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Theoretical method for the analysis of nonlinear effects in weakly dissipative media

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In this paper a general perturbation method is developed for a kinetical analysis of the role and the influence of collisions on the nonlinear effects occurring in weakly or partially ionized gaseous media. The nature of the collision operators leads to an expansion of the electron distribution function in a basis of irreducible Cartesian tensors and to the study of the coupled differential system thus established. It is shown that this system, coupled with Maxwell's equations, can be uncoupled only in weakly dissipative media, where multiple time and space scales can be defined and where the distribution function as well as the electric field can be expanded as a function of a suitably chosen small parameter. The procedure is applied to different physical situations characterized by the order of magnitude of the collisional parameters. It is shown that three cases may be distinguished: collisional, weakly collisional, and an intermediate case. In each of these cases, the propagation of a single wave (longitudinal or transverse) is studied and it is shown how these techniques allow one to determine the orders of magnitude of various nonlinear effects (such as collisional heating and generation of harmonics) as well as the time or space scales at which they occur. Finally, there is a brief discussion of other applications and extensions of the method (such as the influence of a plasma inhomogeneity, the role of the Coulomb collisions, the coupling of several waves, and the low-frequency approximation).

I. INTRODUCTION

In this paper, a method of kinetic analysis is proposed for various nonlinear effects linked with the propagation of electromagnetic waves in collisional plasmas, weakly or partially ionized. Beside purely theoretical considerations, the interest of such a method lies also in a synthesis of the many papers of these latest years devoted to the study of the nonlinear behavior of different plasmas in which collisions may not be neglected, whether being predominant in the analyzed phenomenon or perturbing it notably.

In the first case—collision-dominated plasma—the dielectric properties of the medium are modified by the dissipation of the electromagnetic waves. Then, collisional terms appear from the linear approximation and the nonlinear phenomena are caused by a plasma heating due to the energy drawn from the electric field by the electrons through collisions. There have been many experimental and theoretical works devoted to the analysis of harmonic generation and nonlinear mixing of frequencies in collisional plasmas weakly or partially ionized,[1–12] in addition to those dealing with the study of nonlinearities in semiconductors and gaseous discharges.[13]

In the second case—weakly collisional plasmas—the rise of nonlinear effects (coupling of modes, etc.) is not due to dissipative processes; but these can greatly perturb the studied phenomena, although they can be neglected in the linear approximation. This is the case, for example, in the resonant three-wave interaction: here, the nonlinear coupling is due to the Lorentz force, but the importance of dissipative effects has been made evident and discussed in different works,[14–18] especially for the stabilization of media where certain (explosive) instabilities grow in the absence of collisions; on the other hand, another type of coupling may also be expected.[19,43,44] Likewise, the influence of Coulomb collisions on the behavior of a turbulent plasma cannot be neglected and may give rise to appreciable changes in the turbulent spectra.[20]

As is well known, a phenomenological description of the collisions is insufficient to account in detail for these various phenomena, so that we are led to do a kinetic analysis. So, we have to solve the coupled system of Maxwell's equations and kinetic equations in which occur cumbersome collisional operators. On account of the great complexity of this problem, we can only obtain approximate solutions which must be adapted to each particular problem. But, as there is a large diversity of possible physical situations, the various corresponding approximations ought to be founded upon a general perturbation theory in which the expansion parameters are made evident.

In the present paper, such a method is proposed for weakly or partially ionized media with collisional operators of imperfectly Lorentzian[21,22] or Fokker-Planck[23] types. On account of the properties of these operators, we are led to expand the electronic distribution function f_e according to the well-suited basis of irreducible Cartesian tensors of the velocity space $\overleftrightarrow{f}^{(l)}$. We obtain thus for the $\overleftrightarrow{f}^{(l)}$, instead of the kinetic equation, an infinite system of equations which is itself coupled to the Maxwell's equations.

In Sec. II, we give these general equations, point out the physical parameters, and discuss their ordering.

Then, generalizing the results of previous works,[21a] we seek solutions of this system by expanding the electric field and the electronic distribution function according to the powers of a small parameter characterizing the nonlinearity of the medium. As the utilization of such expansions makes secularities appear in the successive approximations, we are led to use the "multiple time and space scales method" of Krylov-Bogolioubov-Frieman and Sandri,[24-27] by introducing time and space scales associated with the powers of the selected small parameter. It can then be shown that one has to distinguish different kinds of behavior of the plasma according to the values of the ratio $\overline{\nu}/\omega$. We so define the socalled collisional ($\overline{\nu}/\omega \simeq 1$) and weakly collisional ($\overline{\nu}/\omega \simeq \epsilon^2$) cases, for which we analyze, in Sec. III, the initial-value and boundary-value problems. We thus see that explicit solutions can be obtained in the stationary case, only if we assume the medium weakly dissipative, a notion which is introduced in Sec. III C. In Secs. IV and V, we apply our general perturbation techniques to particular cases for which simplified expansions can be used, and we give the essential results so obtained for longitudinal and transverse waves. Finally, we discuss in Sec. VI the various possible extensions and applications as well as the domain of validity of these methods.

Thus we obtain a coherent and systematic procedure for the analysis of various collisional media which includes, as particular cases, all the approximations proposed in other papers for special problems. This technique allows one to determine accurately the order of magnitude of each term and to obtain exact kinetic expressions for the nonlinear effects of successive orders, as well as the time and space scales at which they occur. We are thus able to discuss thoroughly the role played by the collisions in various types of weakly dissipative media.

In view of the lengthly algebra involved by these techniques, this paper is only devoted to the analysis of the basic formulas of our theoretical method for a simple model (an initially homogeneous and isotropic plasma in the absence of waves), while the various particular applications—some of them are quoted in Secs. IV A, IV B 1, and VI—will be the object of further publications. Nevertheless, in order to make clearer the purposes of this paper, we wish to emphasize now several significant results:

(a) In the stationary case, our method allows one to uncouple the field and the kinetic equations at zero order and to obtain the equations verified by the isotropic part of the electronic distribution function, $F_{(0)}^{(0)}$, in which is taken into account the reaction (thermoeffect) of the waves crossing the plasma [see, for instance, Eqs. (4.3) and (4.11)]. These equations, which were never obtained before, to my knowledge, are essential for determining exactly the state of the plasma, because all the successive approximations are expressed in terms of $F_{(0)}^{(0)}$. We see thus that $F_{(0)}^{(0)}$ becomes inhomogeneous under the effect of the wave (dependence on $\vec{x}_2$ in our examples) and that we can so determine the longitudinal stationary field $\vec{e}_{(2)}''$ induced by this inhomogeneity (see Secs. IV A and IV B 2).

(b) We can then calculate, in the stationary case, the complete expression of the electric field at second (and higher) order. We so obtain different expressions for collisional and weakly collisional media [see Eq. (4.5) and the results quoted in Secs. IV A and IV B 2], which one allow to describe nonlinear effects such as the generation of harmonics and mixing of frequencies for several transverse waves interacting through the plasma.

(c) In the weakly collisional case and for an initial-value problem for instance, we can also calculate the higher-order approximations (ϵ^4 order) of the electric field for the three-wave resonant coupling. It is thus possible to obtain exact kinetic formulas for the various collisional contributions to the so-called third-order terms; we are thus able to analyze the role played by the collisions in stabilizing processes of explosively unstable media (Sec. IV B 1).

(d) Finally, we see that our method is well adapted to describe the different behaviors of the medium according to the order of magnitude of the collisional parameter $\overline{\nu}/\omega$. [Note for instance the difference between Eqs. (4.3) and (4.11).] Beside the collisional ($\overline{\nu}/\omega \simeq 1$) and weakly collisional ($\overline{\nu}/\omega \simeq \epsilon^2$) cases studied in Sec. IV, we are thus led to discuss in Sec. V, as an example, an intermediary situation (with $\overline{\nu}/\omega \simeq \epsilon$) for which we obtain new formulas for the various phase shifts [see Eq. (5.4)].

II. GENERAL EQUATIONS

Here plasmas weakly or partially ionized are studied such that we can restrict ourselves to consider the kinetic equation for the electrons only. (The conditions under which this is realized are discussed in the Appendix.) It is also assumed that this plasma contains wave phenomena characterized by a frequency and a wavelength, whose order of magnitude is ω and λ_0 respectively. Under these conditions, the evolution of such a plasma and of the electromagnetic fields propagating in it depends essentially on the following characteristic quantities: the time scale $t_\omega = 1/\omega$ and the length scale λ_0 associated with the regarded periodic phenomenon, the thermal velocity of the electrons in the plasma at rest $\overline{v} = (3kT/m_e)^{1/2}$, the electronic gyrofrequency ω_{B_0} associated to the external magnetic field B_0 if any, and finally the order of magnitude E_0 of the electric field in the plasma. As we are interested in perturbation methods, we define with these quantities reduced variables by

$$\tau = \omega t\,,\quad \vec{\mathrm{x}} = \vec{\mathrm{r}}/\lambda_0\,,\quad \vec{\mathrm{w}} = \vec{\mathrm{v}}/\overline{v}\,,\quad F_e = f_e \overline{v}^3\,,$$
$$\vec{\mathrm{e}} = \vec{\mathrm{E}}/E_0 = \vec{\Gamma}/\Gamma_0 \quad (\text{with } \vec{\Gamma} = e\vec{\mathrm{E}}/m_e)\,, \tag{2.1}$$

where $F_e(\vec{\mathrm{w}}, \vec{\mathrm{x}}, \tau)$ is the reduced electronic distribution function and $\vec{\mathrm{e}}(\vec{\mathrm{x}}, \tau)$ the reduced electric field in the plasma.

