omega$ grows versus the lower right-hand corner. The approximate directions of $\delta_j \vec{E}(k)$ are also marked; $+\perp\vec{B}$ means "$\delta\vec{E}$ normal to $\vec{B}$, rotating around $\vec{B}$ in the Larmor sense".

II. DISPERSION RELATION

We will study waves in a plasma which satisfies the following four conditions: (1) The plasma consists of only *one component* (i.e., is 100% charge separated); (2) *only electromagnetic forces* are acting (i.e., gravity is neglected); (3) the plasma is *cold* (i.e., peculiar motions of particles are neglected); (4) the unperturbed plasma is homogeneous and stationary.

Condition (4) is not strictly compatible with conditions (1) and (2); but we will show later in the deduction that its necessary violation is negligibly small.

In an inertial system, such a medium is described by Maxwell's equations:

$$F^{\alpha\beta}{}_{,\beta} = (4\pi/c)\, j^{\alpha}, \quad F_{[\alpha\beta,\gamma]} = 0\,; \tag{1}$$

by the conduction law,

$$j^{\alpha} = \rho_0 u^{\alpha}\,; \tag{2}$$

and by the equation of motion,

$$m_0 c\, u^{\alpha}{}_{,\beta} u^{\beta} = e F^{\alpha\beta} u_{\beta}\,, \tag{3}$$

where four-current j^{α}, four-velocity u^{α}, rest charge density ρ_0, and field strength $F^{\alpha\beta}$ are related to their familiar three-vector equivalents via:

$$\begin{aligned} &j^{\alpha} = (\vec{j}, c\rho), \quad u^{\alpha} = c\gamma(\vec{\beta}, 1), \quad \rho = \rho_0\gamma\,, \\ &\vec{\beta} \equiv \vec{v}/c, \quad \gamma \equiv (1-\beta^2)^{-1/2}, \\ &F^{\alpha\beta}u_{\beta} = c\gamma(\vec{E} + \vec{\beta}\times\vec{B}, \vec{\beta}\cdot\vec{E})\,. \end{aligned} \tag{4}$$

In three-vector language, Eqs. (1)–(3) read

$$\begin{aligned} &\nabla\times\vec{E} = -c^{-1}\partial_t\vec{B}, \qquad \nabla\cdot\vec{E} = 4\pi\rho \\ &\nabla\times\vec{B} = 4\pi c^{-1}\vec{j} + c^{-1}\partial_t\vec{E}, \quad \nabla\cdot\vec{B} = 0\,, \end{aligned} \tag{5}$$

and

$$\begin{aligned} &\vec{j} = c\rho\vec{\beta}\,, \\ &(c\vec{\beta}\cdot\nabla + \partial_t)\gamma\vec{\beta} = (e/m_0c)(\vec{E} + \vec{\beta}\times\vec{B})\,. \end{aligned} \tag{6}$$

Weak waves are solutions of the first-order perturbed system (5), (6). Using $\delta_\gamma = \gamma^3\vec{\beta}\cdot\delta\vec{\beta}$ we get

$$\begin{aligned} &\nabla\times\delta\vec{E} = -c^{-1}\partial_t\,\delta\vec{B}, \qquad \nabla\cdot\delta\vec{E} = 4\pi\,\delta\rho \\ &\nabla\times\delta\vec{B} = 4\pi c^{-1}\,\delta\vec{j} + c^{-1}\partial_t\,\delta\vec{E}, \quad \nabla\cdot\delta\vec{B} = 0\,, \end{aligned} \tag{7}$$

and

$$\begin{aligned} &\delta\vec{j} = c(\delta\rho\vec{\beta} + \rho\delta\vec{\beta})\,, \\ &(c\vec{\beta}\cdot\nabla + \partial_t)[\gamma\delta\vec{\beta} + \gamma^3\vec{\beta}(\vec{\beta}\circ\delta\vec{\beta})] + c(\delta\vec{\beta}\cdot\nabla)\gamma\vec{\beta} \\ &\qquad = (e/m_0c)[\delta\vec{E} + \delta\vec{\beta}\times\vec{B} + \vec{\beta}\times\delta\vec{B}]\,. \end{aligned} \tag{8}$$

