PHYSICAL REVIEW A VOLUME 12, NUMBER 5 NOVEMBER 1975

Source-field approach to radiative corrections and semiclassical radiation theory

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(Received 26 March 1975)

A relativistic source-field approach in radiation theory is examined and is shown to predict the radiative corrections which normally require second quantization of electromagnetic fields. The connection between this approach and conventional quantum electrodynamics is pointed out and discussed. For most practical situations, the source-field effects are negligible, and semiclassical radiation phenomena are described by the relativistic wave equation which we derive from the total Hamiltonian. This wave equation is then transformed to an exact multipolar form which is suitable for application to many radiation problems. Its nonrelativistic limit is obtained and interaction terms are interpreted.

I. INTRODUCTION

In quantum electrodynamics, part of the energy of a system containing charged particles is understood to arise from its interaction with the quantized radiation field. This is commonly referred to as radiative corrections. The most obvious examples are the atomic Lamb shift and the anomalous magnetic moment of the electron.

The quantum field theory of the interaction of radiation with matter, where both matter and radiation are second quantized, describes these corrections satisfactorily, but the theory is beset by infinities which may suggest that its basic foundations are doubtful. A number of physicists believe that electrodynamics is still incomplete and that its difficulties are of such a profound nature that they can be removed only by a drastic change in the structure of the theory. Despite many attempts to surmount these difficulties, no one has yet succeeded in satisfactorily modifying the theory without abandoning principles like Lorentz invariance, the probabilistic interpretation of state vectors, and the local nature of the interaction.

In recent years, Jaynes and co-workers[1] have emphasized that a great deal of insight into the difficulties inherent in the structure of quantum electrodynamics may be gained by working on a semiclassical radiation theory where radiative corrections are based on the intuitively clearer classical concept of radiation reaction.

In contrast to quantum electrodynamics, the sources in this semiclassical theory are described by ordinary (first-quantized) Schrödinger quantum mechanics and the electromagnetic fields are taken as classical c-number fields. In this nonrelativistic approach vacuum fluctuations are nonexistent and radiative phenomena are attributed to radiation reaction. Quantum electrodynamics attributes these effects to the interactions with the vacuum radiation field. The concept of the vacuum radiation field has been demonstrated in an argument by Welton,[2] who succeeded in obtaining the nonrelativistic formula for the Lamb shift but failed to account for the correct sign of the anomalous magnetic moment of the electron. The Casimir attraction[3] can be taken as another example supporting the belief that this concept is not without foundation.

Ackerhalt *et al.*[4] have suggested that the conventional explanation of spontaneous decay in terms of the vacuum field need not be adopted and that it is the atom's own radiation field that modifies the atom's characteristics in such a way as to produce a finite decay rate and a shift of the unperturbed transition frequency.

Recent work[5-7] seems to concentrate on this so-called source-field approach to radiative corrections, still working within the framework of nonrelativistic semiclassical radiation theory. It has been suspected[8] that such an approach should lead to calculations of the correct atomic level shifts, but that it is not completely straightforward.[5,6] Its advantage may be in providing an alternative description of radiative corrections which is conceptually closer to classical ideas and which may shed light as to where the shortcomings of quantum electrodynamics originate.

In this paper we examine a relativistic theory in which the source-field concept plays the central role in the computation of radiative corrections. A relativistic starting point is essential, because some of the radiative corrections, in particular, the anomalous magnetic moment of the electron, have failed to be accounted for by nonrelativistic approaches.[9] In Sec. II we construct the most general Hamiltonian for a system of charged particles in interaction with classical, static or time-dependent, external electromagnetic fields. Self-interactions are included by invoking classical arguments. The sources give rise to an electromagnetic field which acts back on these sources, giving rise to a self-interaction Hamiltonian from which radiative corrections can be calculated. In the total Hamiltonian, *the only dynamical variable is the spinor field.* Standard methods are then used to obtain the only field equation in the formalism. It is a nonlinear Dirac equation containing in a complicated fashion all the information in the total Hamiltonian. To establish contact with conventional quantum electrodynamics, we consider in detail the simple case of an unbound electron and calculate its self-energy as well as the famous $\alpha/2\pi$ radiative correction to its magnetic moment. This is described in Sec. III. In Sec. IV we take the point of view that the Dirac equation, ignoring source-field effects, is adequate for the description of most of the phenomena in which strong external fields are involved. This equation is then canonically transformed to an exact "multipolar" form without need to second quantize the external fields.[10] In this form the theory is suitable for many applications. In particular, it is useful for the description of nonlinear phenomena and other intense-field effects, as exhibited in laser interactions with matter. Interactions are further given familiar physical significance by obtaining the nonrelativistic limit of the transformed wave equation. This is done by following a Foldy-Wouthuysen procedure.

II. TOTAL HAMILTONIAN AND FIELD EQUATION

Our first goal is to construct the most general Hamiltonian which is capable of describing a wide spectrum of processes for a system of Dirac particles (unbound or bound to an atom). This system is taken to be in interaction with itself and with externally applied, static or time-dependent, *classical* electromagnetic fields.