With these definitions, the electronic kinetic equation takes the form

$$\frac{\partial F_e}{\partial \tau} + \eta'^{1/2}\vec{\mathrm{w}}\cdot\vec{\nabla}_x F_e + \alpha'^{1/2}\vec{\mathrm{e}}\cdot\vec{\nabla}_w F_e + \left(\frac{\vec{\omega}_B}{\omega}\times\vec{\mathrm{w}}\right)\cdot\vec{\nabla}_w F_e = \frac{1}{\omega}\left(\frac{\delta F_e}{\delta\tau}\right)_{\text{coll.}}\,, \tag{2.2}$$

where we have put $\vec{\omega}_B = \vec{\omega}_{B_0} + \vec{\omega}_b$, $\vec{\omega}_b$ being the electronic gyrofrequency associated with the magnetic field of the wave. As in previous works,[21] we have introduced in the left-hand side of (2.2) the two dimensionless parameters α' and η' defined by

$$\alpha'^{1/2} = \Gamma_0/\overline{v}\omega\,,\quad \eta'^{1/2} = \overline{v}/\lambda_0\omega\,. \tag{2.3}$$

It will be seen that they characterize the nonlinear behavior of the medium, α' being linked to the effects due to the electric field intensity, and η' to the inhomogeneity produced by the wave propagating through the plasma. The physical meaning of these two parameters has been discussed in detail in Ref. 21a, in which we had also introduced the parameters $\alpha = \Gamma_0^2/\overline{v}^2\overline{\nu}^2$ and $\eta = \overline{v}^2/\lambda_0^2\overline{\nu}^2$, where $\overline{\nu}$ characterizes in fact the order of magnitude of electron-neutral collision frequency (see the Appendix). We remind the reader only that α' (or α) is equal to the square of the ratio of the electronic velocity increase due to the electric field during a period (or a mean free path), to the thermal velocity of the electrons with the temperature T (of neutrals); likewise η' (or η) is equal to the square of the ratio of the mean path during a period (or of the mean free path $l = \overline{v}/\overline{\nu}$) of the electron, to the inhomogeneity length λ_0.

On the right-hand side of (2.2) is the collisional operator which is generally made up of two terms.

The first is an imperfectly Lorentzian operator describing the electron-neutral (e-n) interactions in which occur two characteristic times: a time of mean free path, $t_\nu \simeq 1/\overline{\nu}$, associated with the momentum transfer (or with the transfer of higher-order anisotropies) and a relaxation time, $t_r \simeq 1/\delta\overline{\nu}$, relative to the energy transfer between electrons and the heavy component (with $\delta = 2m/M$, M being the mass of heavy particles); for more precise definitions, see Appendix formulas (A2) and (A3).

The second is a Fokker-Planck type operator which describes the Coulomb interactions between charged particles, including the electron-ion (e-i) and electron-electron (e-e) collisions: For the situations studied in this paper, the exact operators can be replaced by the simplified expressions (A11) and (A12) of the Appendix. By these formulas, it is seen that a Coulomb characteristic time $t_C \simeq \overline{v}^3/NY$ occurs, and that the relative importance of Coulomb collisions with respect to e-n collisions is determined by the dimensionless parameter $NY/\overline{v}^3\overline{\nu}$.

The meanings of the different terms of the kinetic equation (2.2) being thus specified, we have the following remarks:

(a) It is seen that this equation depends on four characteristic times, viz.: t_ω associated with the period of the wave, the two characteristic times t_ν and t_r which occur in the imperfectly Lorentzian term and the Coulomb characteristic time t_C which proceeds from the Fokker-Planck terms. In the following, it will be assumed that the degree of ionization is weak enough to have $NY/\overline{v}^3\overline{\nu} \simeq \delta$, so that the state of such a plasma will depend essentially on the two ratios $\overline{\nu}/\omega$ and $\delta\overline{\nu}/\omega$. Besides, it will be assumed in this work that $\omega \geq \overline{\nu}$, whence $\omega \gg \delta\overline{\nu}$, so that one will be able to use the "high-frequency approximation" [see Ref. 21(a)] for calculating the approximate solutions of the kinetic equation.

(b) In (2.2), only one inhomogeneity length λ_0 has been introduced. In the case of an inhomogeneous plasma, at least two characteristic lengths would be needed: one, L, linked to the density gradient of the medium; the other, λ_0, to the wavelength of the considered wave, the ratio λ_0/L being then essential. In the following, we shall study only initially homogeneous plasmas, which become inhomogeneous only in the presence of one or several waves: in this case, the inhomogeneity length is merely equal to the characteristic wavelength λ_0 which determines the parameter η'. However, with the effects of wave dissipation, a new inhomogeneity length λ_0' occurs, which is defined by means of the absorption coefficient of the wave (or a new characteristic time t_a associated with the attenuation coefficient). As the collisional part of the absorption (or attenuation) coefficient is proportional to $\overline{\nu}/\omega$, it is seen that the value of the ratio λ_0'/λ_0 is closely linked with the analysis of the influence of collisions.

(c) By Maxwell's equations, the contribution to the Lorentz force of the wave magnetic field, $(\vec{\omega}_b/\omega)\times\vec{w}$, takes the form

$$\frac{\vec{\omega}_b}{\omega}\times\vec{w} = \alpha'^{1/2}\eta'^{1/2}\left(\int^{\tau}(\vec{\nabla}_x\times\vec{e})\,d\tau'\right)\times\vec{w}\,; \tag{2.4}$$

in the absence of an external magnetic field, this is the only contribution to the Lorentz force, and one has $\vec{\omega}_B=\vec{\omega}_b$. Otherwise, if there is an external magnetic field $\vec{B}_0$, the plasma is anisotropic, and one has to introduce the additional parameter ω_{B_0} which is, as is well known, essential for the description of plasma properties. To simplify the analysis, it will be assumed in this paper that there is no external magnetic field, i.e., $\omega_{B_0}=0$ and $\vec{\omega}_B=\vec{\omega}_b$; but the methods here employed can also be applied to an anisotropic plasma, whose properties depend also on the dimensionless parameter ω_{B_0}/ω.

Finally, in this paper, a homogeneous plasma is considered (in the absence of the wave), isotropic ($\omega_{B_0}=0$), weakly or partially ionized (but with a rather weak degree of ionization), for which the electronic kinetic equation takes the form (2.2) with the Lorentz force given by (2.4). Of course this equation must be coupled to Maxwell's equations for the electric and magnetic fields.

We thus have a nonlinear integrodifferential coupled system to determine the electronic distribution function $F_e(\vec{w},\vec{x},\tau)$ and the electric field $\vec{e}(\vec{x},\tau)$ in the plasma. The complexity of this system, already very great in the absence of collisions, is increased even more in our problem by the presence of the collisional terms of the right-hand side of (2.2). Therefore, to handle this system, one is led to seek for F_e an expansion well adapted to the mathematical properties of these collisional operators. Referring to the results established in the Appendix, it is seen that it is convenient to expand the distribution function F_e in the basis of the irreducible Cartesian tensors $(\vec{w}\vec{w}\cdots\vec{w})^{(l)}$ of the velocity space [see Refs. 21(a), 22, and 23]. We thus put

$$F_e = F^{(0)}(w,\vec{x},\tau)+\vec{w}\cdot\vec{F}^{(1)}(w,\vec{x},\tau) + \cdots(\vec{w}\vec{w}\cdots\vec{w})^{(l)}:\vec{F}^{(l)}(w,\vec{x},\tau)+\cdots\,, \tag{2.5}$$

where $F^{(0)}(w,\vec{x},\tau)$ and the irreducible tensors $\vec{F}^{(l)}(w,\vec{x},\tau)$ are, respectively, the isotropic part and the anisotropies of the electronic distribution function. Carrying this expansion into (2.2), one obtains for $F^{(0)}$ and $\vec{F}^{(l)}$ after some algebra (cf. Refs. 21 and 22) the following coupled system:

$$\frac{\partial F^{(0)}}{\partial\tau}+\eta'^{1/2}\frac{w^2}{3}\vec{\nabla}_x\cdot\vec{F}^{(1)}+\frac{\alpha'^{1/2}}{3w^2}\frac{\partial}{\partial w}(w^3\vec{e}\cdot\vec{F}^{(1)})=\frac{1}{\omega}\left(\frac{\delta F^{(0)}}{\delta\tau}\right)_{\text{coll.}}, \tag{2.6}$$

$$\frac{\partial\vec{F}^{(l)}}{\partial\tau}+\eta'^{1/2}(\vec{\nabla}_x\vec{F}^{(l-1)})^0+\frac{\alpha'^{1/2}}{w}\left(\vec{e}\frac{\partial\vec{F}^{(l-1)}}{\partial w}\right)^0+\frac{l+1}{2l+3}\left(\eta'^{1/2}w^2\vec{\nabla}_x\cdot\vec{F}^{(l+1)}+\frac{\alpha'^{1/2}}{w^{2l+2}}\frac{\partial}{\partial w}(w^{2l+3}\vec{e}\cdot\vec{F}^{(l+1)})\right)$$
$$-\,l\alpha'^{1/2}\eta'^{1/2}\left(\int^{\tau}(\vec{\nabla}_x\times\vec{e})\,d\tau'\right)\times\vec{F}^{(l)}=\frac{1}{\omega}\left(\frac{\delta\vec{F}^{(l)}}{\delta\tau}\right)_{\text{coll.}},\quad l\geq 1, \tag{2.7}$$

where the notation $(\cdots)^0$ means a symmetric divergenceless tensor. To complete this system, we have to specify the collisional terms on the right-hand sides. Under conditions which are made more precise in the Appendix, these terms are given, for the first values of l, by the expressions (A11) and (A12) set up for partially ionized media. In this rather general case, we have thus an infinite integrodifferential coupled system for determining $F^{(0)}$ and the various $\vec{F}^{(l)}$.