This linear system of first-order perturbations is solved by Fourier analysis, i.e., by assuming the vectors $\delta\vec{E}, \delta\vec{B}, \delta\vec{\beta}$ to be (real parts of complex vectors which are) proportional to $\exp[i(\vec{k}\cdot\vec{x} - \omega t)]$; we find, using (6) with $\vec{E} + \vec{\beta}\times\vec{B} = 0$,

$$\begin{aligned} &\vec{n}\times\delta\vec{E} = \delta\vec{B}, \quad \vec{n}\cdot\delta\vec{E} = -(4\pi i c/\omega)\delta\rho\,, \\ &\vec{n}\times\delta\vec{B} = -\delta\vec{E} + \vec{\beta}(\vec{n}\cdot\delta\vec{E}) - (4\pi i c\rho/\omega)\delta\vec{\beta}\,, \\ &\gamma(\vec{\beta}\cdot\vec{n} - 1)[\delta\vec{\beta} + \gamma^2\vec{\beta}(\vec{\beta}\cdot\delta\vec{\beta})] - (ic/\omega)(\delta\vec{\beta}\cdot\nabla)\gamma\vec{\beta} \\ &\qquad = -(ie/m_0c\omega)[\delta\vec{E} + \delta\vec{\beta}\times\vec{B} + \vec{n}(\vec{\beta}\cdot\delta\vec{E}) - (\vec{\beta}\cdot\vec{n})\delta\vec{E}]\,, \end{aligned} \tag{9}$$

where

$$\vec{n} \equiv c\vec{k}/\omega$$

is the dimensionless wave vector. Elimination of $\delta\vec{B}$ and $\delta\vec{\beta}$ results in one vector equation for $\delta\vec{E}$, and explicit expressions for $\delta\vec{B}$ and $\delta\vec{\beta}$ in terms of $\delta\vec{E}$:

$$\delta\vec{B} = \vec{n}\times\delta\vec{E}\,, \tag{10}$$

$$\delta\vec{\beta} = i\hat{\omega}B^{-1}[\delta\vec{E} - n^2\delta_\perp\vec{E} - \vec{\beta}(\vec{n}\cdot\delta\vec{E})]\,, \tag{11}$$

$$[\nu\tilde{\omega}^2(1 + \gamma^2\vec{\beta}\vec{\beta}\cdot) + i\hat{\omega}\vec{b}\times][\delta\vec{E} - n^2\delta_\perp\vec{E} - \vec{\beta}(\vec{n}\cdot\delta\vec{E})] + (ic\tilde{\omega}/\gamma\omega_p)[\delta\vec{E} - n^2\delta_\perp\vec{E} - \vec{\beta}(\vec{n}\cdot\delta\vec{E})]\cdot\nabla\gamma\vec{\beta} = \nu\delta\vec{E} + \vec{n}(\vec{\beta}\cdot\delta\vec{E})\,, \tag{12}$$

or

$$0 = \{\delta_{ab}(1-\tilde{\omega}^2) + \nu^{-1}n_a\beta_b + \tilde{\omega}^2[n^2h_{ab} + (n^2-1)\gamma^2\beta_a\delta_b] - i\nu^{-1}\hat{\omega}\epsilon_{acd}b^c(\delta^d_b - n^2h^d{}_{\cdot b}) - ic\tilde{\omega}(\nu\gamma\omega_p)^{-1}(\delta^c_b - n^2h^c{}_{\cdot b})(\gamma\beta_a)_{,c}\}\,\delta E^b\,, \tag{13}$$

where

$$\begin{aligned} &\hat{\omega} \equiv \omega/\omega_r, \quad \omega_r \equiv 4\pi\rho c/B \sim \text{rotation frequency}\,, \\ &\tilde{\omega} \equiv \omega/\omega_p, \quad \omega_p \equiv (4\pi e\rho/m)^{1/2} = \text{relativistic plasma frequency}\,, \\ &m \equiv \gamma m_0, \quad \nu \equiv 1 - \vec{n}\cdot\vec{\beta}, \quad \vec{b} \equiv \vec{B}/B\,, \\ &\delta_\perp\vec{E} \equiv n^{-2}\vec{n}\times(\delta\vec{E}\times\vec{n}) = \text{projection of } \delta\vec{E} \text{ normal to } \vec{n}\,, \\ &h_{ab} \equiv \delta_{ab} - n^{-2}(n_a - \beta_a)n_b \approx \text{projection three-tensor normal to } \vec{n} \text{ for } \beta \ll 1\,, \\ &\delta_b \equiv \beta_b + n_b(\nu - \gamma^{-2})(n^2-1)^{-1} = \text{a vector of magnitude } \beta \text{ for } |n-1| \text{ not too small}\,. \end{aligned} \tag{14}$$