In the absence of external fields and ignoring, for the time being, self-interactions, the Hamiltonian for the system can be written as[11]

$$H_0 = \int d^3x\, \overline{\psi}\,(-i\vec{\gamma}\cdot\vec{\nabla} + m + \gamma^0 V)\psi \ , \tag{2.1}$$

where V represents the Coulomb potential responsible for binding the system to a nucleus. The latter is taken to be infinitely massive and centered at the space origin of some inertial frame. If we set $V=0$, Eq. (2.1) reduces to the Hamiltonian describing a system of free Dirac particles.

In the presence of externally applied fields the Hamiltonian is given by

$$H_1 = \int d^3x\, \overline{\psi}\,(-i\vec{\gamma}\cdot\vec{\nabla} - e\vec{\gamma}\cdot\vec{A} + m + \gamma^0 V)\psi \ , \tag{2.2}$$

where the external fields are represented by the *classical* vector potential $\vec{A}$; in the Coulomb gauge

$$\vec{\nabla}\cdot\vec{A} = 0 \ . \tag{2.3}$$

For application to various physical situations, this vector potential may be taken to represent static electric or magnetic fields or an arbitrary time-dependent "laser" field. A combination of these situations may also be considered.

The inclusion of self-interactions in the total Hamiltonian can now proceed according to the following steps.

Associated with the source operators

$$J^\mu = e\overline{\psi}\gamma^\mu\psi \tag{2.4}$$

is a four-vector potential a^μ. This is called the source field and it satisfies the equations

$$\Box a^\mu = -J^\mu \ , \tag{2.5}$$

$$\partial_\mu a^\mu = 0 \ . \tag{2.6}$$

The solution of Eq. (2.5) is well known. We can write

$$a^\mu(x) = \int \Theta(x,x')J^\mu(x')\,dx' \ . \tag{2.7}$$

The function $\Theta(x,x')$ is usually taken to be the retarded Green's function.[1,8] However, Eq. (2.5) admits another solution involving the advanced Green's function, so that a proper combination of the two solutions is also permissible. Physical arguments and correspondence with quantum electrodynamics should give us clues for the correct choice of the function $\Theta(x,x')$.

The self-interactions contribution to the total Hamiltonian is then given by the minimal coupling of the source field a^μ to the Dirac sources which produce it. Thus we write

$$H_{\rm self} = \int J_\mu(x)a^\mu(x)\,d^3x \ . \tag{2.8}$$

Substituting for a_μ from (2.7), we obtain

$$H_{\rm self} = \tfrac{1}{2}\int d^3x \int dx'\, J_\mu(x)J^\mu(x')\Theta(x,x') \ . \tag{2.9}$$

A factor $\frac{1}{2}$ has been introduced in Eq. (2.9) because, by virtue of Eq. (2.4), $H_{\rm self}$ involves noncommuting fields and the order of the double integral is ambiguous. The inclusion of the factor $\frac{1}{2}$ takes care of this ambiguity.

The required total semiclassical Hamiltonian of the system is then obtained by adding (2.9) to (2.2). The result is

$$H = \int d^3x\, \overline{\psi}\,(-i\vec{\gamma}\cdot\vec{\nabla} - e\vec{\gamma}\cdot\vec{A} + m + \gamma^0 V)\psi + \tfrac{1}{2}\int d^3x \int dx'\,\Theta(x,x')J_\mu(x)J^\mu(x') . \tag{2.10}$$

This can be written as

$$H = H_0 + H_{\text{self}} + H_{\text{int}} \ , \tag{2.11}$$

where

$$H_{\text{int}} = -e \int \bar{\psi} \vec{\gamma} \cdot \vec{A} \psi \, d^3x \ . \tag{2.12}$$

From the Hamiltonian (2.10) one can describe a wide range of phenomena including self-interaction effects.

Before we proceed to obtain the field equation we remark at this stage that this total Hamiltonian can be extended to the case of external classical fields interacting with material media (many-atom systems). This can be done by associating with each atom in the medium an electron field which is kinematically independent of fields associated analogously with other atoms. Such a procedure will correspond in many-body formalism to imposing the Pauli principle for electrons on each atom, but ignoring exchange effects in interatomic interactions. This procedure can be arranged so as to simply introduce a summation over the atoms in the basic relations of this paper.

