Eikonal theory of charged-particle scattering in the presence of a strong electromagnetic wave*

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The first Born approximation calculation of Kroll and Watson is extended by a time-dependent eikonal calculation. The resulting amplitude returns to their first Born approximation in the extreme limit of a weak scattering potential or small momentum transfer. In the ultrastrong-field limit, the total cross section reduces to the field-free eikonal result.

I. INTRODUCTION

Atomic scattering processes in the presence of a strong electromagnetic wave (laser) have become of great interest recently. The first step towards the understanding of such processes was taken by Kroll and Watson,[1] who studied the scattering of a structureless charged particle by a potential in the presence of a single-mode laser field. Among other results they showed that when the potential is weak enough for the first Born approximation to hold, the cross section is given by

$$\frac{d\sigma}{d\Omega}=\frac{q(\nu)}{q_0}J_\nu^2(\chi)\frac{d\sigma_B}{d\Omega}(Q)\,, \tag{1.1}$$

where $d\sigma_B/d\Omega$ is the scattering cross section in the absence of the field, and where $\vec{q}_0$ is the initial momentum, $\vec{q}(\nu)$ is the final momentum, $\vec{Q}=\vec{q}(\nu)-\vec{q}_0$ is the momentum transfer, and ν is the number of photons emitted during scattering, i.e.,

$$\hbar\omega\nu=(1/2m)[q_0^2-q^2(\nu)]\,. \tag{1.2}$$

The parameter χ is the Bessel function is $\chi=-\vec{\alpha}\cdot\vec{Q}$, where α is the amplitude of the oscillation of the charged particle (electron) in the presence of the laser field alone. They also show that $\chi\hbar\omega$ represents the electromagnetic energy emitted in the corresponding classical collision.

First we shall extend their results here by an eikonal treatment instead of a Born treatment of the scattering process. Our results return to Eq. (1.1) when the momentum transfer is large compared to some measure of the scattering potential, but for the case of strong potentials our results are quite different. In the process of our derivation we are forced to make some approximations as follows:

(a) If a_0 is the range of the scattering potential, then we require that

$$\frac{\alpha}{a_0}=\frac{eE}{mc\omega^2a_0}\ll 1\,, \tag{1.3}$$

that is, that the amplitude of the electron's oscillation in the laser field be small compared to the potential range. Kroll and Watson[1] also derive a result similar to Eq. (1.1) in the low-frequency limit. Their result is that Eq. (1.1) holds to all orders in the scattering potential provided that the Born approximation is replaced by the exact result for the elastic scattering, evaluated at some prescribed energy. Our approximation (1.3) invalidates our discussion at low frequencies, so no comparison with their result is possible in that domain.

(b) If v is the velocity of the incident electron, then we require that

$$\alpha\omega/v\ll 1\,, \tag{1.4}$$

or that the ratio of the amplitude of the oscillating velocity to the incident velocity be small. This approximation, while it arises as a specific requirement of our analysis, is also required by the eikonal (or Born) approximation itself. It is simply the requirement that the instantaneous particle velocity should never be small.

(c) We also require that the fractional energy transfer be small enough so that

$$\frac{n\omega\hbar}{q_0^2/2m}\left(\frac{a_0q}{\hbar}\right)\ll 1\,. \tag{1.5}$$

In the usual eikonal development one is forced to neglect the momentum transfer along some direction close to the incident direction. Equation (1.5) arises in a similar manner.

We next proceed to study the strong-field limit, and relax the restrictions imposed by Eqs. (1.3) and (1.4). We find that the total cross section tends to the field-free result in the ultrastrong-field limit. A formula for the correction to the total cross section is also derived in this asymptotic limit.