Note that the Larmor frequency $\omega_L = eB/mc$ and the classical Alfvén velocity $c\beta_A$ are related to ω_p and ω_r by

$$\omega_L = \omega_p^2/\omega_r , \tag{15}$$

$$\beta_A = \omega_p/\omega_r = \omega_L/\omega_p = B(4\pi\mu c^2)^{-1/2} , \tag{16}$$

where $\mu \equiv m\rho/e$ is the mass density; i.e., the plasma frequency is the geometric mean of ω_L and ω_r. ω_L and ω_r have the same sign as the charge (according to our definition). Note further that the last (inhomogeneity) term in (13) is of order $i\beta\tilde{\omega}/\nu\beta_A$ because β varies on the length scale c/ω_r. It has negligible influence on all results discussed below; we therefore drop it from now on, but give explicit estimates in the Appendix.

Equation (13) has the form

$$0 = \tilde{\omega}^2(n^2\delta_{ab} - n_a n_b - \tilde{\epsilon}_{ab})\,\delta E^b , \tag{17}$$

where $\tilde{\epsilon}_{ab}(\omega, \vec{n})$ can be shown to be related to the (for $\beta = 0$ wave-vector independent!) dielectric tensor $\epsilon_{ab}(\omega)$ of the plasma by

$$x_a^{\cdot c}(n^2\delta_{cb} - n_c n_b - \tilde{\epsilon}_{cb}) = (n^2\delta_{ab} - n_a n_b - \epsilon_{ab}) \tag{18a}$$

with, for $\beta = 0$:

$$x_{ab} = b_a b_b + (1 - \breve{\omega}^{-2})^{-1}({}_\perp\delta_{ab} + i\breve{\omega}^{-1}\epsilon_{abc}b^c) ,$$
$${}_\perp\delta_{ab} \equiv \delta_{ab} - b_a b_b , \quad \breve{\omega} \equiv \omega/\omega_L . \tag{18b}$$

ϵ_{ab} is Hermitian, and has the eigenvalues mentioned in the Introduction.

The *dispersion relation* is obtained by equating to zero the determinant of $\{\cdots\}$ in (13). Either by straightforward calculation, or by first Lorentz transforming to the local rest system of the plasma we obtain (ordered with respect to magnetic, and nonmagnetic effects):

$$\hat{\omega}^2(n^2-1)[n_\parallel^2(1-\beta_\perp^2) - \nu_\perp^2 - \tilde{\omega}^2(n^2-1)\nu^2\gamma^2(1-\beta_\perp^2)] = \nu^2\{1 + \tilde{\omega}^2[2(n^2-1) - \nu^2\gamma^2] + \tilde{\omega}^4(n^2-1)(n^2-1-2\nu^2\gamma^2) - \tilde{\omega}^6(n^2-1)^2\nu^2\gamma^2\} \tag{19}$$

with $\parallel$, $\perp$ referring to the magnetic field $\vec{B}$, $\nu_\perp \equiv 1 - \vec{n}\cdot\vec{\beta}_\perp$. The result (19) can be written in a manifestly four-covariant form (after multiplication by $\omega^2\gamma^2$):

$$(\tilde{k}^2/4\pi\mu_0 c^2)[(k^{\alpha *}F_{\alpha\beta}u^\beta)^2 - \tfrac{1}{2}F_{\alpha\beta}F^{\alpha\beta}(k\cdot u)^2(1+\tilde{k}^2)] = (k\cdot u)^2[1 - (\tilde{k}\cdot u)^2](1+\tilde{k}^2)^2 , \tag{20}$$

where

$$k^\alpha \equiv (\vec{k}, (\omega/c)), \quad u^\alpha \equiv c\gamma(\vec{\beta}, 1); \quad {}^*F_{\alpha\beta} \equiv \tfrac{1}{2}\epsilon_{\alpha\beta\gamma\delta}F^{\gamma\delta};$$
$$k\cdot u \equiv k^\alpha u_\alpha, \quad k^2 \equiv k^\alpha k_\alpha; \quad \tilde{k}^\alpha \equiv \omega_p^{-1}k^\alpha; \quad \mathrm{sgn}(g_{\alpha\beta}) = (+++,-) . \tag{21}$$