But in the particular case where the ionization degree of the plasma is weak enough to have $NY/\overline{v}^3\overline{\nu}\ll 1$, it is seen in the Appendix that the collisional terms between charged particles may be neglected except in the equation for $F^{(0)}$ and that we can use simpler expressions (see Appendix). Then the integrodifferential equation (2.2) is replaced by the infinite system (2.6) and (2.7) in which the right-hand sides must be written

$$\frac{1}{\omega}\left(\frac{\delta F^{(0)}}{\delta\tau}\right)_{\text{coll.}}=\frac{\delta\overline{\nu}}{2\omega}\frac{1}{w^2}\frac{\partial}{\partial w}\left[\nu_1'w^2\left(wF^{(0)}+\frac{1}{3}\frac{\partial F^{(0)}}{\partial w}\right)\right]+\frac{NY}{\overline{v}^3\omega}\frac{1}{w^2}\frac{\partial}{\partial w}\left(F^{(0)}I_0'^{\,0}+\frac{w}{3}(I_2'^{\,0}+J_{-1}'^{\,0})\frac{\partial F^{(0)}}{\partial w}\right)$$
$$\equiv\frac{\delta\overline{\nu}}{\omega}I(F^{(0)})+\frac{NY}{\overline{v}^3\omega}C_{ee}(F^{(0)}),$$
$$\frac{1}{\omega}\left(\frac{\delta\vec{F}^{(l)}}{\delta\tau}\right)_{\text{coll.}}=-\frac{\overline{\nu}}{\omega}\nu_l'\vec{F}^{(l)},\quad l\geq 1, \tag{2.8}$$

where the $\nu_l'(v)$ are defined by (A4).

In any case, the system (2.6) and (2.7) must be coupled with Maxwell's equations which, after having eliminated the magnetic field, may be written in the form

$$\frac{1}{\lambda_0^2}\vec{\nabla}_x\times(\vec{\nabla}_x\times\vec{e})+\frac{\omega^2}{c^2}\frac{\partial^2\vec{e}}{\partial\tau^2}=-\frac{4\pi e}{mc^2\bar{v}}(\alpha')^{-1/2}\frac{\partial\vec{j}}{\partial\tau}, \quad (2.9)$$

$$\alpha'^{1/2}\eta'^{1/2}\omega^2\vec{\nabla}_x\cdot\vec{e}=\frac{4\pi e}{m}\rho, \quad (2.10)$$

with the equation of continuity

$$\frac{\partial\rho}{\partial\tau}+\frac{\eta'^{1/2}}{\bar{v}}\vec{\nabla}_x\cdot\vec{j}=0. \quad (2.11)$$

With these notations, the electronic charge and current densities are defined by

$$\rho=4\pi e\int_0^\infty w^2F^{(0)}\,dw-Ne=e(n-N),$$
$$\vec{j}=\frac{4\pi e\bar{v}}{3}\int_0^\infty w^4\vec{F}^{(1)}\,dw, \quad (2.12)$$

where N is the density of the positive background.

The equations (2.6)–(2.12) constitute the fundamental system of this problem; its complexity is so great that it cannot be rigorously solved in the general case and that it is necessary to use perturbation expansions. The presence of the two parameters α' and η', which couple between them the equations of (2.6) and (2.7), suggests that one expands the $\vec{F}^{(l)}$ according to these parameters which are assumed small; this procedure will have a precise physical meaning by virtue of the definitions of α' and η'. As, on the other hand, this fundamental system depends also on the three other parameters δ, $\bar{\nu}/\omega$, and $NY/\bar{v}^3\omega$, it is seen that very different physical situations can occur depending on the respective orders of magnitude of these five parameters. Thus, for a particular problem, it must at first be decided which is the small parameter ϵ adequate for the considered situation, by ordering between them the five parameters α', η', δ, $\bar{\nu}/\omega$, and $NY/\bar{v}^3\omega$.

In order to make easier the statement of our perturbation method, it will be always assumed in the following that the degree of ionization is sufficiently small to have $NY/\bar{v}^3\bar{\nu}\simeq\delta$. With this condition, the differential system (2.6)–(2.8) depends only on four parameters: α' and η' on the one hand; $\bar{\nu}/\omega$ and $\delta\bar{\nu}/\omega$ on the other hand which are respectively associated with the two characteristic times $t_\nu\simeq1/\bar{\nu}$ and $t_r\simeq1/\delta\bar{\nu}$. More precisely, it is seen that the equations for the anisotropies $\vec{F}^{(l)}$ depend on the ratio $\bar{\nu}/\omega=t_\omega/t_\nu$, which is a measure of the order of magnitude of the momentum transfer, while the equation for the isotropic part $F^{(0)}$ depends on $\delta\bar{\nu}/\omega=t_\omega/t_r$, which characterizes the energy transfer for the e-n collisions.

Now, for the parameters α' and η', the following situations may be distinguished:

(i) $\eta'\ll\alpha'$, to which corresponds the condition $\bar{v}^2\ll\Gamma_0\lambda_0$: This is the case for cold plasmas with relatively long wavelengths or relatively strong electric fields; in this case, the terms in $\vec{\nabla}_x$ can be neglected in the first approximation in the system (2.6) and (2.7) which becomes thus homogeneous.

(ii) $\alpha'\ll\eta'$, corresponding to the condition $\Gamma_0\lambda_0\ll\bar{v}^2$: This case is realistic for warm plasmas with relatively short waves or weak electric fields; in this case, the effects of the plasma inhomogeneity are dominant, and no harmonics occur in the first approximation.

(iii) $\alpha'\simeq\eta'\ll1$: This is the case studied in this paper, in which the effects due to the intensity and to the inhomogeneity of the electric field are of the same order of magnitude.

The orders of magnitude of α' and η' being thus fixed, one has still to compare them to δ. In the Secs. III–V, only the case $\alpha'\simeq\eta'\simeq\delta$ will be studied, for which the order of magnitude of nonlinear effects and of energy exchanges between electrons and neutrals can be compared. As will be seen, this condition is satisfied in warm plasmas (with $\bar{v}^2/c^2\simeq\delta$) and for wave amplitude and frequency satisfying the relation $e^2E_0^2/3mkT\omega^2\simeq\delta$. In Sec. VI, the other physical situations will be briefly discussed.

Finally, one has to consider the collision parameter $\bar{\nu}/\omega$, which is essential for the analysis of the dissipative effects of the waves in weakly ionized plasmas. According to the value of this parameter, the "collisional" case, with $\bar{\nu}/\omega\simeq1$, and different "weakly collisional" cases, with $\bar{\nu}/\omega\ll1$, will be successively studied. In these latter cases, one has to compare the orders of magnitude of $\bar{\nu}/\omega$ and δ; we shall consider, in Sec. IV B, the case $\bar{\nu}/\omega\simeq\delta$ which is called precisely "weakly collisional" and, in Sec. V, the "intermediary" case $\bar{\nu}/\omega\simeq\delta^{1/2}$, which may be interesting for the study of the coupling of several waves.

III. PERTURBATION METHODS IN THE CASE $\alpha'\simeq\eta'\simeq\delta$, $NY/\bar{v}^3\bar{\nu}\simeq\delta$

This section is devoted to the analysis of the perturbation method proposed for solving the fundamental system (2.6)–(2.12) in the case $\alpha'\simeq\eta'\simeq\delta$ and $NY/\bar{v}^3\bar{\nu}\simeq\delta$. It is discussed for two values of the parameter $\bar{\nu}/\omega$, which may be of the order of unity (collisional plasma) or of the order δ (weakly collisional plasma). It will be seen later that to solve effectively the equations, especially for continuous waves, it must be assumed that the absorption (or attenuation) coefficient of the waves is small, that will lead to the definition of weakly dissipative media and to the analysis in detail of the actual effects of the collisions.

By virtue of the structure of (2.6) and (2.7), one is induced to choose, as the small parameter, $\epsilon = \alpha'^{1/2} \simeq \delta^{1/2}$, and to expand $F^{(0)}$ and $\vec{F}^{(l)}$, on the one hand, and the electric field $\vec{e}$ on the other hand, according to successive powers of ϵ. But with such a procedure, it is known that secularities appear in the calculation from the second approximation. Thus, by relying upon the existence of the two time scales t_ν and t_r, with $t_\nu \simeq \epsilon^2 t_r$, one is led to use the well-known method of "multiple time and space scales" of Krylov-Bogolioubov-Frieman-Sandri[24-27] which proceeds from nonlinear dynamics; this method, which has been largely used in statistical mechanics and plasma physics, especially for the study of mode coupling,[28-31] has been applied for the first time to the weakly ionized plasmas by Caldirola *et al.*[32]

According to this method, we at first define time scales $\tau_0, \tau_1, \tau_2, \ldots$, as well as the corresponding space scales $\vec{x}_0, \vec{x}_1, \vec{x}_2, \ldots$ by putting

$$\begin{aligned}&\tau \equiv \tau_0, \quad \tau_1 = \epsilon\tau_0, \quad \tau_2 = \epsilon^2\tau_0, \ldots;\\ &\vec{x} \equiv \vec{x}_0, \quad \vec{x}_1 = \epsilon\vec{x}_0, \quad \vec{x}_2 = \epsilon^2\vec{x}_0, \ldots;\end{aligned} \tag{3.1}$$

it follows that we have for the derivatives $\partial/\partial\tau$ and $\vec{\nabla}_x \equiv \partial/\partial\vec{x}$:

$$\begin{aligned}&\frac{\partial}{\partial\tau} = \frac{\partial}{\partial\tau_0} + \epsilon\frac{\partial}{\partial\tau_1} + \epsilon^2\frac{\partial}{\partial\tau_2} + \cdots,\\ &\frac{\partial}{\partial\vec{x}} = \frac{\partial}{\partial\vec{x}_0} + \epsilon\frac{\partial}{\partial\vec{x}_1} + \epsilon^2\frac{\partial}{\partial\vec{x}_2} + \cdots,\end{aligned} \tag{3.2}$$

and that the unknown functions $F^{(0)}(\tau, \vec{x}, w)$, $\vec{F}^{(l)}(\tau, \vec{x}, w)$, and $\vec{e}(\tau, \vec{x})$ of this problem become functions of these new variables, viz.,

$$\begin{aligned}&F^{(0)}(\tau_0\vec{x}_0, \tau_1\vec{x}_1, \ldots; w),\\ &\vec{F}^{(l)}(\tau_0\vec{x}_0, \tau_1\vec{x}_1, \ldots; w),\\ &\vec{e}(\tau_0\vec{x}_0, \tau_1\vec{x}_1, \ldots).\end{aligned}$$