The field equation associated with the Hamiltonian (2.10) is obtained by means of the Heisenberg equation of motion,

$$\begin{aligned} \dot{\psi} &= i[H, \psi] \\ &= i[H_1, \psi] + i[H_{\text{self}}, \psi] \ . \end{aligned} \tag{2.13}$$

The computation of the first commutator is standard. We need to use only the equal-time anticommutation relations

$$\{ \psi(x), \bar{\psi}(x') \}_{x_0 = x_0'} = \gamma^0 \delta(\vec{x} - \vec{x}') \ . \tag{2.14}$$

We obtain

$$i[H_1, \psi(x)] = -\gamma^0 (\vec{\gamma} \cdot \vec{\nabla} - e\vec{\gamma} \cdot \vec{A} + im + i\gamma^0 V)\psi(x) \ . \tag{2.15}$$

For the second commutator in (2.13) we shall also use the rules

$$\{\psi(x), \bar{\psi}(x')\} = -iS_F(x, x') \ , \tag{2.16}$$

where $S_F(x, x')$ is the complete Feynman electron-propagator function for the case described by the Hamiltonian (2.10). For a free Dirac field with $V = 0 = \vec{A}$, we have

$$S_F(x, x') \to S_F(x - x') = (i\not{\nabla} + m)\Delta_F(x - x', m) \ , \tag{2.17}$$

where $\Delta_F(x - x', m)$ is the usual invariant delta function.

Substituting from (2.4) and (2.9) we obtain

$$\begin{aligned} i[H_{\text{self}}, \psi(x)] = \tfrac{1}{2} i e^2 \int d^3x_1 \int dx_2 \Theta(x_1, x_2) \\ \times [\bar{\psi} \gamma_\mu \psi(\vec{x}_1, t) \bar{\psi} \gamma^\mu \psi(\vec{x}_2, t_2), \psi(x)] \ . \end{aligned} \tag{2.18}$$

The commutator in (2.18) can be evaluated using (2.14) and (2.16). The result can be written as[12]

$$i[H_{\text{self}}, \psi(x)] = -\gamma^0 (ie/2) [\not{a}(x) + \not{b}(x)] \ , \tag{2.19}$$

where a_μ is given by (2.7) while b_μ is defined by

$$\begin{aligned} b_\mu = -ie \int d^3x_1 \int dx_2 \Theta(x_1, x_2) \bar{\psi} \gamma_\mu \psi(\vec{x}_1, t) \gamma^0 \\ \times S_F(x, x_2) \gamma^0 \ . \end{aligned} \tag{2.20}$$

Adding (2.15) to (2.19) we get

$$\begin{aligned} \dot{\psi}(x) = -\gamma^0 [\vec{\gamma} \cdot \vec{\nabla} - ie\vec{\gamma} \cdot \vec{A} + im \\ + i\gamma^0 V + \tfrac{1}{2} ie(\not{a} + \not{b})] \psi(x) \ , \end{aligned} \tag{2.21}$$

which can finally be written as

$$(i\not{\nabla} - m - \gamma^0 V)\psi(x) = [e\vec{\gamma} \cdot \vec{A} + \tfrac{1}{2} e(\not{a} + \not{b})] \psi(x) \ . \tag{2.22}$$

Equation (2.22) is the equation of motion for the Dirac field in the presence of the external fields and taking account of self-interactions by means of the source field method. We note that, by virtue of (2.7) and (2.20), this equation is nonlinear. In Sec. III we approximately remove this nonlinearity for the simple case of a free electron by working only to the first order in self-interaction, and describe how radiative corrections, to this order, can be obtained.

III. SELF-INTERACTIONS TO FIRST ORDER: THE FREE ELECTRON

We shall approximate the total Hamiltonian (2.10) by taking account only of the term of order α responsible for radiative corrections. Using covariant perturbation theory we first define the S matrix:

$$S = T \exp\left(-i \int dt \, H_{\text{self}}\right) \ . \tag{3.1}$$

Retaining only the first-order term, we then have

$$\begin{aligned} S^{(1)} &= -i \int dt \, T(H_{\text{self}}) \\ &= -\tfrac{1}{2} i e^2 \int dx \int dx' \, \Theta(x, x') T[\bar{\psi} \gamma_\mu \psi(x) \bar{\psi} \gamma^\mu \psi(x')] \ . \end{aligned} \tag{3.2}$$

The expansion of (3.2) into normal products will give rise to four terms. The first term represents scattering processes of the Moller type. The second and the third terms involve single contractions of the fields. These terms are equal in magnitude, and they combine to give

$$S_R^{(1)} = -ie^2 \int dx \int dx' \,\overline{\psi}(x)\Theta(x,x')\gamma_\mu \times i S_F(x,x')\gamma^\mu \psi(x') . \quad (3.3)$$

This expression will be shown to be responsible for radiative corrections. The fourth term involves two contractions of the fields and will therefore cause no transitions.

Specializing now in the free-electron case and setting $V=0$, we can then transform Eq. (3.3) into a more convenient form. We begin by the Fourier decomposition of the fields,

$$\overline{\psi}(x') = \int dp'\, e^{ip'x'}\,\Phi(p'),$$
$$\psi(x) = \int dp\, e^{-ipx}\Phi(p), \quad (3.4)$$

and introduce new variables by

$$\xi = x' - x, \quad X = \tfrac{1}{2}(x'+x) ,$$
$$P = \tfrac{1}{2}(p'+p), \quad Q = p' - p . \quad (3.5)$$