II. EIKONAL APPROXIMATION

Our starting point is the time-dependent Schrödinger equation[2]

$$\left(i\frac{\partial}{\partial t}+\tfrac{1}{2}\nabla^2 - iA(t)\nabla - V(r)\right)\Psi = 0\,, \tag{2.1}$$

where the usual dipole approximation for A has been made and the A^2 term removed by a contact transformation. The form

$$\Psi = \exp\{i\vec{k}_0\cdot\vec{r} - i[\tfrac{1}{2}k_0^2 t - \vec{k}\cdot\vec{\alpha}(t)] + i\Lambda(\vec{r},t)\}\,, \tag{2.2}$$

where

$$\vec{\alpha}(t) = \int^t dt'\vec{A}(t') \tag{2.3}$$

is substituted into (2.1) with the result

$$-\dot{\Lambda} - \vec{k}_0\cdot\nabla\Lambda + \dot{\vec{\alpha}}\cdot\nabla\Lambda - V = \tfrac{1}{2}(\nabla\Lambda)^2 - \tfrac{1}{2}i\nabla^2\Lambda\,. \tag{2.4}$$

When the potential and therefore the phase Λ are slowly varying in a wavelength, the terms on the right are dropped as in the conventional eikonal-approximation derivation. The remaining linear first-order equation may be simplified by the coordinate transformation

$$\vec{r} = \vec{\rho} + \vec{k}_0\tau - \vec{\alpha}(\tau)\,,\quad t=\tau\,, \tag{2.5}$$

and the equation may then be integrated with the result

$$\begin{aligned}\Lambda &= -\int_{-\infty}^{\tau} d\tau'\, V(\vec{\rho}+\vec{k}_0\tau - \vec{\alpha}(t'))\\ &= \int_{-\infty}^{t} d\tau'\, V(\vec{r}-\vec{k}_0(t-\tau')+\vec{\alpha}(t)-\vec{\alpha}(\tau'))\\ &= -\int_{-\infty}^{z/k_0} dt'\, V(\vec{b}+\vec{k}_0 t' - \vec{\alpha}(t+t'-z/k))\,,\end{aligned} \tag{2.6}$$

where $\vec{k}_0$ is taken in the z direction[3] and $\vec{b}$ is the component of $\vec{r}$ perpendicular to $\vec{k}_0$.

The S matrix for scattering from wave number $\vec{k}_0$ to $\vec{k}'$ is

$$S_{\vec{k}'\vec{k}_0} = -i\int_{-\infty}^{\infty} dt\int d\vec{r}\,\chi^*_{\vec{k}'}V\Psi_{\vec{k}_0}\,, \tag{2.7}$$

where

$$\chi_{\vec{k}'}(\vec{r},t) = \exp[i\vec{k}'\cdot\vec{r} - \tfrac{1}{2}ik'^2 t + i\vec{k}'\cdot\vec{\alpha}(t)]\,. \tag{2.8}$$

Substitution of (2.2), (2.6), and (2.8) into S results in

$$S_{\vec{k}'\vec{k}_0} = -i\int_{-\infty}^{\infty} dt\int d\vec{r}\exp\{-i\vec{Q}\cdot[\vec{r}+\vec{\alpha}(t)] - \tfrac{1}{2}i(k'^2-k_0^2)t\}\times V(\vec{r})e^{i\Lambda(\vec{r},t)}\,, \tag{2.9}$$

where the momentum transfer is

$$\vec{Q} = \vec{k}' - \vec{k}_0\,. \tag{2.10}$$

Simple transformations of the time integration variable in Λ and the z component of $\vec{r}$ allows S to be rewritten as

$$S = -i\int_{-\infty}^{\infty} dt\int d\vec{r}\exp[-i\vec{Q}\cdot\vec{r} - \tfrac{1}{2}i(k'^2-k_0^2)t]\left(\frac{\partial}{\partial t}+\vec{k}_0\cdot\nabla\right)\exp\left(-i\int_{-\infty}^{z/k} dt'\, V(\vec{b}+\vec{k}_0 t - \vec{\alpha}(t+t'-z/k_0))\right)\,. \tag{2.11}$$

The eikonal phase is now seen to be the usual $\int dt\, V$ integral except that the path is not a straight-line motion but instead is that determined by the classical equations of motion in the electromagnetic field. In this form some simple limits are obvious: (a) If the external field vanishes ($\alpha=0$), then the time derivative vanishes, the time integral yields the energy-conservation condition, and the remaining factor is the conventional eikonal T matrix; and (b) if the phase $\int V dt$ is small, then the exponential can be expanded and the leading term is identical with the first Born approximation of Kroll and Watson.[1]