We prefer the three-space form (19), and order it with respect to powers of ω:

$$x^3 - x^2\{\beta_A^2(1-\beta_\perp^2) - (n^2-1)^{-1}2\nu^2 + \gamma^{-2}\} + x(n^2-1)^{-1}\gamma^{-2}\{\beta_A^2[n_\parallel^2(1-\beta_\perp^2) - \nu_\perp^2] + (n^2-1)^{-1}\nu^4\gamma^2 - 2\nu^2\} - (n^2-1)^{-2}\nu^4\gamma^{-2} = 0 , \tag{22}$$

where $x \equiv (\nu\tilde{\omega})^2$. β_A^2 is the ratio between magnetic and plasma energy density, which should be large for stationary charge separation:

$$\beta_A^2 = B^2/4\pi\mu c^2 = eB/m_0 c\omega_r\gamma = 10^{18}B_{12}m_{-27}^{-1}\omega_{r,1}^{-1}\gamma_0^{-1} \gg 1 \tag{23}$$

holds for typical[3] values of a pulsar magnetosphere near the stellar surface (at least in a negative, corotating charge sector). We therefore assume $\beta_A^2 \gg 1$ in what follows.

The dispersion diagram $\omega_j(\vec{k})$ will be constructed as follows: Unless $|n^2 - 1| \lesssim \beta_A^{-2}$, the third-degree polynomial (22) is well approximated by dropping all terms in the curly brackets except the first ones. This truncated polynominal has two large coefficients; its zeros x_j are approximately obtained by equating consecutive terms; they give

$$\nu\tilde{\omega}_j \approx \pm\left\{\beta_A(1-\beta_\perp^2)^{1/2}, \left(\frac{1-\beta^2}{1-\beta_\perp^2}\right)^{1/2}\left(\frac{n_\parallel^2(1-\beta_\perp^2) - \nu_\perp^2}{n^2-1}\right)^{1/2}, \frac{\beta_A^{-1}\nu^2}{\{(n^2-1)[n_\parallel^2(1-\beta_\perp^2) - \nu_\perp^2]\}^{1/2}}\right\}. \tag{24}$$

For $|n^2 - 1| \ll \beta_A^{-2}$, one finds analogously

$$\tilde{\omega}_j \approx \pm\left\{\left(\frac{1}{1-n^2}\right)^{1/2}, \left(\frac{1}{1-n^2}\right)^{1/2}, (\nu\gamma)^{-1}\right\}. \tag{25}$$

Note that all roots x_j of (22) turn out to be real, though only positive x_j lead to dispersion branches (real $\vec{k}$, arbitrary ω). In the n interval defined by

$$n > 1, \quad n_\parallel^2(1-\beta_\perp^2) - \nu_\perp^2 < 0$$

there is only one positive x_j (i.e., only one dispersion branch).

Figure 1 gives a dispersion diagram $\omega_j(k)$ for fixed angles. Such a diagram is determined by the five parameters $\beta_A, \beta, \beta_\perp, |\cos(\vec{n},\vec{B})|, |\cos(\vec{n},\vec{\beta})|$, one of which sets the scale. For $\beta \ll 1$, only two of them determine the shape of the branches, which do not change vigorously when the parameters vary. The most severe differences are obtained when the straight line $n_\parallel^2(1-\beta_\perp^2)=\nu_\perp^2$ "moves" towards $n=1$, or approaches $\nu=0$ (corresponding to wave normals parallel to $\vec{B}$, and normal to $\vec{B}$ respectively). An exceptional case is $\beta=0$. The diagram is obtained by starting with the limiting cases $n\ll 1$ and $n \gg 1$, and using (25) for the immediate neighborhood of $n=1$ at $\tilde{\omega} \gtrsim 1$. Special attention is needed for the possibility $\nu \to 0$ for which opposite signs of $\cos(\vec{n},\vec{\beta})$ give markedly different branches drawn as broken and dotted lines, respectively. Note that for fixed n, there are one or three branches.

There are no complex branches ($\vec{k}$ = real, ω = complex), as expected on physical grounds. This can be proved by noting that our diagram is complete: A general initial disturbance $[\delta\vec{E}(\vec{x}), \delta\vec{B}(\vec{x}), \delta\vec{\beta}(\vec{x})]$ at $t=0$ can be Fourier transformed, and then expanded with respect to the proper modes $[\delta_j\vec{E}(\vec{k}), \delta_j\vec{B}(\vec{k}), \delta_j\vec{\beta}(\vec{k})]$ to be obtained (uniquely up to a common factor) for the respective dispersion branches (numbered by j):

$$\delta\vec{E}(\vec{k}) = \sum_j a_j(\vec{k})\delta_j\vec{E}(\vec{k}),$$
$$\delta\vec{B}(\vec{k}) = \sum_j a_j(\vec{k})\delta_j\vec{B}(\vec{k}), \tag{26}$$
$$\delta\vec{\beta}(\vec{k}) = \sum_j a_j(\vec{k})\delta_j\vec{\beta}(\vec{k}).$$

There have to be as many free amplitudes $a_j(\vec{k})$ as independent equations in the system (26); the latter number equals 8 (in our case) as there is just one *primary constraint* $\vec{k}\cdot\delta\vec{B}(\vec{k})=0$. The reader confirms that for each k, there are exactly 8 branches $\pm\omega_j(k)$.