Substituting in (3.3) we get

$$S_R^{(1)} = -i \int dp \int dp' \int dX\, e^{iQX}\Phi(p')\Big(ie^2 \int d\xi\, e^{iP\xi}\,\gamma_\mu S_F(X,\xi)\gamma^\mu\Theta(X,\xi)\Big)\Phi(p)$$
$$= -i \int dp \int dp' \int dX\, e^{iQX}\,\Phi(p')\Delta M^e\,\Phi(p) , \quad (3.6)$$

where

$$\Delta M^e = ie^2 \int d\xi\, e^{iP\xi}\gamma_\mu S_F(X,\xi)\gamma^\mu\Theta(X,\xi) . \quad (3.7)$$

Hence by virtue of (3.4) we obtain from (3.6)

$$S_R^{(1)} = -i \int dX\, \overline{\psi}(X)\Delta M^e \psi(X) . \quad (3.8)$$

Thus under this approximation the interaction Hamiltonian for the unbound Dirac field can be written as

$$H_{\text{self}} \sim \int d^3x\, \overline{\psi}\Delta M^e\psi , \quad (3.9)$$

and the total Hamiltonian is now

$$H \sim \int d^3x\, \overline{\psi}\,(-i\vec{\gamma}\cdot\vec{\nabla} - e\vec{\gamma}\cdot\vec{A} + m + \Delta M^e)\psi , \quad (3.10)$$

from which we obtain the field equation

$$(i\not\nabla - m - \Delta M^e)\psi \sim e\vec{\gamma}\cdot\vec{A}\psi . \quad (3.11)$$

Our next step is to show that ΔM^e contains the self-energy as well as radiative corrections to the magnetic moment of the electron.

To be able to account for the anomalous magnetic moment of the electron, we shall assume that the external potential represents a classical static magnetic field $\vec{B}$. This is defined by means of the relations

$$F_{\mu\nu} = \partial_\nu A_\mu - \partial_\mu A_\nu ,$$
$$A_\mu = (0, \vec{A}) = \tfrac{1}{2} F_{\mu\nu} X^\nu , \quad (3.12)$$
$$F_{\mu 0} = 0, \quad F_{ij} = \epsilon_{ijk} B_k .$$

The self-energy of the electron in this magnetic field is now given by

$$\Delta M^e = ie^2 \int d\xi\, e^{iP\xi}\,\gamma_\mu S_F^e(X,\xi)\gamma^\mu\Theta(X,\xi) , \quad (3.13)$$

where $S_F^e(X,\xi)$ is the electron propagator in the presence of the magnetic field. To order e this is explicitly given by[13]

$$S_F^e(X,\xi) = \frac{1}{(2\pi)^4} \int dq\, e^{-iq\xi} S(q,X) , \quad (3.14)$$

where

$$S(q,X) = \frac{\not q - e\not A + m}{q^2 - m^2} + \frac{2eqA(\not q + m)}{(q^2 - m^2)^2} - \frac{\frac{1}{4}e(2m\sigma F + \{\sigma F, \not q\}_+)}{(q^2 - m^2)^2} . \quad (3.15)$$

In Eq. (3.15) we have used the abbreviation

$$\sigma F = \sigma_{\mu\nu}F^{\mu\nu} = 2\vec{\sigma}\cdot\vec{B} . \quad (3.16)$$

At this stage we remark that in order to obtain quantum-electrodynamical results from this semiclassical theory, we shall have to recognize the function $\Theta(X,\xi)$ [originally encountered in solving the source-field equation (2.5)] as identical to the "photon" Feynman propagator. In this case $\Theta(X,\xi)$ is given by a proper combination of the advanced and retarded Green's functions.[14] Thus we write

$$\Theta(X,\xi) = D_F(X,\xi) = D_F(\xi) = \frac{1}{(2\pi)^4} \int ds \frac{e^{-is\xi}}{s^2 + i\epsilon} . \quad (3.17)$$

Equation (3.7) now gives

$$\Delta M^e = \frac{ie^2}{(2\pi)^8}\int d\xi \int dq \int ds\, e^{i(P-q-s)\xi}\frac{\gamma_\mu S_F^e(q,X)\gamma^\mu}{s^2+i\epsilon}, \tag{3.18}$$

where we have made use of Eq. (3.14) and (3.17).

In the absence of the external magnetic field

$$S_F^e(q,X)|_{A_\mu\equiv 0} = (\not{q}+m)/(q^2-m^2). \tag{3.19}$$

In this case Eq. (3.18) reduces to

$$\Delta M^e(A_\mu \equiv 0) = \frac{ie^2}{(2\pi)^8}\int d\xi \int dq \int ds\, \frac{e^{i(P-q-s)\xi}}{s^2+i\epsilon} \times \frac{\gamma_\mu(\not{q}+m)\gamma^\mu}{q^2-m^2}, \tag{3.20}$$

which yields

$$\Delta M^e(A_\mu \equiv 0) = \frac{ie^2}{(2\pi)^4}\int ds\, \frac{\gamma_\mu(\not{P}-\not{s}+m)\gamma^\mu}{[(P-s)^2-m^2+i\epsilon]} \times \frac{1}{(s^2+i\epsilon)}. \tag{3.21}$$

This is identical to the quantum-electrodynamic result for the self-energy of the electron obtained from conventional quantum electrodynamics.