This being the case, we seek for the solutions of the system (2.6)–(2.12) expansions of the form

$$F^{(0)} = F^{(0)}_{(0)} + \epsilon F^{(0)}_{(1)} + \epsilon^2 F^{(0)}_{(2)} + \cdots, \tag{3.3}$$

$$\vec{F}^{(l)} = \vec{F}^{(l)}_{(0)} + \epsilon\vec{F}^{(l)}_{(1)} + \epsilon^2\vec{F}^{(l)}_{(2)} + \cdots, \quad l > 0, \tag{3.4}$$

$$\vec{e} = \vec{e}_{(0)} + \epsilon\vec{e}_{(1)} + \epsilon^2\vec{e}_{(2)} + \cdots, \tag{3.5}$$

where each approximation, $F^{(0)}_{(n)}$, $\vec{F}^{(l)}_{(n)}$, or $\vec{e}_{(n)}$ is a function of $\tau_0\vec{x}_0, \tau_1\vec{x}_1, \ldots$. Putting these expansions into (2.6)–(2.12), this fundamental system is split into a sequence of equations which can be solved step by step, by annulling the secularities in each of these equations. One can thus calculate the successive approximations of the charge density ρ and current density $\vec{j}$; by (3.3) and (3.4), they are written

$$\begin{aligned}\rho &= \rho_{(0)} + \epsilon\rho_{(1)} + \epsilon^2\rho_{(2)} + \cdots\\ &= e(n_{(0)} - N) + \epsilon e n_{(1)} + \epsilon^2 e n_{(2)} + \cdots,\\ \vec{j} &= \vec{j}_{(0)} + \epsilon\vec{j}_{(1)} + \epsilon^2\vec{j}_{(2)} + \cdots.\end{aligned} \tag{3.6}$$

Of course, one has to distinguish several cases according to the order of magnitude of $\bar{\nu}/\omega$; in the following, the "collisional case," $\bar{\nu}/\omega \simeq 1$, and the "weakly collisional case," $\bar{\nu}/\omega \simeq \delta \simeq \epsilon^2$, are successively studied.

A. Collisional case: $\bar{\nu}/\omega \simeq 1$

This case is essentially characterized by the following properties: (i) the anisotropies relax at the time scale τ_0; (ii) the field equations, at zero order, include dissipative terms which depend on $F^{(0)}_{(0)}$; (iii) for continuous transverse waves, these equations and the equation for $F^{(0)}_{(0)}$ are not uncoupled, except if the absorption coefficient is small. To verify these different points, one has to consider the general equations obtained by putting (3.3)–(3.6) into the whole system (2.6)–(2.12). We cannot give here all the calculations, which are long and tedious; so we limit ourselves to pointing out the essential results and stages of argumentation (for a more detailed discussion, see Ref. 33).

Let us first consider the kinetic equations (2.6) and (2.7); starting with the equations of zero order

$$\frac{\partial F^{(0)}_{(0)}}{\partial\tau_0} = 0, \quad \frac{\partial\vec{F}^{(l)}_{(0)}}{\partial\tau_0} = -\frac{\nu_l}{\omega}\vec{F}^{(l)}_{(0)}, \quad \frac{\partial\rho_{(0)}}{\partial\tau_0} = 0, \tag{3.7}$$

it is seen that $F^{(0)}_{(0)}$ and $\rho_{(0)}$ are independent of τ_0, while the variation of $\vec{F}^{(l)}_{(0)}$, at the scale τ_0, is given by the exponential $e^{-(\nu_l/\omega)\tau_0}$ which goes to zero when $\tau_0 \to \infty$. Thus, one can write, by (3.7),

$$\begin{aligned}&F^{(0)}_{(0)} = F^{(0)}_{(0)}(\vec{x}_0, \tau_1\vec{x}_1, \ldots; w),\\ &\rho_{(0)} = \rho_{(0)}(\vec{x}_0, \tau_1\vec{x}_1, \ldots),\\ &\vec{F}^{(l)}_{(0)} = \vec{F}^{(l)}_{(0,0)} e^{-(\nu_l/\omega)\tau_0},\end{aligned} \tag{3.7$'$}$$

so that one has $\vec{F}^{(l)}_{(0)}(\tau_0 \to \infty) = 0$; so, we check the relaxation of anisotropies of zero order at time scale τ_0.[22,34]

Then, using the successive equations of higher orders and assuming that the medium is homogeneous in the absence of the wave (viz., $\vec{F}^{(l)}_{(0,0)}$ independent of $\vec{x}_0$), it can be shown more generally that

$$\vec{F}^{(l)}_{(p)}(\tau_0 \to \infty) = 0, \quad p < l; \tag{3.8}$$

so, all the anisotropies $\vec{F}^{(l)}_{(p)}$ (with $p < l$) relax also at the time scale τ_0; therefore, it follows from this result that $\vec{F}^{(l)}_{(l)}$ is the first term different from zero in the expansion (3.4), if the transient terms

$\vec{F}^{(l)}_{(p)}$ $(p<l)$ are neglected.

This being the case, it can still be shown that $F^{(0)}_{(0)}$ depends only on τ_2 and eventually on other time scales of higher order, and that $\vec{F}^{(1)}_{(1)}$ and $\vec{e}_{(0)}$ can be split into two components, according to

$$\begin{aligned}\vec{F}^{(1)}_{(1)} &= \vec{F}^{(1)\prime}_{(1)}(\tau_0\vec{x}_0, \vec{x}_1, \tau_2\vec{x}_2, \ldots ; w) \\ &\quad + \vec{F}^{(1)\prime\prime}_{(1)}(\vec{x}_0, \vec{x}_1, \tau_2\vec{x}_2, \ldots ; w) , \\ \vec{e}_{(0)} &= \vec{e}'_{(0)}(\tau_0\vec{x}_0, \vec{x}_1, \tau_2\vec{x}_2, \ldots) \\ &\quad + \vec{e}''_{(0)}(\vec{x}_0, \vec{x}_1, \tau_2\vec{x}_2, \ldots) ,\end{aligned} \tag{3.9}$$

where $\vec{F}^{(1)\prime\prime}_{(1)}$ and $\vec{e}''_{(0)}$ do not depend on the time variable τ_0; $\vec{e}''_{(0)}$ is the zero-order stationary field (at time scale τ_0) generated by the wave $\vec{e}'_{(0)}$ crossing the plasma. Note that one has also such decompositions for anisotropies and approximations of higher order.

Considering now the field equations (2.9) and (2.10) for which we seek periodic solutions, we obtain for the oscillating electric field $\vec{e}'_{(0)}$ at zero order the two equations

$$\omega^2 \frac{\partial^2 \vec{e}'_{(0)\parallel}}{\partial \tau_0^2} = -i\omega_p^2 \omega_1 \sigma(\omega_1) \vec{e}'_{(0)\parallel} , \tag{3.10a}$$

$$\omega^2 \frac{\partial^2 \vec{e}'_{(0)\perp}}{\partial \tau_0^2} - \frac{c^2}{\lambda_0^2} \frac{\partial^2 \vec{e}'_{(0)\perp}}{\partial \vec{x}_0^2} = -i\omega_p^2 \omega_1 \sigma(\omega_1) \vec{e}'_{(0)\perp} , \tag{3.10b}$$

in which $\sigma(\omega_1)$ is the usual conductivity defined by

$$\sigma(\omega_1) = -\frac{4\pi}{3N}\int_0^\infty \frac{w^3}{\nu_1 + i\omega_1} \frac{\partial F^{(0)}_{(0)}}{\partial w} dw , \tag{3.11}$$

and where $\vec{e}'_{(0)\parallel}$ and $\vec{e}'_{(0)\perp}$ are, respectively, the longitudinal and transverse components of $\vec{e}'_{(0)}$.

So, one finds again, at order zero, the usual equations for the wave propagation, but with a dissipative term which proceeds from the real part of $\sigma(\omega_1)$. This term defines either an attenuation, if the shape of the wave is given at an initial time (initial-value problem), or an absorption if the wave amplitude is fixed on a certain surface, for instance the plane $x=0$ (boundary-value problem). In the first case, ω_1 is complex and the wave vector $\vec{K}_0$ real, whereas in the second case it is $\vec{K}_0$ which is complex and ω_1 real.

For an *initial-value problem*, one can put $\omega_1 = \omega_{1R} + i\omega_{1I}$ and $\sigma(\omega_1) = \bar{\nu}\sigma'_R - i\omega_1^*\sigma'_I$, with obvious definitions for σ'_R and σ'_I by (3.11). As the complex conductivity σ is independent of τ_0, it is seen that one can always solve, in this case, the field equations at zero order independently of the evolution of $F^{(0)}_{(0)}$; in fact, the influence of the electric field $\vec{e}'_{(0)}$ on $F^{(0)}_{(0)}$ appears only at the time scale τ_2.

If ω_{1I} is of the same order as ω_{1R}, one has an attenuation at the time scale τ_0; of course, this case is of no physical interest, the field being damped during a wave period. Therefore, only situations for which $\omega_{1I}^2 \ll \omega_{1R}^2$ are considered; in the longitudinal case, this condition entails $\bar{\nu}^2 \ll \omega_{1R}^2$, so that one has the usual plasma oscillations, namely $\omega_{1R} \simeq \omega_p$, with the attenuation coefficient ω_{1I}/ω given by

$$\begin{aligned}\frac{\omega_{1I}}{\omega} &\simeq \frac{\bar{\nu}}{2\omega}\tau'_R , \\ \tau'_R \equiv \bar{\nu}'_1 &= -\frac{4\pi}{3N}\int_0^\infty \nu'_1 w^3 \frac{\partial F^{(0)}_{(0)}}{\partial w} dw ;\end{aligned} \tag{3.12}$$

in the transverse case, it is found with the same condition:

$$\begin{aligned}\frac{\omega_{1I}}{\omega} &\simeq \frac{\omega_p^2}{\omega_{1R}^2}\frac{\bar{\nu}}{2\omega}\sigma''_R , \\ \sigma''_R &= -\frac{4\pi}{3N}\int_0^\infty \frac{\nu'_1 w^3}{1+\nu_1^2/\omega_{1R}^2}\frac{\partial F^{(0)}_{(0)}}{\partial w} dw .\end{aligned} \tag{3.13}$$

It is thus checked that in order to satisfy the condition $\omega_{1I}^2 \ll \omega_{1R}^2$, one has two different situations: (a) for longitudinal waves, one must have necessarily a weakly collisional plasma, with $\bar{\nu}^2 \ll \omega_{1R}^2$; (b) for transverse waves, one can have either a collisional plasma ($\bar{\nu}/\omega \simeq 1$) with waves satisfying the condition $\omega_p^2/\omega_{1R}^2 \ll 1$, or once again a weakly collisional plasma.