The diagram shows that there are three breakoffs (cutoffs) at the frequencies

$$\omega_j = \{\omega_L(1-\beta_\perp^2)^{1/2},\ \omega_p[(1-\beta^2)/(1-\beta_\perp^2)]^{1/2},\ \omega_r\}, \tag{27}$$

no resonances, and three falling branches leading to three static modes at the wave numbers k_j given by

$$ck_j = \left[\omega_L \frac{(1-\beta_\perp^2)^{1/2}}{\beta\,|\cos(\vec{n},\vec{\beta})|},\ \omega_p\left(\frac{1-\beta^2}{1-\beta_\perp^2}\right)^{1/2} \frac{[\cos^2(\vec{n},\vec{B})(1-\beta_\perp^2)-\beta_\perp^2\cos^2(\vec{n},\vec{\beta}_\perp)]^{1/2}}{\beta\,|\cos(\vec{n},\vec{\beta})|},\right.$$
$$\left.\omega_r \frac{\beta\,|\cos(\vec{n},\vec{\beta})|}{[\cos^2(\vec{n},\vec{B})(1-\beta_\perp^2)-\beta_\perp^2\cos^2(\vec{n},\vec{\beta}_\perp)]^{1/2}}\right]. \tag{28}$$

These static modes correspond to (phasewise) counter moving disturbances at exactly the (negative) rotation speed, at wavelengths roughly of order $2\pi c(\omega_L^{-1}, \omega_p^{-1}, \omega_r^{-1})$. We also observe that electromagnetic waves with frequencies as low as the rotation frequency can be propagated.

The group velocity of a perturbation is given by

$$\vec{V}_{gr} = \nabla_{\vec{k}}\omega \tag{29}$$

as follows from the Fourier representation of an arbitrary space-time function $f(\vec{x},t)$:

$$f(\vec{x},t) = \int d^3k f(\vec{k}) e^{i(\vec{k}\cdot\vec{x}-\omega t)}$$
$$\approx e^{i(\vec{k}_0\cdot\vec{x}-\omega_0 t)} \int d^3k f(\vec{k}) e^{i\delta\vec{k}(\vec{x}-\nabla_{\vec{k}}\omega\cdot t)}$$
$$\approx e^{i(\vec{k}_0\cdot\vec{x}-\omega_0 t)} g(\vec{x}-\nabla_{\vec{k}}\omega\cdot t). \tag{30}$$

It gives the velocity of energy propagation whenever the latter can be reasonably defined.

From Eqs. (24), (25), or (19) we find for $\vec{V}_{gr} = c\vec{\beta}_{gr}$, approximately for the (pieces of) branches numbered $1, 2, \ldots,$ the expressions of $\vec{\beta}_{gr}$ shown in Table I. The case distinction for branch 6 is necessary because the approximation in the first line gets bad when the denominator in f_2 vanishes; the case $n_\parallel \approx n$ was treated more carefully by starting from the exact relation (19) [rather than (24)], which is necessary if one wants to differentiate the result once more (in order to get the frequency-dependent signal delay). Note that an energy flow almost in the direction of the wave vector occurs only along the velocity-of-light branches 4, 5, and 6. Along the neighboring branches 8 and 9, the energy propagates in some direction in the $(\vec{n},\vec{B})$ plane.

For branches 6 and 8 respectively, we find for $n_\parallel \approx n$

$$\omega \approx ck\left(1-\frac{\beta_\perp^2}{4(1-\tilde{\omega}^2\nu^2\gamma^2)}\right) + \frac{\omega_r^2\nu^2}{4ck} \pm \frac{\omega_r\nu}{2}\left(1+\frac{(\omega_r\nu)^2}{8c^2k^2}\right), \tag{31}$$

whence

$$\omega\partial_\omega\beta_{gr} \approx \frac{1}{2}\left[\frac{\nu(1-3\vec{n}\cdot\vec{\beta})-\beta^2/2}{\tilde{\omega}^2} - 3\left(\frac{\nu\gamma\beta_\perp\tilde{\omega}}{1-\tilde{\omega}^2\nu^2\gamma^2}\right)^2\right]. \tag{32}$$

TABLE I. $\vec{\beta}_{gr}$ for various branches. An upper index "0" at a vector denotes unit norm, $(|\vec{n}^0|=1)$. For branches 5 and 6, "lin(⋯)" means a linear combination of the vectors in parentheses; the corresponding terms are lengthy, and of magnitude comparable to the small term in braces, so that their contribution to the norm is in general small.