We proceed now to calculate the lowest-order corrections of the anomalous magnetic moment of the electron from Eq. (3.18). On the substitution from Eq. (3.15), one can separate the ensuing expression into three parts:

$$\Delta M^e = \Delta M_1^e + \Delta M_2^e + \Delta M_3^e, \tag{3.22}$$

where

$$\Delta M_1^e = \frac{ie^2}{(2\pi)^4}\int ds\, \gamma_\mu \frac{\not{P}-\not{s}+m-e\not{A}}{(s^2+i\epsilon)[(P-s)^2-m^2]}\gamma^\mu, \tag{3.23}$$

$$\Delta M_2^e = \frac{ie^2}{(2\pi)^4}\int ds\, \gamma_\mu \frac{(\not{P}-\not{s})2e(P-s)A}{(s^2+i\epsilon)[(P-s)^2-m^2]^2}\gamma^\mu, \tag{3.24}$$

$$\Delta M_3^e = \frac{ie^2}{(2\pi)^4}\int ds\, (-\tfrac{1}{4}e)\gamma_\mu \times \frac{2m\sigma F + \{\sigma F, (\not{P}-\not{s})\}_+}{(s^2+i\epsilon)[(P-s)^2-m^2]^2}\gamma^\mu. \tag{3.25}$$

These terms can be dealt with individually using the standard Feynman tricks of combining denominators and shifting the origin of the s-integration, and by making extensive use of the Dirac algebra. The calculation is straightforward, but tedius.[13] Keeping terms up to an over-all order αe, we obtain eventually

$$\Delta M_1^e = \frac{ie^2}{(2\pi)^4}\int ds' \int_0^1 dy\left(\frac{2[m(1+y)+e\not{A}y]}{(s'^2-m^2y^2)^2} + \frac{2my(1-y^2)e\sigma F - 8y(1-y)[m(1+y)eAP]}{(s'^2-m^2y^2)^3}\right), \tag{3.26}$$

$$\Delta M_2^e = \frac{ie^2}{(2\pi)^4}\int ds' \int_0^1 dy\left(\frac{8eAPm(1-y^2)}{(s'^2-m^2y^2)^3} - \frac{8eyAs'\not{s}'}{(s'^2-m^2y^2)^3}\right), \tag{3.27}$$

$$\Delta M_3^e = \frac{ie^2}{(2\pi)^4}\int ds' (-\tfrac{1}{4}e)\int_0^1 \frac{8y(1-y)\sigma F}{(s'^2-m^2y^2)^3}\,dy. \tag{3.28}$$

Adding up these expressions we obtain

$$\Delta M^e = \frac{ie^2}{(2\pi)^4}\int ds' \int_0^1 dy\left(\frac{2m(1+y)}{(s'^2-m^2y^2)^2} + \frac{2mey^2(1-y)\sigma F}{(s'^2-m^2y^2)^3} + \frac{2e\not{A}y}{(s'^2-m^2y^2)^2} - \frac{8es'A\not{s}'}{(s'^2-m^2y^2)^3}\right). \tag{3.29}$$

Next we perform the s' integrations using standard integrals.[15] The last two terms give zero contribution and we finally obtain

$$\Delta M^e = \frac{\pi^2 e^2}{(2\pi)^4}\int_0^1 dy\left(\frac{2(1+y)}{y} - \frac{e}{my^2}\sigma F y^2(1-y)\right). \tag{3.30}$$

The first term in Eq. (3.30) originates from the expression (3.21) which gives the self-energy.

The second term in (3.30) depends on the external field and can be evaluated easily to give

$$\Delta M^e_{\text{anom}} = \frac{-\pi^2 e^2}{(2\pi)^4 m}\int_0^1 dy\,(1-y)\sigma F = \frac{-e^2}{16\pi^2 m}\vec{\sigma}\cdot\vec{B}, \tag{3.31}$$

which can be put in the form

$$\Delta M^e_{\text{anom}} = \frac{\alpha}{2\pi}\frac{e}{2m}(-\vec{\sigma}\cdot\vec{B}). \tag{3.32}$$

This is precisely the standard result obtained otherwise from conventional quantum electrodynamics.

We finally remark that, although we have only considered in detail the simple problem of the lowest-order radiative corrections for the unbound electron, in principle the source-field approach is capable of describing all radiative effects. Not only can it account for higher-order corrections for this case, but it can predict atomic radiative effects to all orders and under any physical situations, as may be dictated by the external classical fields. It is clear that by making use of equation (3.17), this source-field approach is equivalent to quantum electrodynamics. If a modification of the theory is contemplated, one of the first candidates for thorough investigation is the use of the various Green's functions in the solution of the source-field equation (2.5).[16]

It seems to us that, although previous source-field treatments are formally correct within the context of nonrelativistic quantum mechanics, they are unlikely to shed light as to where quantum electrodynamics fails. A relativistic theory, as we explained earlier, is more likely to do so.