Now we note that the last exponential factor of Eq. (2.11) contains t only through the periodic function α. If we define

$$\overline{V} = \frac{\omega}{2\pi}\int_0^{2\pi/\omega} V(\vec{b}+\vec{k}_0 t' - \vec{\alpha}(t'+t-z/k_0))\,dt\,, \tag{2.12}$$

then the remaining part is periodic in t with period $2\pi/\omega$ and may be expanded in a Fourier series

$$\exp\left(-i\int_{-\infty}^{z/k_0} dt'[V(\vec{b}+\vec{k}_0 t - \vec{\alpha}(t+t'-z/k_0)) - \overline{V}]\right) = \sum_{\nu=-\infty}^{\infty} e^{i\nu\omega t}\int_0^{2\pi}\frac{da}{2\pi}e^{-i\nu a}\exp\left(-i\int_{-\infty}^{z/k_0} dt'[V(\vec{b}+\vec{k}_0 t - \vec{\alpha}(a/\omega+t'-z/k_0)) - \overline{V}]\right)\,. \tag{2.13}$$

After substituting this into Eq. (2.11) and performing the t integral, the result may be written

$$S_{k'k_0}=-2\pi i\sum_{\nu}\delta(\tfrac{1}{2}(k'^2-k_0^2)-\nu\omega)T_\nu(\vec{k}',\vec{k}_0)\,, \tag{2.14}$$

where

$$T_\nu(\vec{k}',\vec{k}_0)=-i\int d\vec{r}\int\frac{da}{2\pi}\exp(-i\nu a-i\vec{Q}\cdot\vec{r})(\vec{k}_0\cdot\nabla+i\nu\omega)\exp\left(-i\int_{-\infty}^{z/k_0}dt'\,V(\vec{b}+\vec{k}_0t-\vec{\alpha}(a/\omega+t'-z/k_0))\right) \tag{2.15}$$

and T_ν may then be interpreted as the T matrix for scattering with the absorption of ν photons.

In order to proceed further with T we shall now expand in powers of α in V, which is the reason for our first assumption, Eq. (1.3):

$$\frac{|\alpha|}{a_0}=\frac{eE}{mc\omega^2a_0}\ll 1\,.$$

We optimize this expansion to some extent by working in the appropriate coordinate system. If we let

$$t'=z/v-\tau'\,,\quad \vec{r}=\vec{r}'+\vec{\alpha}(a/\omega-z'/k_0)\,, \tag{2.16}$$

then the Jacobian of this transformation is

$$J=1-\vec{k}_0\cdot\vec{\alpha}_0(\omega/k_0^2)\cos[a-(\omega/k_0)z']\,, \tag{2.17}$$

and the derivative becomes

$$\vec{k}\cdot\nabla=J^{-1}\{\vec{k}_3\cdot\nabla'+\omega\cos[a-(\omega/k_0)z']\vec{\alpha}_{0\perp}\cdot\nabla'\}\,, \tag{2.18}$$

where we have specialized to a linear polarization

$$\vec{\alpha}(t)=\vec{\alpha}_0\sin\omega t\,, \tag{2.19}$$

and where $\vec{\alpha}_{0\perp}$ is the component perpendicular to $\vec{k}_0$. The quantity $\alpha\omega/k_0$ is the second parameter which we have assumed to be small [Eq. (1.4)] so that Eq. (2.14) may be written

$$T_\nu=-i\int d\vec{r}\int\frac{da}{2\pi}e^{-i\nu a}[\vec{k}_0\cdot\nabla'+i(n\omega+\vec{k}_0\cdot\vec{Q})J]$$
$$\times\exp-i\left\{\vec{Q}\cdot\left[\vec{r}'+\alpha\left(\frac{a}{\omega}-\frac{z'}{k_0}\right)\right]+\int_0^\infty d\tau'\,V\left(\vec{r}'-\vec{k}_0\tau'+\alpha\left(\frac{a}{\omega}-\frac{z'}{k_0}\right)-\alpha\left(\frac{a}{\omega}-\tau'\right)\right)\right\}. \tag{2.20}$$