Branch	$\vec{\beta}_{gr}$
1, 10, 13	$\vec{\beta}$
2	$\vec{b}\beta_{\parallel} + \vec{n}_{\perp}$
3	$-\vec{b}(\beta_{\parallel} - n_{\parallel}) + \vec{n}$
4	$\vec{n}^0$
5	$\vec{n}^0[1-\tilde{\omega}^{-2}f_1(\vec{n})] + \tilde{\omega}^{-2}\,\mathrm{lin}(\vec{\beta},\vec{\beta}_{\perp},\vec{b}),\quad f_1 \equiv \left(\frac{1-\beta^2}{1-\beta_{\perp}^2}\right)^{1/2}\frac{\nu_{\perp}^2 - n_{\parallel}^2(1-\beta_{\perp}^2)}{2\nu^2}$
6	$\vec{n}^0[1-\hat{\omega}^{-2}f_2(\vec{n})] + \hat{\omega}^{-2}\,\mathrm{lin}(\vec{\beta},\vec{\beta}_{\perp},\vec{b}),\quad f_2 \equiv \frac{\nu^2}{2[\nu_{\perp}^2 - n_{\parallel}^2(1-\beta_{\perp}^2)]},$ for $\|[\cdots]\| \not< 1$ $\vec{n}^0\left[1-\frac{\beta_{\perp}^2}{4(1-[\tilde{\omega}\nu\gamma]^2)} + \frac{\vec{n}\cdot\vec{\beta}}{2\hat{\omega}} - \frac{\nu^2}{4\hat{\omega}^2}\right] - \vec{\beta}\,\frac{1}{2\hat{\omega}},$ for $n_{\parallel} \approx n$
8	$\vec{\beta}_{\perp} \pm \vec{b}(1-\beta_{\perp}^2)^{1/2} \pm n_{\parallel}(\gamma\tilde{\omega})^2(1-\beta_{\perp}^2)^{1/2}[(1-\beta_{\perp}^2)^{1/2} \mp \beta_{\parallel}]^2$ $\times\{\vec{\beta}_{\perp}[\vec{n}\cdot\vec{\beta}_{\perp} \pm n_{\parallel}(1-\beta_{\perp}^2)^{1/2}] \pm \vec{b}[(1-\beta_{\perp}^2)^{1/2}(\vec{n}\cdot\vec{\beta}_{\perp} \pm n_{\parallel}(1-\beta_{\perp}^2)^{1/2}) \mp (n^2-1)/2n_{\parallel}] - \vec{n}\}$
9	$\vec{\beta} + \vec{b}n_{\parallel}^{-1}[1-1/2n_{\parallel}^2] + \vec{n}^0 n^{-1}[1-3/2n_{\parallel}^2]$ for $\beta \ll \lvert n_{\parallel}\rvert^{-1} \ll 1$
11	$\vec{\beta} + [\vec{b} - (\vec{n}-\vec{\beta})n_{\parallel}/(n^2-1)]\,\nu n_{\parallel}$
12	$\vec{\beta} + [\vec{b} - (\vec{n}-\vec{\beta})n_{\parallel}/(n^2-1)]\,\nu n_{\parallel}^{-1}$
14	$\vec{\beta} + [\vec{b} - \vec{n}^0 n_{\parallel} n^{-1}]\,\nu n_{\parallel}^{-1}[1-n^2\beta_{\perp}^2 n_{\parallel}^{-2}]^{-1}$
15	$\vec{\beta} + [\vec{b} + \vec{n}^0(n_{\parallel}^2 - 2n^2\beta_{\perp}^2)/n n_{\parallel}]\,\nu n_{\parallel}^{-1}[1-n^2\beta_{\perp}^2/n_{\parallel}^2]^{-1}$

III. POLARIZATION OF PROPER MODES

So far we know the phase and group velocity of all proper modes. An understanding of their properties needs a calculation of their amplitudes $\delta_j\vec{E}(\vec{k}), \delta_j\vec{B}(\vec{k}), \delta_j\vec{\beta}(\vec{k})$ which follow from Eqs. (10), (11), and (13) up to a common (scalar) factor. More precisely, once we know $\delta_j\vec{E}$, we get $\delta_j\vec{B}$ from (10) (by left exterior multiplication with $\vec{n}$), and $\delta_j\vec{\beta}$ from (11) (in a transparent way). We therefore concentrate on solving (13), which will have to be done separately for each branch because in each case, different terms in the matrix in (13) are dominating.