Finally it is important to point out the difference between the source field discussed in this paper and that in a completely semiclassical theory. In a semiclassical theory all fields including the source field are c numbers, while our source field satisifies the operator equations (2.5) and is thus a q number.

IV. TRANSFORMATION OF THE THEORY

Having thus verified that radiative corrections and other self-interactions can be accounted for by using the source-field concept, we next concentrate on situations of practical significance. If we ignore self-interactions in Eq. (2.22), we can then write

$$i\,\frac{\partial\psi}{\partial t}=(\vec{\alpha}\cdot\vec{p}-e\vec{\alpha}\cdot\vec{A}+\beta m+V)\psi\ . \tag{4.1}$$

For most practical situations this wave equation can be taken as the starting point for the computations of external-field effects for a bound-electron system. In this case V can be assumed to represent the Coulomb field binding the electron to an atomic core.

Equation (4.1) can be put in another form which is suitable for applications to a wide range of phenomena, in particular atomic processes where intense fields at optical frequencies are involved. In ordinary quantum mechanics the sources are defined by the operators

$$\vec{J}(\vec{r})=e\vec{\alpha}\,\delta(\vec{r}-\vec{x}),\quad \rho(\vec{r})=e\delta(\vec{r}-\vec{x})\ . \tag{4.2}$$

Another representation of these sources is provided by the polarization operator $\vec{P}$ and the magnetization operator $\vec{M}$:

$$P_i(\vec{r})=e\int_0^1 d\lambda\, x_i\,\delta(\vec{r}-\lambda\vec{x})\ , \tag{4.3}$$

$$M_i(\vec{r})=e\epsilon_{ijk}\int_0^1 d\lambda\, x^j\alpha^k\delta(\vec{r}-\lambda\vec{x})\ . \tag{4.4}$$

These polarization sources are related to the Dirac sources by means of the relations[17]

$$\rho(\vec{r})=e\delta(\vec{r})-\vec{\nabla}\cdot\vec{P}(\vec{r})\ ,\qquad \vec{J}(\vec{r})=\dot{\vec{P}}(\vec{r})+\vec{\nabla}\times\vec{M}(\vec{r})\ . \tag{4.5}$$

The transformation of the theory is achieved by means of the operation $\Lambda=e^{-iS}$, where S is the generating function

$$S=\int \vec{P}\cdot\vec{A}\,d^3r\ . \tag{4.6}$$

Under this transformation we have

$$\psi\to\psi'=e^{-iS}\psi\ . \tag{4.7}$$

The transformed wave function will therefore satisfy the wave equation

$$i\,\frac{\partial\psi'}{\partial t}=\left(e^{-iS}he^{iS}+\frac{\partial S}{\partial t}\right)\psi'\ , \tag{4.8}$$

where

$$h=\vec{\alpha}\cdot(\vec{p}-e\vec{A})+\beta m+V\ . \tag{4.9}$$

In the Coulomb gauge for the external fields,

$$\vec{E}^{\perp}=-\partial\vec{A}/\partial t\,,\quad \vec{B}=\vec{\nabla}\times\vec{A}\,, \tag{4.10}$$

where $\vec{E}^{\perp}$ represents the transverse part of the external electric field vector.

Using Eq. (4.6) and the first equation of (4.10), we get

$$\frac{\partial S}{\partial t}=-\int \vec{P}\cdot\vec{E}^{\perp}\,d^3r\ . \tag{4.11}$$

The expression $e^{-iS}he^{iS}$ can be evaluated by making use of the operator identitv

$$e^{-iS}he^{iS}=h-i\,[S,h]+[(-i)^2/2\,!]\;[S,[S,h]]+\cdots\,, \tag{4.12}$$

together with the commutation relations

$$[x_i\,,p_j]=i\delta_{ij}\ . \tag{4.13}$$

The evaluation is straightforward. Only the first commutator in (4.12) requires evaluation, and all commutators of higher orders vanish. The result is exact and can be written as

$$e^{-iS}he^{iS}=\vec{\alpha}\cdot\vec{p}+\beta m+\gamma^0V-\int \vec{M}\cdot\vec{B}\,d^3r\ . \tag{4.14}$$

In arriving at Eq. (4.14), we have also made use of the distribution identities[18]

$$\delta(\vec{r}-\vec{x}) \equiv \delta(\vec{r}) - \frac{\partial}{\partial r^k}\int_0^1 d\lambda\, x^k \delta(\vec{r}-\lambda\vec{x}) \qquad (4.15)$$

$$\equiv \int_0^1 d\lambda \left(1-\lambda x^i \frac{\partial}{\partial r^i}\right)\delta(\vec{r}-\lambda\vec{x})\,. \qquad (4.16)$$

Equation (4.8) now reads

$$i\frac{\partial\psi'}{\partial t} = \left(\vec{\alpha}\cdot\vec{p}+\beta m+\gamma^0 V - \int \vec{P}\cdot\vec{E}^{\perp}\, d^3r - \int \vec{M}\cdot\vec{B}\, d^3r\right)\psi'\,. \qquad (4.17)$$