As a *boundary-value problem*, we consider the case of continuous transverse waves, with ω_1 real and $\vec{K}_0$ complex. Then, by putting $\sigma(\omega_1) = (1/\omega_1^2) \times (\bar{\nu}\sigma''_R - i\omega_1\sigma''_I)$ with σ''_R and σ''_I defined by (3.11), Eq. (3.10b) yields the usual equation of propagation of transverse waves:

$$\frac{c^2}{\lambda_0^2}\frac{\partial^2 \vec{e}'^{\,0}_{(0)\perp}}{\partial \vec{x}_0^2} + \omega_1^2\left[1 - \frac{\omega_p^2}{\omega_1^2}\left(\sigma''_I + i\frac{\bar{\nu}}{\omega_1}\sigma''_R\right)\right]\vec{e}'^{\,0}_{(0)\perp} = 0 , \tag{3.14}$$

where $\vec{e}'_{(0)} = \vec{e}'^{\,0}_{(0)}(\vec{x}_0, \vec{x}_1, \ldots)e^{i(\omega_1/\omega)\tau_0} + \text{c.c.}$ But, in (3.14), σ''_R and σ''_I are now functions of $\vec{x}_0$ through $F^{(0)}_{(0)}$: Then, it is *not* possible, in the general case, to solve (3.14) independently of the knowledge of $F^{(0)}_{(0)}$.

On the other hand, the equation for $F^{(0)}_{(0)}$ is deduced from the equation (2.6) at second order, in which the terms varying with τ_0 must be separated from the stationary ones. One thus gets an equation in which occurs the wave absorption by the intermediate of the term $\vec{e}'^{\,0}_{(0)\perp} \cdot \vec{e}'^{\,0*}_{(0)\perp}$, so that this equation is coupled with the field equation (3.14). Moreover, the Poisson and continuity equations give two supplementary conditions: first, one has, by (2.10), (3.5), and (3.6), $\rho_{(0)} = \rho_{(1)} = 0$, whence $n_{(0)} = N$, $n_{(1)} = 0$, which determine the normalization of $F^{(0)}_{(0)}$ and show that the electronic density at zero order remains constant and equal to N; second, the continuity equation at second order allows one to determine the longitudinal induced field $\vec{e}''_{(0)}$, which corresponds to a thermoelectric effect of the wave crossing the plasma.

So, in the general case, one has to solve simultaneously this system of coupled equations, which is a very difficult problem. But if $F_{(0)}^{(0)}$ is a "slowly varying" function of $\vec{x}_0$, one can seek solutions for (3.14) of the form of the first approximation of geometrical optics, in which the complex wave vector $\vec{K}_0 = \vec{K}_{0R} - i\vec{K}_{0I}$ is a slowly varying function of $\vec{x}_0$. In such a case, it can be seen by the equation for $F_{(0)}^{(0)}$ that one must have $K_{0I}^2 \ll K_{0R}^2$, which corresponds to a weak absorption. With this condition, one can separately solve Eq. (3.14) and the equation for $F_{(0)}^{(0)}$; one thus gets the usual dispersion equation which yields for K_{0R} and K_{0I} the approximate relations

$$\omega_1^2 \simeq \omega_p^2 \sigma_I'' + \frac{c^2 K_{0R}^2}{\lambda_0^2}, \tag{3.15a}$$

$$K_{0I} \simeq \frac{\omega_p^2}{\omega_1^2} \frac{\bar{\nu}}{\omega_1} \frac{K_{0R}\sigma_R''}{2[1-(\omega_p^2/\omega_1^2)\sigma_I'']}, \tag{3.15b}$$

where σ_R'' is given by (3.13), with $\omega_{1R} = \omega_1$.

By (3.15b), it is seen that the condition for weak absorption can again be fulfilled in two cases: (a) a collisional plasma with waves satisfying the condition $\omega_p^2/\omega_1^2 \ll 1$; (b) a weakly collisional medium, with $\bar{\nu}/\omega_1 \ll 1$.

B. Weakly collisional case: $\bar{\nu}/\omega \simeq \delta \simeq \epsilon^2$

In this case, one has to consider the same successive equations as in Sec. III A, but with collisional terms which are now incorporated into higher-order equations. This fact entails two important consequences: (1) The field equations at zero order have no dissipative terms and the uncoupling between these equations and the $F_{(0)}^{(0)}$ equation is now insured by the condition $\bar{\nu}/\omega \simeq \delta \simeq \epsilon^2$. (2) The $\vec{F}_{(0)}^{(l)}$ do not relax at the time scale τ_0 but at the time scale τ_2, so that these initial anisotropies may not be neglected in the general case, for an initial-value problem.

Indeed, the zero-order equations (3.7) relative to $F_{(0)}^{(0)}$ and $\rho_{(0)}$ are still valid, while the equation for $\vec{F}_{(0)}^{(l)}$ becomes

$$\frac{\partial \vec{F}_{(0)}^{(l)}}{\partial \tau_0} = 0,$$

whence

$$\vec{F}_{(0)}^{(l)} = \vec{F}_{(0)}^{(l)}(\vec{x}_0, \tau_1 \vec{x}_1, \ldots; w), \tag{3.16}$$

so that $\vec{F}_{(0)}^{(l)}$ is now independent of τ_0 as $F_{(0)}^{(0)}$ and $\rho_{(0)}$. Then by applying the usual technique of the multiple time or space scales method to the kinetic equations of first and second order and assuming that the plasma is initially homogeneous (in the absence of waves), it can be shown[33]: (i) that $F_{(0)}^{(0)}$ is independent of τ_1 and that decompositions of the type (3.9) can also be used (but with oscillating terms depending both on τ_0 and τ_1), and (ii) that $\vec{F}_{(0)}^{(l)}$ now relaxes at the scale τ_2, according to

$$\frac{\partial \vec{F}_{(0)}^{(l)}}{\partial \tau_2} = -\nu_l' \vec{F}_{(0)}^{(l)},$$

whence

$$\vec{F}_{(0)}^{(l)} = \vec{F}_{(0,0)}^{(l)} e^{-\nu_l' \tau_2}, \tag{3.17}$$

so that the relations (3.8) are no longer valid.

On the other hand, the field equations at zero order are now given by

$$\omega^2 \frac{\partial^2 \vec{e}'_{(0)\parallel}}{\partial \tau_0^2} = -\omega_p^2 \vec{e}'_{(0)\parallel}, \tag{3.18a}$$

$$\omega^2 \frac{\partial^2 \vec{e}'_{(0)\perp}}{\partial \tau_0^2} - \frac{c^2}{\lambda_0^2} \frac{\partial^2 \vec{e}'_{(0)\perp}}{\partial \vec{x}_0^2} = -\omega_p^2 \vec{e}'_{(0)\perp}, \tag{3.18b}$$

which are independent of the $F_{(0)}^{(0)}$ equation; one thus gets, at zero order, the usual dispersion relations. Then the field equations at first order allow one to calculate the variation of $\vec{e}'_{(0)}$ at the scales τ_1 and $\vec{x}_1$, which is due to the coupling between the oscillating field and the initial anisotropies $\vec{F}_{(0)}^{(1)}$, $\vec{F}_{(0)}^{(2)}$.

For an initial-value problem, it can be shown[33(b)] that there occur frequency shifts of order ϵ which are due to this coupling. But if it is *assumed* that the initial anisotropy is only due to the electric oscillating field, one must put $\vec{F}_{(0)}^{(l)} = 0$ and one can establish, by considering the higher-order equations, the following general results: (a) one has $\vec{F}_{(p)}^{(l)} = 0$, with $p < l$, so that $\vec{F}_{(l)}^{(l)}$ is again the first term different from zero in the expansion (3.4); (b) one can eliminate the odd scales $\tau_1\vec{x}_1$, $\tau_3\vec{x}_3$, etc., so that one will be able to replace (3.3)–(3.5) by expansions according to ϵ^2.

For a boundary-value problem, one must consider the equation for $F_{(0)}^{(0)}$: This one is now deduced from the equation of fourth order (relative to $F^{(0)}$) since, as we have $\delta\bar{\nu}/\omega \simeq \epsilon^4$, it is the time scale τ_4 which is characteristic for the evolution of $F_{(0)}^{(0)}$. But, at this scale, all the anisotropies $\vec{F}_{(p)}^{(l)}$ (with $p < l$) are cancelled by (3.17) and similar relations at higher orders; so, $\vec{F}_{(l)}^{(l)}$ is again the first term different from zero in (3.4), and a stationary state is set up in which $F_{(0)}^{(0)}$ is generally non-Maxwellian (as in the collisional case) and depends only on the even length scales $\vec{x}_2$, $\vec{x}_4$, etc.

C. Weakly dissipative media: Effect of collisions

From the previous results, it follows clearly that the nature of the solutions of our fundamental system (2.6)–(2.12) is closely linked with the order of magnitude of the coefficients of attenuation ω_{1I} or of absorption K_{0I}. One is thus led to define weakly dissipative media, for which these coefficients are small compared to the corresponding real parts ω_{1R} or K_{0R}; particularly, it has been seen that this condition is necessary, for a stationary situation, to uncouple the field equations at zero order and the equation for $F^{(0)}_{(0)}$, and to obtain solutions "slowly-varying" in $\vec{x}_0$. But, for such a case, the methods of geometrical optics are in fact not well adapted because one is dealing with a nonlinear problem. Thus, one is led to use the method of multiple time and space scales and to associate with ω_{1I} or K_{0I} a new characteristic time or length which will be large with respect to t_ω or λ_0.