For economy's sake we content ourselves with the special case of nonrelativistic plasma velocities: $\beta \ll 1$, but include correction terms of order β. In each case, the matrix in (13) is simplified first by row multiplication with factors, and then by linear combination of rows. We express all three-tensors in the following orthonormal triad: $[\vec{n} - \vec{b}(\vec{b}\cdot\vec{n}), \vec{b}\times\vec{n}, \vec{b}]^0$, i.e., in orthonormal components in which $\vec{B}$ points in the three direction, and $\vec{n}$ is normal to the two direction. The matrix in (13) then reads, with $\check{n}_a \equiv n_a - \beta_a$, and with the inhomogeneity term dropped:

$$\begin{bmatrix} 1 - i(\hat{\omega}/\nu)n_1\beta_2 + \tilde{\omega}^2(n_3^2 - \nu_{\perp}) + n_1\beta_1/\nu & -(i\hat{\omega}/\nu)(n^2-1) + n_1\beta_2/\nu & -(i\hat{\omega}/\nu)n_3\beta_2 - \tilde{\omega}^2\check{n}_1 n_3 + n_1\beta_3/\nu \\ i(\hat{\omega}/\nu)(n_3^2 - \nu_{\perp}) + \tilde{\omega}^2 n_1\beta_2 & 1 + \tilde{\omega}^2(n^2-1) & -(i\hat{\omega}/\nu)\check{n}_1 n_3 + \tilde{\omega}^2 n_3\beta_2 \\ -\tilde{\omega}^2\check{n}_3 n_1 + n_3\beta_1/\nu & n_3\beta_2/\nu & 1 + \tilde{\omega}^2(n_1^2 - 1 + n_3\beta_3) + n_3\beta_3/\nu \end{bmatrix}. \tag{33}$$

From here we find the (arbitrarily normalized) approximations for the electric-field amplitudes $\delta\vec{E}(\vec{k})$ (see Table II). For branches 7 and 9, $\delta\vec{E}$ was not evaluated within order β, (hence the sign ~). Note that $\delta\vec{E} \parallel \vec{n}$ holds approximately for branches 7, 10, 11, 13, 14, and 15, while $\delta\vec{E} \perp \vec{n}$ holds for one of the two modes of branches 4 and 5, and $\delta\vec{E} \parallel \vec{B}$ holds for branch 2. Circular proper modes normal to $\vec{B}$ occur for branches 1 and 3, and approximately for branch 9. For branch 8 and $n_1 \to 0$, $\delta\vec{E}$ approaches the second transverse mode (1, 0, 0).

TABLE II. $\delta\vec{E}$ for various branches. The (approximate) dispersion relation is given whenever it was used to simplify matrix (33).

Branch	δE_1	δE_2	δE_3	Dispersion relation
1	1	$-i$	0	
2	0	0	1	
3	1	i	0	
4	$\{0$	1	0	$n^2 \approx 1$
	n_3	0	$-n_1$	
5	$\{n_3$	0	$-n_1$	$n^2 \approx 1$
	0	1	0	
6	0	1	0	
7~	n_1	0	n_3	
8	$\check{n}_1$	$-\beta_2$	$-n_1\beta_1 n_3/\lvert n_3\rvert$	$n_3^2(1-\beta_\perp^2)=\nu_\perp^2$
9~	$i\hat{\omega}(n^2-1)$	1	0	$\hat{\omega}^2(n^2-1)(n_\parallel^2-1)=0$
10, 11	$\check{n}_1$	$-\beta_2$	$\check{n}_3$	$\nu \approx 0$
12	$n_1\beta_2$	-1	$n_3\beta_2$	$\nu \approx 0$
13, 14, 15	n_1	0	n_3.	