Defining an electromagnetic polarization tensor operator $\Pi_{\mu\nu}$ associated with the atomic system by

$$\Pi^{0j} = P^j\,,\quad \Pi^{ij} = \epsilon_{ijk} M_k\,, \qquad (4.18)$$

Eq. (4.17) now takes the form

$$i\frac{\partial\psi'}{\partial t} = \left(\vec{\alpha}\cdot\vec{p}+\beta m+\gamma^0 V + \tfrac{1}{2}\int \Pi_{\mu\nu} F^{\mu\nu} d^3r\right)\psi'\,, \qquad (4.19)$$

where

$$F^{0j} = E^{\perp j}\,,\quad F^{ij} = \epsilon_{ijk} B^k\,. \qquad (4.20)$$

The transformed wave equation is equivalent to the original wave equation. While the Dirac sources in the latter couple to the potentials, the coupling to the external fields in Eq. (4.19) is given in terms of the exact electric and magnetic polarization sources. These couple directly to the gauge-invariant field intensities. The theory is therefore manifestly gauge invariant.

In a previous paper,[17] a transformation, similar in some respects to the one we have just considered, has been applied to the Dirac Hamiltonian and results were obtained only by assuming that the electromagnetic fields are second quantized. In this paper note the absence of the term $\int |P^{\perp}|^2 d^3r$. This term has been shown[19] to be important for self-energies. Its presence was required for the correct computation of the transverse self-energy of the bound electron. Its absence in this semiclassical treatment is therefore not surprising since we have shown that all radiative corrections stem from the source-field terms which we have not considered in this section.

A transformation similar to Eq. (4.6) was also suggested by Reiss[20] in an attempt to devise a nonperturbative approach to intense-field radiation problems. Reiss's transformation was, however, confined to the electric dipole and the nonrelativistic approximations. The results of this section constitute a relativistic extension of Reiss's treatments and without resort to any multipolar approximations.

The nonrelativistic limit of Eq. (4.17) can be obtained by using a Foldy-Wouthuysen-type transformation. First we note that odd operators are contained in the expression

$$Q = \vec{\alpha}\cdot\vec{p} - \int \vec{M}\cdot\vec{B}\, d^3r\,. \qquad (4.21)$$

This can be written

$$Q = \vec{\alpha}\cdot(\vec{p}-e\vec{W})\,, \qquad (4.22)$$

where

$$W_k(\vec{x}) = \epsilon_{ijk}\int d^3r \int_0^1 d\lambda\, \lambda x^j \delta(\vec{r}-\lambda\vec{x}) B^i(\vec{r})\,. \qquad (4.23)$$

Thus Eq. (4.17) becomes

$$i\frac{\partial\psi'}{\partial t} = \left(\vec{\alpha}\cdot(\vec{p}-e\vec{W}) + \beta m + eV - \int \vec{P}\cdot\vec{E}^{\perp}\, d^3r\right)\psi'\,. \qquad (4.24)$$

Following Foldy and Wouthuysen,[21] we now define the transformation

$$\Lambda_2 = e^{-i\beta Q/2m}\,. \qquad (4.25)$$

By taking identical steps to the familiar procedure, one will eventually obtain

$$i\frac{\partial\psi'}{\partial t} \sim \left[\beta\left(m+\frac{(\vec{p}-e\vec{W})^2}{2m} - \frac{p^4}{8m^3}\right) + V - \int \vec{P}\cdot\vec{E}^{\perp} d^3r - \frac{e}{2m}\beta\vec{\sigma}\cdot\vec{\nabla}\times\vec{W} - \frac{ie}{8m^2}\vec{\sigma}\cdot\vec{\nabla}\times\vec{E} - \frac{e}{4m^3}\vec{\sigma}\cdot\vec{E}\times\vec{p} - \frac{e}{8m^2}\vec{\nabla}\cdot\vec{E}\right]\psi'\,. \qquad (4.26)$$

In arriving at this result, we have also made use of the distribution identities (4.15) and (4.16).

Apart from the terms involving $\vec{W}$ and $\vec{P}$, the individual terms in Eq. (4.26) have their usual physical interpretation. To be able to identify the other terms, we first consider the expansion

$$\frac{1}{2m}(\vec{p}-e\vec{W})^2 = \frac{p^2}{2m} + \frac{e^2}{2m}W^2 - \int \vec{\mathfrak{M}}\cdot\vec{B}\, d^3r\,, \qquad (4.27)$$

where

$$\vec{\mathfrak{M}} = \frac{e}{2m}\int_0^1 d\lambda\, [\vec{l}\lambda\delta(\vec{r}-\lambda\vec{x}) + \lambda\delta(\vec{r}-\lambda\vec{x})\vec{l}\,] \qquad (4.28)$$

is the nonrelativistic magnetization operator,[22] and

$$\vec{l} = -i\vec{x}\times\vec{\nabla} \qquad (4.29)$$

is the orbital angular momentum operator.