So for an initial-value problem, one introduces the new time scale $t'_\omega \simeq t_\omega \omega_{1R}/\omega_{1I} \gg t_\omega$ and one has to compare the ratio ω_{1I}/ω_{1R} with the small parameter ϵ of the problem. Likewise, for a boundary-value problem, one associates with K_{0I} the new characteristic length $\lambda'_0 \simeq \lambda_0 K_{0R}/K_{0I}$ and one must compare the ratio K_{0I}/K_{0R} with ϵ. Then, by the approximate formulas (3.12) and (3.13), it is seen that the ratios K_{0I}/K_{0R} and ω_{1I}/ω_{1R} have the following properties: (a) they are proportional to τ'_R or σ''_R and thus closely dependent on the $(e\text{-}n)$ interaction law by means of the reduced collision frequency relative to the momentum transfer; (b) their order of magnitude is determined by that of $\bar{\nu}/\omega$ in the two cases, and by that of ω_p^2/ω_1^2 (or ω_p^2/ω_{1R}^2) in the case of transverse waves.

Finally, one can study weakly dissipative media such as either K_{0I}/K_{0R} (or $\omega_{1I}/\omega_{1R}) \simeq \epsilon = \alpha'^{1/2} \simeq \delta^{1/2}$, or K_{0I}/K_{0R} (or $\omega_{1I}/\omega_{1R}) \simeq \epsilon^2 = \alpha' \simeq \delta$. In the latter case, it follows from the previous arguments that one has to consider the two following cases:

(a) Weakly dissipative collisional media, with transverse waves such as $\bar{\nu}/\omega \simeq 1$, $\omega_p^2/\omega^2 \simeq \epsilon^2 (\simeq \alpha')$. Here, it is the condition $\omega_p^2/\omega^2 \simeq \epsilon^2$ which allows one to uncouple the $F^{(0)}_{(0)}$ equation and the field equations at zero order, for a boundary-value problem. Moreover, this condition entails that the solutions of the system depend only on the even time or space scales. It follows that, if one neglects the vanishing transient terms (at the scale τ_0), one can replace (3.3)–(3.5) by expansions in ϵ^2 which may be written

$$\vec{F}^{(l)}(\tau_0\vec{x}_0, \tau_2\vec{x}_2, \ldots, w) = \epsilon^l (\vec{F}^{(l)}_{(l)} + \epsilon^2 \vec{F}^{(l)}_{(l+2)} + \cdots), \quad l = 0, 1, 2, \ldots, \qquad (3.19)$$

$$\vec{e}(\tau_0\vec{x}_0, \tau_2\vec{x}_2, \ldots) = \vec{e}_{(0)} + \epsilon^2 \vec{e}_{(2)} + \cdots. \qquad (3.20)$$

(b) Weakly collisional media, for which one has $\bar{\nu}/\omega \simeq \epsilon^2 (= \alpha')$. For a boundary-value problem, this condition is sufficient to get uncoupled equations at zero order and it has been seen that one can use expansions in ϵ^2 of the type (3.19) and (3.20). For an initial-value problem, this is only possible if one *assumes* that the initial anisotropies are due to the existence of the oscillating field $\vec{e}'_{(0)}$; when this is not true, one must use the more general expansions (3.3)–(3.5).

For plasmas with a dissipation of order ϵ, one can consider an "intermediary case" characterized by the condition $\bar{\nu}/\omega \simeq \epsilon$ $(= \alpha'^{1/2})$, from which it follows that ω_{1I}/ω_{1R} (or $K_{0I}/K_{0R}) \simeq \epsilon$ in the absence of any assumption on the magnitude of ω_p^2/ω^2. The analysis of this situation is outlined in Sec. V and can be made with a method similar to that employed for the weakly collisional case.

IV. EXPANSIONS IN ϵ^2: COLLISIONAL AND WEAKLY COLLISIONAL CASES

In view of the previous discussion, one can use the expansions (3.19) and (3.20) to study the collisional case with transverse waves fulfilling the condition $\omega_p^2/\omega^2 \simeq \epsilon^2$, and the weakly collisional case when the initial anisotropy of the medium is only due to the oscillating field $\vec{e}'_{(0)}$. Of course, one must then put (3.19) and (3.20) in the general system (2.6) and (2.7), and solve step by step the successive equations so obtained, by eliminating at each stage of the calculation the secularities which appear. We do not give here the details of these calculations (as are shown in Refs. 33 and 35), and we note in this section only the main results that we can so obtain.

A. Collisional case

If one makes appear in the source term of (2.9) the ratio $\omega_p^2/\omega^2 \simeq \epsilon^2$, it is seen that the field equation at zero order is nothing other than the field equation in the vacuum; this means that, at this scale, the electric field is not modified by the plasma, whose effects occur in the higher-order equations. We consider in the following a boundary-value problem in which the transverse wave is given on the plane $x = 0$ and propagates along $0x$ in the half-plane $x > 0$; thus, one has only one time variable $\tau = \tau_0$, so that one has to cancel all the terms $\partial/\partial\tau_2, \partial/\partial\tau_4, \ldots$ in the successive equations.

This being the case, one finds for the electric oscillating field $\vec{e}'_{(0)\perp}$ at zero order:

$$\vec{e}'_{(0)\perp} = \vec{u}_{(0)\perp} e^{i\varphi} e^{-\beta(\vec{x}_2) + i\varphi_1(\vec{x}_2)} + \text{c.c.} \qquad (4.1)$$

with $\varphi = (\omega_1/\omega)\tau_0 - \vec{K}_0 \cdot \vec{x}_0 + \varphi_0$, $\vec{K}_0 \cdot \vec{u}_{(0)\perp} = 0$, where φ_0 is an initial phase, and $K_0 = |\vec{K}_0| = \lambda_0 \omega_1/c$ is the real wave vector at zero order. In (4.1), the absorption coefficient $\beta(\vec{x}_2)$ and the phase shift $\varphi_1(\vec{x}_2)$ are given by

$$\beta(\vec{x}_2)=\frac{K_0}{2}\frac{\omega^2}{\omega_1^2}\frac{\bar{\nu}}{\omega_1}\int^{x_2}\sigma_R''\,dx_2',$$
$$\varphi_1(\vec{x}_2)=\frac{K_0}{2}\frac{\omega^2}{\omega_1^2}\int^{x_2}\sigma_I''\,dx_2', \tag{4.2}$$

in which σ_R'' and σ_I'', defined in Sec. III, are functions of $\vec{x}_2$ through $F_{(0)}^{(0)}$. The phase shift $\varphi_1(\vec{x}_2)$ allows one to obtain the dispersion equation at ϵ^2 order for transverse waves.

On the other hand, the equation for $F_{(0)}^{(0)}$ is deduced from the equation (2.6) at second order by using the method outlined in Sec. III A. At first, it is seen that one gets for the stationary components $\vec{e}_{(0)}''=\vec{F}_{(1)}^{(1)''}=0$, while the Poisson equation yields $n_{(0)}=N$, by virtue of the transversality of the waves. Then, one gets the equation for $F_{(0)}^{(0)}$ of the form:

$$I(F_{(0)}^{(0)})+C_{ee}(F_{(0)}^{(0)})=-\frac{2\gamma'\omega^2u_{(0)\perp}^2e^{-2\beta(\vec{x}_2)}}{3w^2}\times\frac{\partial}{\partial w}\left(\frac{\nu_1'w^2}{\nu_1^2+\omega_1^2}\frac{\partial F_{(0)}^{(0)}}{\partial w}\right), \tag{4.3}$$

where $\alpha'/\delta=\gamma'\simeq O(1)$; so $F_{(0)}^{(0)}$ is now a function of $\vec{x}_2$: it is determined by (4.3) to which one has to add the definition of σ_R'', given by (3.13) with $\omega_{1R}=\omega_1$, and the condition of normalization,

$$4\pi\int_0^\infty w^2F_{(0)}^{(0)}\,dw=N. \tag{4.4}$$

One sees that, for a sufficiently weakly ionized plasma, one can neglect on the left-hand side of (4.3) the Coulomb term after $I(F_{(0)}^{(0)})$; one thus finds for $F_{(0)}^{(0)}(\vec{x}_2;w)$ a usual expression.[33] Finally, it is seen that $F_{(0)}^{(0)}(\vec{x}_2;w)$ is determined by two functional relations deduced from (4.3) and (3.13).

In conclusion, one thus sees that the plasma becomes inhomogeneous at zero order and at the space scale $\vec{x}_2$ under the effect of the electric oscillating field $\vec{e}_{(0)\perp}'$; it follows that, even at the "linear stage," the conductivity σ is, in fact, a nonlinear function of the field amplitude. Moreover, as the electronic density remains constant at this approximation, it results that $\vec{e}_{(0)\perp}$ produces essentially an increase of the electronic temperature given by the term $\gamma'\omega^2u_{(0)\perp}^2e^{-2\beta}/\nu_1^2+\omega_1^2$: it is a thermoeffect associated with the crossing of the wave in the plasma. It can also be shown, with the next approximation, that this inhomogeneity induces a longitudinal stationary field of ϵ^2 order, $\vec{e}_{(2)\parallel}''(\vec{x}_2)$.

Because of the length of the calculations, one cannot here discuss the second approximation. One notes only that, by calculating the various terms $F_{(2)}^{(0)}$, $\vec{F}_{(2)}^{(2)}$, $\vec{F}_{(3)}^{(1)}$, ..., one gets the expression of the nonlinear source term $\partial\vec{J}_{(3)}/\partial\tau_0$; thus, it can be shown that the electric field $\vec{e}_{(2)}$ at ϵ^2 order is of the form (cf. Ref. 33)

$$\vec{e}_{(2)}=[{}^1\vec{e}_{(2)\perp}'^0(\vec{x}_2)e^{i\varphi}+{}^3\vec{e}_{(2)}'^0(\vec{x}_2)e^{3i\varphi}+\text{c.c.}]+\vec{e}_{(2)\parallel}''(\vec{x}_2). \tag{4.5}$$

Note that these results will be the subject of a further publication devoted to the study of the propagation of continuous waves in weakly ionized media.[35]

B. Weakly collisional case

In this case, the collisional terms of the kinetic equations are carried over to the next order equations. On the other hand, as we do not make any assumption concerning the ratio ω_p^2/ω^2, the field equations at zero order are now given by (3.18). Let us consider now longitudinal and transverse waves successively.