In a pulsar magnetosphere, ω_p^2 is believed[3] to be of order

$$\omega_p^2 \approx eB\omega_r/mc \approx (10^{21}\ \mathrm{sec}^{-1}) B_{12}\omega_{r,1} m_{-27}^{-1},$$

so that for typical radio pulses (in the range $\omega \approx 10^8$–$10^{11}\ \mathrm{sec}^{-1}$), $\tilde{\omega}$ ranges from fractions of a percent to unity or more (depending on the strength of the magnetic field, the type of the charged particles, and their average γ factor). For such regimes branches 6, 8, and 5 are of special interest. In particular for $\vec{n} \to \vec{b}$, branch 8 approaches branch 6, and the two polarization states (with $\delta\vec{E} \perp \vec{B}$) can interfere to yield Faraday rotation of the plane of polarization (spanned by $\vec{n}$ and $\delta\vec{E}$). The difference ψ in phase increment (for the 2 modes with wave vectors $\vec{k}_j$) along a path of line element $d\vec{x}$ is described by

$$\psi \equiv \int_{\mathrm{path}} d\vec{x}\,(\vec{k}_1 - \vec{k}_2)\,. \tag{34}$$

For $\vec{k}_j \parallel \vec{B}$ we get from Eq. (31):

$$\frac{ck\pm}{\omega} \approx \left[1 \mp \frac{\nu}{2\hat{\omega}}\left(1+\frac{\nu^2}{8\hat{\omega}^2}\right)\right]\left(1-\frac{\beta_\perp^2}{4(1-\tilde{\omega}^2\nu^2\gamma^2)}\right)^{-1}, \tag{35}$$

whence

$$\psi \approx \int \frac{dx}{c}\,\omega_r \nu\left(1+\frac{\beta_\perp^2}{4(1-\tilde{\omega}^2\nu^2\gamma^2)}+\frac{\nu^2}{8\hat{\omega}^2}\right), \tag{36}$$

and

$$\partial_\omega \psi \approx \int \frac{dx}{c}\,\frac{\nu^2}{2\hat{\omega}}\left[\left(\frac{\gamma\beta_1\tilde{\omega}}{1-\tilde{\omega}^2\nu^2\gamma^2}\right)^2(1-2\vec{n}\cdot\vec{\beta}) - \frac{1+\vec{n}\cdot\vec{\beta}}{2\hat{\omega}^2}\right] \tag{37}$$

for the differential change of ψ with ω; which is smaller by a factor $\ll 10^{-10}$ than the differential Faraday rotation[3] caused by the interstellar medium.

As an outlook, let us mention that to follow the pulsar radiation through the magnetosphere we must—as a next step—take into account the curvature of the magnetic field lines, and also possibly the variation of field strength. At the velocity-of-light cylinder and beyond, the plasma is likely to become more and more neutral and sheared, and the propagation of low-frequency waves through this part of the magnetosphere poses additional difficulties.[4]

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APPENDIX

We promised to estimate the effect of the last (inhomogeneity) term in Eq. (13) on the dispersion relations. For $1 < \hat{\omega} < \beta_A^2$, the leading correction to the determinant (22) is obtained by keeping the last two terms in (13); it adds to the right-hand side of (22) the small imaginary term

$$ix^{3/2}\beta_A\delta, \quad \delta \equiv (c/\gamma\omega_r)(\vec{b}\cdot\nabla)\gamma\beta_\parallel, \tag{A1}$$

where δ is of order β. This leading correction, and all the other imaginary ones vanish for $\vec{B}\cdot\vec{\beta}=0$.

Writing $x \equiv x_{\mathrm{hom}} + \Delta x$ in (22), we find for the leading first-order corrections to the dispersion branches (24), due to (A1):

$$\frac{\Delta\omega}{\omega} = \frac{1}{2}\frac{\Delta x}{x} \approx i\frac{\delta}{2\beta_A} \times \begin{cases} \dfrac{1}{\beta_A(1-\beta_\perp^2)^{3/2}} \\ -\dfrac{1}{\nu\tilde{\omega}(1-\beta_\perp^2)} \\ \dfrac{\nu^2\gamma^2(n^2-1)^{1/2}}{\beta_A[n_\parallel^2(1-\beta_\perp^2)-\nu_\perp^2]^{3/2}} \end{cases}, \tag{A2}$$

or

$$\Delta\tilde{\omega} \approx i\frac{\delta}{2\beta_A} \times \begin{cases} \dfrac{1}{\nu(1-\beta_\perp^2)} \\ -\dfrac{1}{\nu(1-\beta_\perp^2)} \\ \dfrac{\nu^3\gamma^2}{\beta_A^2[n_\parallel^2(1-\beta_\perp^2)-\nu_\perp^2]^2} \end{cases}. \tag{A3}$$

$\Delta\omega/\omega$ and $\Delta\tilde{\omega}$ are imaginary, hence correspond to damping, or antidamping. They are, however, small, of order $\beta_\parallel\beta_A^{-b}$, where b varies between 1 and 3.

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