The term quadratic in e in Eq. (4.27) represents an exact expression for the nonrelativistic diamagnetic energy shift. In the magnetic dipole approximation this term gives

$$\frac{e^2}{2m}W^2 \sim \frac{e^2}{8m}[\vec{x}\times\vec{B}(\vec{0})]^2 . \qquad (4.30)$$

The spin magnetic moment does not couple directly to the magnetic field but to $\vec{\nabla}\times\vec{W}$. Only in the dipole approximation do we get

$$-\frac{e}{2m}\vec{\sigma}\cdot\vec{\nabla}\times\vec{W} \sim -\frac{e}{2m}\vec{\sigma}\cdot\vec{B}(\vec{0}) . \qquad (4.31)$$

We also note that the term involving the electric polarization retains its form in the relativistic equation. Thus the electric polarization operator, unlike the magnetic polarization operator, has the same form in both relativistic and nonrelativistic formulations.

Keeping terms up to the order $1/m$, one obtains the Schrödinger equation

$$i\frac{\partial\Phi}{\partial t} = \left(\frac{p^2}{2m} + V - \int \vec{P}\cdot\vec{E}^{\perp} d^3r - \int \vec{\mathfrak{M}}\cdot\vec{B}\, d^3r + \frac{e^2}{2m}W^2 - \frac{e}{2m}\vec{\sigma}\cdot\vec{\nabla}\times\vec{W}\right)\Phi . \qquad (4.32)$$

This equation is also manifestly gauge invariant and takes account of all contributions from the entire electric multipole series as well as the nonrelativistic contributions from the magnetic multipoles. In addition, spin contributions and diamagnetic energy effects are properly included to all orders.

ACKNOWLEDGMENTS

I am grateful to Dr. P. L. Knight for valuable comments and for numerous discussions. I am also grateful to the University of Sussex for support while the final version of the manuscript was prepared.

[1] E. T. Jaynes and F. W. Cummings, Proc. IEEE 51, 89 (1963); M. D. Crisp and E. T. Jaynes, Phys. Rev. 179, 1253 (1969); C. R. Stroud and E. T. Jaynes, Phys. Rev. A 1, 106 (1970); 2, 1613(E) (1970).

[2] T. A. Welton, Phys. Rev. 74, 1157 (1948).

[3] H. A. Casimir, Proc. Natl. Acad. Sci. USA 60, 793 (1948).

[4] J. R. Ackerhalt, P. L. Knight, and J. H. Eberly, Phys. Rev. Lett. 30, 456 (1973); J. R. Ackerhalt and J. H. Eberly, Phys. Rev. D 10, 3350 (1974).

[5] R. K. Nesbet, Phys. Rev. Lett. 27, 553 (1971); Phys. Rev. A 4, 259 (1971).

[6] W. L. Lama and L. Mandel, Phys. Rev. A 6, 2247 (1972).

[7] G. W. Series, in *Optical Pumping and Atomic Line Shape*, edited by T. Skaliniski (Panstowowe Wdawnictowo Naukowe, Warszawa, 1969), p. 25; Bull. Am. Phys. Soc. 18, 1523 (1973).

[8] J. R. Ackerhalt and J. H. Eberly, Ref. 4.

[9] V. Arunasalam, Am. J. Phys. 37, 887 (1969). Here the author fails to take proper account of mass renormalization.

[10] M. Babiker, E. A. Power, and T. Thirunamachandran, Proc. R. Soc. A 338, 235 (1974); M. Babiker, Phys. Rev. A 11, 308 1975.

[11] We use natural units $\hbar = 1 = c$. Electronic mass $= m$ and electronic charge $e = -|e|$. Our notations and conventions are those of J. D. Bjorken and S. D. Drell, *Relativistic Quantum Fields* (McGraw-Hill, New York, 1965).

[12] Strictly speaking, the Dirac sources should be defined as the limit of expressions in which the fields are taken at separated space-time points. See J. Schwinger, Phys. Rev. Lett. 3, 2969 (1959).

[13] M. Babiker and G. Barton, Proc. R. Soc. A 326, 277 (1972); J. Gehenieau and F. Villars, Helv. Phys. Acta 23, 178 (1950).

[14] J. Hamilton, *The Theory of Elementary Particles* (Oxford U. P., Oxford, England, 1959), p. 452.

[15] See, for instance, J. J. Sakurai, *Advanced Quantum Mechanics* (Addison-Wesley, Reading, Mass., 1967), Appendix E.

[16] A. O. Barut, Phys. Rev. D 10, 3335 (1974).

[17] M. Babiker, Ref. 10.

[18] M. Babiker, Proc. R. Soc. Lond. A 342, 113 (1975).

[19] E. A. Power and S. Zienau, Philos. Trans. R. Soc. Lond. A 251, 427 (1959).

[20] H. R. Reiss, Phys. Rev. A 1, 803 (1970).

[21] L. L. Foldy and S. A. Wouthuysen, Phys. Rev. 78, 29 (1950).

[22] M. Babiker, E. A. Power, and T. Thirunamachandran, Ref. 10.