The expansion of V is now an expansion (under the τ' integral) in powers of $\alpha(a/\omega-z'/k_0)-\alpha(a/\omega-\tau')$ which vanishes at $\tau'=z'/k_0$. But this is the point of maximum contribution in the τ' integral, so that the expansion is optimized. We now expand, keeping only terms linear in α in the exponential. Note that this requires that α/a_0 be small, but we may still have $(\alpha_0/a_0)(\overline{V}a_0/k_0)=\alpha_0\overline{V}/k_0$ large. ($\overline{V}$ is the average of V along the direction of integration.) The result may be written

$$T_\nu=-i\int d\vec{r}'\frac{da}{2\pi}e^{-i\nu a}[\vec{k}_0\cdot\nabla'+i(\nu\omega+\vec{k}_0\cdot\vec{Q})J]\exp[-i\vec{Q}\cdot\vec{r}'-i\nu(\omega/k_0)z]$$
$$\times\exp[-i\overline{\Phi}_0(b',z')-i\alpha_0\overline{\Omega}(b',z')\sin(a-\overline{\eta})+(i\nu\omega/k_0)z']\,, \tag{2.21}$$

where

$$\overline{\Phi}_0(b,z)=\int_0^\infty d\tau\,V(\vec{r}-\vec{k}_0\tau)\,,$$

$$\overline{\Phi}_1(b,z)=\hat{\alpha}_0\cdot\nabla\overline{\Phi}_0\,,$$

$$\overline{\Omega}=(\{\overline{\Phi}_s-[\sin(\omega z/k_0)](\overline{\Phi}_1+\hat{\alpha}_0\cdot\vec{Q})\}^2+\{\overline{\Phi}_0-[\cos(\omega z/k_0)](\overline{\Phi}_1+\hat{\alpha}_0\cdot\vec{Q})\}^2)^{1/2}\,,$$

$$\tan\overline{\eta}=\frac{\overline{\Phi}_s-[\sin(\omega z/k_0)](\overline{\Phi}_1+\hat{\alpha}_0\cdot\vec{Q})}{\overline{\Phi}_0-[\cos(\omega z/k_0)](\overline{\Phi}_1+\hat{\alpha}_0\cdot\vec{Q})}\,, \tag{2.22}$$

$$\overline{\Phi}_s(b,z)=\hat{\alpha}_0\cdot\nabla\int_0^\infty d\tau'\,\sin\omega\tau'\,V(\vec{r}-\vec{k}_0\tau')\,,$$

$$\overline{\Phi}_c(b,z)=\hat{\alpha}_0\cdot\nabla\int_0^\infty d\tau'\,\cos\omega\tau'\,V(\vec{r}-\vec{k}_0\tau')\,,$$

$$\overline{\Phi}_\pm(b,z)=\overline{\Phi}_c\pm i\overline{\Phi}_s\,.$$

At this stage in the usual eikonal method one neglects the momentum transfer along the line of motion, or better, arranges the direction of integration in the eikonal to be perpendicular to the momentum transfer.[3] In our case the factor

$$\hat{k}_0\cdot\vec{Q}+\nu\omega/k_0\approx 2\nu\omega/k_0$$

in the exponential plays that role. We neglect it in the exponent and in the coefficient of J. This is our third assumption, Eq. (1.5). Then the a integration may be performed by using

$$\int\frac{da}{2\pi}\exp[-i\nu a-i\alpha_0\Omega\sin(a-\eta)]=(-)^\nu J_\nu(\alpha_0\Omega)e^{-i\nu\eta}\,, \tag{2.23}$$

and the z integral can now be done.

$$T_\nu = -ik_0 \int d^2b e^{-i\vec{Q}\cdot\vec{b}}[e^{-i\Phi_0}J_\nu(\alpha_0\Omega)e^{-i\nu\eta} - J_\nu(\vec{\alpha}_0\cdot\vec{Q})](-)^\nu , \quad (2.24)$$

where Φ_0 is the usual eikonal[4]

$$\Phi_0(b) = \int_{-\infty}^{\infty} V(\vec{b}+\vec{k}t)\,dt ,$$

and

$$\Phi_1 = \hat{\alpha}_0\cdot\nabla\Phi_0 ,$$

$$\Phi_\pm = [\hat{\alpha}_{0\perp}\cdot\nabla \pm i(\omega/k_0)\hat{\alpha}_0\cdot\hat{k}_0]\int_{-\infty}^{\infty} dt e^{\mp i\omega t}V(\vec{b}+\vec{k}t) ,$$

$$\Omega = |\Phi_- - \Phi_1 - \vec{Q}\cdot\hat{\alpha}_0| ,$$

$$e^{-i\eta} = (\Phi_1 + \vec{Q}\cdot\hat{\alpha}_0 - \Phi_-)/\Omega . \quad (2.25)$$