1. Longitudinal waves

If we investigate an initial-value problem for which the wave is given at an initial time $\tau=0$, one then has only one space scale, $\vec{x}=\vec{x}_0$, and one must cancel in the successive equations all the terms in $\partial/\partial\vec{x}_2$, ... etc.

With the equations of orders 0 and 2, one gets for the longitudinal field at zero order

$$\vec{e}_{(0)\parallel}'=\vec{e}_{(0)\parallel}'^0(\tau_2)\exp\{i[(\omega_p/\omega)\tau_0-\vec{K}_0\cdot\vec{x}_0+\varphi_0]\}+\text{c.c.}, \tag{4.6}$$

with

$$\vec{e}_{(0)\parallel}'^0(\tau_2)=\vec{u}_{(0)\parallel}\exp(-\tfrac{1}{2}\bar{\nu}_1'\tau_2)\exp\left(i\frac{\omega}{\omega_p}\frac{K_0^2}{2}\tau_2\right), \tag{4.6'}$$

where $\vec{u}_{(0)\parallel}$ is the wave amplitude at $\tau_0=0$, φ_0 an initial phase, and $\bar{\nu}_1'$ is given by (3.12). It is thus seen that the variation of $\vec{e}_{(0)\parallel}'$ at the scale τ_2 includes two terms: an attenuation, $e^{-\bar{\nu}_1'\tau_2/2}$, due to the collisions, and a phase shift

$$\frac{\omega^2}{\omega_p^2}\frac{K_0^2}{2}\tau_2=\frac{\bar{v}^2}{2}\frac{k_0^2}{\omega_p^2}$$

(since $\epsilon^2=\eta'=\bar{v}^2/\lambda_0^2\omega^2$) which gives the first correction to the dispersion relation; so, with this term, one finds again the dispersion equation of plasma waves at the electronic temperature T.[36] Using the same equations, it can also be shown that, for an initially homogeneous plasma, one has $\vec{e}_{(0)}''=0$, $\vec{e}_{(2)}''=\vec{F}_{(1)}^{(1)''}=0$, for the stationary components; one finds in addition the first even harmonic ${}^2\vec{e}_{(2)\parallel}'$, which may be written:

$$ {}^2\vec{e}'_{(2)\parallel} = -i\,\frac{\omega^2}{\omega_p^2}\,\vec{K}_0\cdot\left\{\vec{e}'^{\,0}_{(0)\parallel}\,\vec{e}'^{\,0}_{(0)\parallel}\exp\left[2i\left(\frac{\omega_p}{\omega}\,\tau_0-\vec{K}_0\cdot\vec{x}_0+\varphi_0\right)\right]-\text{c.c.}\right\}. \tag{4.7} $$

With the fourth-order equations, it is first seen that $F^{(0)}_{(0)}$ depends only on the time scale τ_4, so that it is not modified by the presence of the field $\vec{e}'_{(0)\parallel}$ (attenuated at the scale τ_2); thus, by the previous results, it is seen that the only effect of the collisions at order ϵ^2 is a damping of the plasma wave. But the advantage of this kinetic formulation lies essentially in its application to the analysis of wave-wave interaction, in which the longitudinal modes are coupled with the transverse ones. Indeed, the ϵ^4-order equations allow one to calculate the nonlinear current[33(a)] $\partial\vec{j}_{(5)}/\partial\tau_0$; then, it can be shown with the fourth-order field equation that the coupling coefficients (at third approximation) include new contributions due to the collisions (cf. Ref. 37).

2. *Transverse waves*

If we consider the same boundary-value problem as in Sec. IV A, the equations of orders 0 and 2 give for the electric field at zero order:

$$ \vec{e}'_{(0)\perp} = \vec{u}_{(0)\perp}\exp\left[i\left(\frac{\omega_1}{\omega}\,\tau_0-\vec{K}_0\cdot\vec{x}_0+\varphi_0\right)\right] \times\exp\left[-\beta(\vec{x}_2)+i\,\psi_1(\vec{x}_2)\right]+\text{c.c.}, \tag{4.8} $$

with $\vec{K}_0\cdot\vec{u}_{(0)\perp}=0$ and where one now gets the usual dispersion relation for transverse waves:

$$ K_0^2=\frac{\lambda_0^2\omega_1^2}{c^2}\left(1-\frac{\omega_p^2}{\omega_1^2}\right). \tag{4.9} $$

In (4.9), the absorption $\beta(\vec{x}_2)$ and the phase shift $\psi_1(\vec{x}_2)$ are then given by

$$ \begin{aligned} \beta(\vec{x}_2) &= \frac{K_0}{2}\,\frac{\omega_p^2}{\omega_1^2-\omega_p^2}\,\frac{\omega}{\omega_1}\int^{x_2}\bar{\nu}'_1\,dx'_2\,,\\ \psi_1(\vec{x}_2) &= \frac{K_0}{2}\,\frac{\omega_p^2}{\omega_1^2-\omega_p^2}\,\frac{\omega^2}{\omega_1^2}\,\frac{K_0^2}{3}\int^{x_2}\frac{T'_e}{T}\,dx'_2\,, \end{aligned} \tag{4.10} $$

in which $\bar{\nu}'_1$ [given by (3.12)] and the electronic temperature T'_e are functions of $\vec{x}_2$ through $F^{(0)}_{(0)}(\vec{x}_2;w)$. As previously, the phase shift allows one to define the ϵ^2-correction to the wave vector K_0 and to find thus the dispersion formula of transverse waves in temperate plasmas with an electronic temperature such as $\bar{v}^2/c^2\simeq\epsilon^2$.[38] With the equations of the same order, one also gets: (i) the expression of the longitudinal stationary field, $\epsilon^2\vec{e}''_{(2)}(\vec{x}_2)$, in term of $\vec{\nabla}_{x_2}F^{(0)}_{(0)}$; (ii) the first even harmonic, of ϵ^2 order and of longitudinal type, which is induced by $\vec{e}'_{(0)\perp}$.[33(a)]

Then, the ϵ^4-order equations give various higher-order terms, such as the variation at the scale $\vec{x}_2$ of the first odd harmonic ${}^3\vec{e}'^{\,0}_{(2)}(\vec{x}_2)$, and the equation for $F^{(0)}_{(0)}$ which may be written

$$ I(F^{(0)}_{(0)})+C_{ee}(F^{(0)}_{(0)})=\gamma'\,\frac{w^2}{3}\,\vec{\nabla}_{x_2}\cdot\vec{F}^{(1)\prime\prime}_{(1)}+\frac{\gamma'}{3w^2}\,\frac{\partial}{\partial w}\left(w^3\vec{e}''_{(2)}\cdot\vec{F}^{(1)\prime\prime}_{(1)}-\frac{2\omega^2u^2_{(0)\perp}e^{-2\beta}}{\omega_1^2}\,w^2\nu'_1\,\frac{\partial F^{(0)}_{(0)}}{\partial w}\right) \tag{4.11} $$

in which $\vec{F}^{(1)\prime\prime}_{(1)}$ must be expressed in terms of $\vec{\nabla}_{x_2}F^{(0)}_{(0)}$ and $\vec{e}''_{(2)}(\vec{x}_2)$, and where $F^{(0)}_{(0)}(\vec{x}_2;w)$ must satisfy the normalization condition (4.4); one sees again that the medium becomes inhomogeneous under the effect of the transverse field $\vec{e}'_{(0)\perp}$ which generates a thermoeffect in the plasma.

V. INTERMEDIARY CASE: $\bar{\nu}/\omega=\epsilon(=\alpha'^{1/2})$

The essential difference with the previous cases is that one cannot use here the expansion (3.19) and (3.20) in ϵ^2. Indeed, as we have $\bar{\nu}/\omega=\epsilon$, the attenuation or absorption coefficients are of order ϵ, so that the odd scales τ_1 and $\vec{x}_1$ must now be kept. This taken into account, one can do the same analysis as in Sec. III B and show that $\vec{F}^{(l)}$ is also of order ϵ^l if the initial anisotropy of the medium is only due to the electric oscillating field. For such a situation, one can then replace (3.3)–(3.5) by the expansions:

$$ \vec{F}^{(l)}(\tau_0\vec{x}_0,\tau_1\vec{x}_1,\ldots;w)=\epsilon^l(\vec{F}^{(l)}_{(l)}+\epsilon\vec{F}^{(l)}_{(l+1)}+\cdots),\quad l=0,1,2,\ldots, \tag{5.1} $$

$$ \vec{e}(\tau_0\vec{x}_0,\tau_1\vec{x}_1,\ldots)=\vec{e}_{(0)}+\epsilon\vec{e}_{(1)}+\cdots. \tag{5.2} $$

By putting (5.1) and (5.2) in the fundamental system (2.6)–(2.12), one again gets a system of coupled equations that one has to solve step by step. As the involved techniques are essentially the same as previously,[33(a)] we give only the main results for longitudinal and transverse waves and point out the essential differences with those of the Sec. IV B.

A. Longitudinal waves

Considering again the same initial-value problem as in Sec. IV B, the equations of the two first orders allow to determine the electric field at zero order and its variation at the time scale τ_1. One thus gets

$$\vec{e}'_{(0)\parallel} = \vec{u}_{(0)\parallel} e^{-(\bar{\nu}'_1/2)\tau_1} \exp\{i[(\omega_p/\omega)\tau_0 - \vec{K}_0\cdot\vec{x}_0 + \varphi_0]\} + \text{c.c.}, \qquad (5.3)$$

where $\bar{\nu}'_1$ and $\vec{u}_{(0)\parallel}$ have the same meaning as in Sec. IV. If one compares (5.3) with (4.6), it is seen that one gets the same attenuation factor, but on the time scale τ_1, while the phase shift of (4.6) is now missing, since it is of order ϵ^2, and is removed in the next order equation. One also gets in the present case (homogeneous plasma) for the stationary components: $\vec{e}''_{(0)} = \vec{e}''_{(1)} = 0$, whence $\vec{F}^{(1)\prime\prime}_{(1)} = 0$.