If V is a spherically symmetric potential, then these may all be written in terms of one real integral

$$I(b,\omega) = \int_{-\infty}^{\infty} dt e^{i\omega t}V(\vec{b}+\vec{k}t) = \frac{2b}{k}\int_0^\infty ds[\cos(\omega bs/k)]V(b(1+s^2)^{1/2}) , \quad (2.26)$$

and then

$$\Phi_0(b) = I(b,0) ,$$

$$\Phi_1(b) = \hat{\alpha}_0\cdot\hat{b}I'(b,0) , \quad (2.27)$$

$$\Phi_-(b) = \hat{\alpha}_0\cdot\hat{b}I'(b,\omega) - (i\omega/k)\hat{\alpha}_0\cdot\hat{k}I(b,\omega) ,$$

where

$$I'(b,\omega) = \frac{\partial}{\partial b}I(b,\omega) . \quad (2.28)$$

Note that when the momentum-transfer term in Ω dominates the potential terms

$$\hat{\alpha}_0\cdot\vec{Q} > |\Phi_- - \Phi_1| , \quad (2.29)$$

T_2 becomes just $J_\nu(\vec{\alpha}_0\cdot\vec{Q})$ times a "zero-field" eikonal result with the momentum transfer evaluated at $\vec{Q} = \vec{k}'(\nu) - \vec{k}_0$. This is analogous to the result obtained by Kroll and Watson[1] for the first Born approximation.

The perpendicular component of the momentum transfer is given classically by $\hat{b}\cdot\nabla\int_{-\infty}^{\infty}dt\,V(\vec{b}+\vec{k}t)$, which is essentially Φ_1, so that we may expect that in general Eq. (2.29) will not hold. We may crudely estimate $I \sim bV/k$ by using Eq. (1.5) so that the inequality (2.29) is then

$$|\hat{\alpha}_0\cdot Q| > |\hat{\alpha}_0\cdot\hat{k}\,b\omega V/k^2| ,$$

which is not true for strong-enough potentials. Examination of Eq. (2.21) shows that the Φ_1 and Φ_0 terms arise from the modification of the conventional (zero-field) eikonal due to the field. As in the eikonal formulation of potential scattering, we see that the modification due to the eikonal is not negligible for large-enough potentials.

In the limit where the momentum transfer along $\vec{\alpha}_0$ is small, we obtain

$$T_\nu = -ik_0\int d^2b e^{-i\vec{Q}\cdot\vec{b}}[e^{-i\Phi_0}J_\nu(\alpha\Omega)e^{-i\nu\eta} - \delta_{\nu 0}](-)^\nu , \quad (2.30)$$

where Ω and η now are independent of $\vec{Q}$, and Ω is proportional to the potential strength. Now in the limit of weak laser fields, T_ν is proportional to α^ν and to the potential strength to the νth power. The νth-order radiative absorption is described as (at least) ν weak scatterings with a photon absorbed during each process.

The optical theorem may be used to evaluate the total cross section, elastic ($\nu = 0$) plus inelastic, from the imaginary part of $T_0(Q=0)$.

$$\sigma_{\text{tot}} = (2/k_0)\,\text{Im}\,T_0(Q=0) . \quad (2.31)$$

From Eq. (2.15),

$$\sigma_{\text{tot}} = 4\int d^2b\int\frac{da}{2\pi}\sin^2\left(\frac{1}{2}\int_{-\infty}^{\infty}dt\,V(\vec{b}+\vec{k}t - \vec{\alpha}(a/\omega+t))\right), \quad (2.32)$$

or after the expansion in α, from Eq. (2.24),

$$\sigma_{\text{tot}} = 2\int d^2b[1 - \cos\Phi_0 J_0(\alpha\Omega_0)] , \quad (2.33)$$

where Ω_0 is Ω evaluated at $\vec{Q} = 0$.

If we use Eqs. (2.25), (2.26), and (2.27) to estimate the relative magnitude of $\alpha_0\Omega$ and Φ_0, we see that since $\omega a_0 < k_0$ [Eq. (1.5)], the ratio $\alpha_0\Omega/\Phi_0$ is estimated to be the larger of

$$[(\vec{\alpha}_0\cdot\hat{b})/a_0](\omega a_0/k_0)^2 \text{ and } [(\vec{\alpha}_0\cdot\hat{k})/a_0](\omega a_0/k) .$$

Both of these are small because of Eqs. (1.3) and (1.5). This does not mean that J_0 can be replaced by unity, since $\alpha_0\Omega_0$ is not necessarily small.

We shall now simply quote the analogous results for circular polarization. We define a plane perpendicular to the incident momentum and a plane of polarization. The x axis is taken as the line of intersection of the two planes and the angle Θ as the angle of intersection of the two planes. Then

$$\vec{\alpha}(t) = \alpha_0[\hat{a}_x\cos\omega t + (\hat{a}_y\cos\Theta + \hat{a}_z\sin\Theta)\sin\omega t] , \quad (2.34)$$

and the result analogous to Eq. (2.24) is

$$T = -i(-)^\nu k_0\int d^2b e^{-i\vec{Q}\cdot\vec{b}}\{e^{-i\Phi_0}J_\nu(\alpha_0\tilde{\Omega})e^{-i\nu\tilde{\eta}} - J_\nu[\alpha_0\tilde{\Omega}(-\infty)]e^{i\nu\tilde{\eta}(-\infty)}\} , \quad (2.35)$$

where Φ_0 is the unperturbed eikonal (2.25) and where

$$\begin{aligned}
&\tilde{\Omega} = |(\hat{a}_\perp + i\hat{a}_x)\cdot(\vec{Q} + \vec{\chi}_1 - \vec{\chi}_-)| , \\
&e^{-i\tilde{\eta}} = (\hat{a}_\perp + i\hat{a}_x)\cdot(\vec{Q} + \vec{\chi}_1 - \vec{\chi}_-)/\tilde{\Omega} , \\
&\tilde{\Omega}(-\infty) = |(\hat{a}_\perp + i\hat{a}_x)\cdot\vec{Q}| , \\
&e^{-i\tilde{\eta}(-\infty)} = (\hat{a}_\perp + i\hat{a}_x)\cdot\vec{Q}/\tilde{\Omega}(-\infty) .
\end{aligned} \tag{2.36}$$

The integrals χ_1 and χ_- are defined analogously to Φ_1 and Φ_-.

$$\begin{aligned}
&\chi_1 = \nabla\Phi_0 , \\
&\chi_- = \left(-i\frac{\omega}{k}\hat{a}_z + \nabla\right)\int_{-\infty}^{\infty} dt\, e^{i\omega t} V(\vec{b} + \vec{k}t) .
\end{aligned} \tag{2.37}$$

The total cross section can also be obtained in the form of Eq. (2.33), except that now $\tilde{\Omega}_0 = \tilde{\Omega}(Q=0)$ occurs instead of Ω_0.

III. ULTRASTRONG-FIELD LIMIT

In this section we explore the scattering that occurs in the presence of a very strong laser field. It will be assumed that the classical ac velocity is much larger than the range of the free-particle speed, so instead of Eq. (14) we have

$$\alpha\omega \gg v . \tag{3.1}$$

Rather than repeat the analysis for the partial cross sections we elect to study only the total cross section for the linearly polarized light case. Thus we study Eq. (2.32) in the strong-field limit.

We expand $V(\vec{r})$ as a Fourier integral

$$V(\vec{r}) = \int d\vec{q}\, \tilde{V}(\vec{q}) e^{i\vec{q}\cdot\vec{r}} , \tag{3.2}$$

and carry out the time integration to obtain

$$\sigma_{\text{tot}} = 4\int d^2b \int_0^{2\pi} \frac{da}{2\pi} \sin^2\left[\frac{\pi}{k}\sum_{n=-\infty}^{\infty}\int d^2q_\perp e^{i\vec{q}_\perp\cdot\vec{b}} J_n\left(\frac{n\omega\alpha_0}{k}\right) e^{-ina}\tilde{V}(\vec{q})\Big|_{q_z = n\omega/k}\right] . \tag{3.3}$$

Alternatively this may be written as

$$\sigma_{\text{tot}} = 4\int d^2b \int_0^{2\pi} \frac{da}{2\pi} \sin^2\left[\frac{1}{2k}\sum_{n=-\infty}^{\infty} J_n\left(\frac{n\omega\alpha_0}{k}\right) e^{-ina}\int_{-\infty}^{\infty} V(\vec{r}) e^{-i(n\omega/k)z} dz\right] . \tag{3.4}$$

We break the sum into two parts—one the contribution from $n=0$ and the other the contribution from the $n\neq 0$ terms. If we adopt Eq. (3.1) we may use the asymptotic formula for the Bessel function for the latter sum

$$J_n\left(\frac{n\omega\alpha_0}{k}\right) \to \left(\frac{2k}{\pi\omega\alpha_0|n|}\right)^{1/2}\cos\left(\frac{n\omega\alpha_0}{k} - (n+\tfrac{1}{2})\frac{\pi}{2}\right). \tag{3.5}$$

Thus we may write Eq. (3.4) as

$$\sigma_{\text{tot}} = 4\int d^2b \int_0^{2\pi}\frac{da}{2\pi}\sin^2\left(\frac{\Phi_0}{2} + \Phi_1\right) , \tag{3.6}$$

where

$$\Phi_0 = \frac{1}{k}\int_{-\infty}^{\infty} dz\, V(\vec{r}) \tag{3.7}$$

is just the usual eikonal for the unperturbed scattering, and where Φ_1 is an asymptotic correction term given by

$$\Phi_1 = \frac{1}{2k}\sum_{\substack{n \\ n\neq 0}}\left(\frac{2k}{\pi\omega\alpha_0|n|}\right)^{1/2} e^{-ina}\cos\left(\frac{n\omega\alpha_0}{k} - (n+\tfrac{1}{2})\frac{\pi}{2}\right) \times\int_{-\infty}^{\infty} dz\, V(\vec{r}) e^{-in\omega z/k} . \tag{3.8}$$

For reasonably well-behaved potentials, we expect the Fourier transform appearing in Eq. (3.8) to fall off rapidly with increasing n. For example, for a potential which goes as

$$V(\vec{r}) \sim -c/r^\nu , \tag{3.9}$$

we have

$$\int dz\, V(\vec{r}) e^{-in\omega z/k} = -\frac{2c\pi^{1/2}(|n|\omega/k)^{[(\nu-1)/2]}}{\Gamma(\frac{1}{2}\nu)(2b)^{[(\nu-1)/2]}} \times K_{[(\nu-1)/2]}(b|n|\omega/k) , \tag{3.10}$$

which behaves like

$$\int dz\, V(\vec{r}) e^{-in\omega z/k} \xrightarrow[n\to\infty]{} n^{\nu/2-1} e^{-nb\omega/k} . \tag{3.11}$$

Thus the sum in Eq. (3.8) is bounded by a constant $\Phi_1 \leq \text{const}/\alpha_0^{1/2}$ in the strong-field limit. Thus as a first approximation we retain only the Φ_0 term in Eq. (3.6) (which is independent of a) and regain the field-free limit.

If we make the further assumption that $\Phi_1 \ll 1$ we may expand the sine function appearing in Eq. (3.6) and obtain an explicit formula

$$\sigma_{\text{tot}} = \sigma_{\text{free}} + 4\int d^2b \langle\Phi_1^2\rangle_a \cos 2\Phi_0 , \tag{3.12}$$

where σ_{free} is the field-free cross section. Insertion of Eqs. (3.7) and (3.8) into Eq. (3.12) leads to

$$\sigma_{\text{tot}} = \sigma_{\text{free}} + \sum_{\substack{n \\ n\neq 0}} \frac{2}{\pi \omega k \alpha_0 |n|} \cos^2\left(\frac{\omega n \alpha_0}{k} - (n+\tfrac{1}{2})\frac{\pi}{2}\right) \int d^2 b \cos\left(\frac{1}{k}\int_{-\infty}^{\infty} V(\vec{r})dz\right) \left|\int_{-\infty}^{\infty} dz\, V(\vec{r}) e^{in\omega z/k}\right|^2 . \tag{3.13}$$

We see that the correction to σ_{free} goes asymptotically as the reciprocal of the electric field strength. A similar behavior of the multiphoton ionization cross section has been found in the high-field limit.[4]

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