By using the equations of second order, one first gets the expression of the first even harmonic which is of order ϵ^2 and is given by an expression identical to (4.7). One then finds for the amplitude $\vec{e}'^{0}_{(1)\parallel}$ of the first perturbation of the electric field:

$$\vec{e}'^{0}_{(1)\parallel} = \vec{u}_{(0)\parallel}\tau_1 e^{-(\bar{\nu}'_1/2)\tau_1} \frac{i}{2}\frac{\omega}{\omega_p}[K_0^2 - (\overline{\nu_1'^2} - \tfrac{3}{4}\bar{\nu}_1'^2)], \qquad (5.4)$$

with

$$\overline{\nu_1'^2} = -\frac{4\pi}{3N}\int_0^\infty \nu_1'^2 w^3 \frac{\partial F^{(0)}_{(0)}}{\partial w}\,dw. \qquad (5.5)$$

It is seen by (5.4) that $\epsilon\vec{e}'^{0}_{(1)\parallel}$ includes a term of the form $i\epsilon^2(\omega/\omega_p)(K_0^2/2)$ which is nothing other than the first term of the expansion in ϵ of the phase shift $\exp[i(\omega/\omega_p)\epsilon^2(K_0^2/2)\tau_0]$ in Eq. (4.7); but in the present case, one has also the supplementary term $(\bar{\nu}^2/\omega_p^2)(\overline{\nu_1'^2} - \frac{3}{4}\bar{\nu}_1'^2)$, which brings new collisional effects characteristic of the intermediate case and which modified the dispersion relation which may be deduced from (4.10). One finds lastly that one has to put, at this approximation, $\vec{e}''_{(2)} = 0$ and $\vec{F}^{(1)\prime\prime}_{(2)} = 0$.

Naturally, we may also consider the next approximations which allow one to calculate the higher-order harmonics. Particularly, it is seen by Eq. (2.6) at ϵ^3 order that $F^{(0)}_{(0)}$ depends only on the time scale τ_3 and is not influenced by the electric field $\vec{e}'_{(0)\parallel}$ (which is attenuated at the time scale τ_1), as in the weakly collisional case.

B. Transverse waves

We consider here the same boundary-value problem as in Sec. IV A; as before, one gets for the electric field $\vec{e}'_{(0)\perp}$:

$$\vec{e}'_{(0)\perp} = \vec{u}_{(0)\perp} e^{-\beta(\vec{x}_1)} \exp\{i[(\omega_1/\omega)\tau_0 - \vec{K}_0\cdot\vec{x}_0 + \varphi_0]\} + \text{c.c.} \qquad (5.6)$$

in which K_0 satisfies the dispersion equation (4.9) and where $\beta(\vec{x}_1)$ is given by a relation analogous to (4.10). But, now, the absorption $\beta(\vec{x}_1)$ depends on the space scale $\vec{x}_1$, since $\bar{\nu}/\omega_1$ is of order ϵ; thus, $\bar{\nu}'_1$ is also a function of $\vec{x}_1$ through $F^{(0)}_{(0)}$ which is determined by Eq. (2.6) at ϵ^3 order. One finds again, by the Poisson equation, $n_{(0)} = N$, $n_{(1)} = n_{(2)} = 0$, while the stationary field $\vec{e}''_{(1)}$ may be determined in term of $\vec{\nabla}_{x_1}F^{(0)}_{(0)}$.

With the equations of second order, one finds a first even harmonic of order ϵ^2, $\vec{e}'_{(2)\parallel}$, which is of longitudinal type and generated by the Lorentz force. It is also possible to obtain the linear equation verified by the first perturbation $\vec{e}'^{0}_{(1)\perp}(\vec{x}_1)$ of the oscillating electric field: it includes a term corresponding to the phase shift ψ_1 defined in (4.10), a collisional term

$$\left(\overline{\nu_1'^2} + \frac{1}{4}\frac{\omega_p^2}{\omega_1^2 - \omega_p^2}\bar{\nu}_1'^2\right)\vec{e}'_{(0)\perp}$$

analogous to that of (5.4) and a contribution due to the nonlinear conductivity associated with $F^{(0)}_{(1)}$.

The equation for $F^{(0)}_{(0)}$ is deduced from the equations of order ϵ^3. One thus gets an equation identical to (4.11) in weakly collisional case, but depending on the space variable $\vec{x}_1$: as previously, its solution is generally a non-Maxwellian distribution $F^{(0)}_{(0)}(\vec{x}_1; w)$. At the same order, one also finds the first odd transverse harmonic which is of order ϵ^2 and includes collisional contributions characteristic of the intermediate case.

VI. POSSIBLE EXTENSIONS, DISCUSSION AND CONCLUSION

In order to illustrate these methods, we have considered here simple physical situations characterized by the following properties: (i) the plasma is homogeneous and, for weakly collisional media, initially isotropic in the absence of the wave; (ii) one has $\alpha' \simeq \eta' \simeq \delta$, so that the effects linked to the inhomogeneity and to the amplitude of the wave are of the same order and equal to those due to the e-n energy exchanges; (iii) one also has $NY/\bar{v}^3\bar{\nu} \simeq \delta$, so that the electronic kinetic equation is reducible (under conditions made precise in the Appendix) to an infinite differential system of coupled equations; then, the collisions between charged particles give contributions only in the $F^{(0)}$ equation; (iv) finally, it has been assumed that one has only a single wave, of longitudinal or transverse type, with a frequency ω of the same order of magnitude as $\bar{\nu}$, at least ("high-frequency approximation"). Of course, these perturbation techniques can be applied to other and more sophisticated problems that we now discuss briefly.

(a) Firstly, they can be applied to initially inhomogeneous plasmas, in which the positive background becomes itself inhomogeneous so that N and T are now functions of $\vec{x}$. One then has to associate to this inhomogeneity a new characteristic length L that one has to compare to λ_0; after having ordered λ_0 and L, one can apply again these perturbation methods. For a weak inhomogeneity, one can consider, for example, situations such as $\lambda_0/L \simeq \epsilon^2$ for which the effects of the positive background inhomogeneity appear at the scale $\vec{x}_2$; in this case, one can use the ϵ^2 expansions. When the directions of wave propagation and of inhomogeneity gradient are orthogonal, one obtains an even harmonic of transverse type due to the coupling between the wave field and the inhomogeneous medium.[7, 10, 12]

It is seen, on the other hand, from the arguments of the Sec. III B, that one can treat the case of an initially anisotropic plasma by using the complete expansions (3.3)–(3.5); for a weakly ionized plasma, one thus finds coupling effects between the wave field and the initial anisotropy of the medium. It can be observed that this method can be extended to the study of wave propagation in presence of an external electrostatic field; in this case, in effect, the static field maintains anisotropies (independent of the wave field) such as $\vec{F}^{(1)}_{(0)}, \vec{F}^{(1)\prime\prime}_{(1)}$, etc., which are now different from zero. Naturally, the present perturbation techniques may also be applied to the important case where the plasma anisotropy is due to an external static magnetic field $\vec{B}_0$; one then has to introduce the supplementary parameter ω_{B_0}/ω that one must compare (and order) with the other ones.

(b) We can also consider physical situations characterized by other relative values of the three parameters α', η', and δ; in this case, they must be necessarily reordered in view of defining the new small parameter relevant to the problem. For example, we can have the following situations:

(i) $\eta' \ll \alpha' = \epsilon^2 \simeq \delta$. Here one has a "cold plasma" in which the effects of the thermal motion are smaller than those due to the wave amplitude. Indeed, from the results of Sec. IV B, the condition $\eta' \ll \delta$ is equivalent to $\overline{v}^2/c^2 \ll \delta$, for transverse waves, and to $l_D^2/\lambda_0^2 \ll \delta$ (l_D is the Debye length at temperature T) for longitudinal waves. Of course, our methods can then be applied and one finds equations in which the terms in $\vec{\nabla}_{x_0}$ now give higher-order contributions, while the dissipative effects due to the collisions remain unchanged.

(ii) $\eta' \simeq \alpha' \ll \delta$. One has, in this case, a weak perturbed system in which the nonlinear effects are smaller than those due to the $(e\text{-}n)$ energy exchanges: In this case, the term $I(F^{(0)}_{(0)}) + C_{ee}(F^{(0)}_{(0)})$ is dominant compared to nonlinear terms so that one has, for a boundary-value problem, a system weakly perturbed around the Maxwellian state: then, the first perturbation is of order $\gamma' \simeq \alpha'/\delta$, which is no longer of first order [see Ref. 21(a)]; moreover, the nonlinear contributions of $\vec{F}^{(1)}_{(3)}$ (and $\vec{J}_{(3)}$), being of order $\alpha' \simeq \eta'$, are consequently smaller than δ, so that they can often be neglected.

(iii) $\delta \ll \alpha' \simeq \eta' = \epsilon^2 \ll 1$. This is the inverse situation for which the nonlinear effects become dominant compared to those due to the $(e\text{-}n)$ energy exchanges. It can easily be seen that, for an initial-value problem, the characteristic time of the $(e\text{-}n)$ energy exchanges is yet of higher order than the attenuation time of the wave; then the results of Secs. IV B and V remain valid for weakly collisional plasmas, while one has, for the collisional plasmas of Sec. IV A, a system of two coupled equations which determine the evolution of $F^{(0)}_{(0)}$ and $\vec{e}\,'_{(0)\perp}$ at the time scale τ_2.[33(a)] For a boundary-value problem, the heating terms in $F^{(0)}_{(0)}$ equation being greater than those due to the energy exchanges, one no longer gets stationary solutions; then, the decompositions (3.9) are not valid. Nevertheless, it can be shown that one gets a new stationary state if a reduced velocity $w' = w/\gamma'$ is introduced (now with $\gamma' \gg 1$), corresponding to a new scale for the electronic thermal motion.

(c) Of course, the previous methods may also be applied to partially ionized gases for which one has to use the more general expressions (A11) and (A12) for the collisional terms of the kinetic equations (2.6) and (2.7). For example, if one assumes $NY/\overline{v}^3\overline{\nu} \simeq 1$, it can be easily proved that (i) for weakly collisional plasmas, the essential results of Secs. IV B and V are little modified, the Coulomb terms giving contributions only in the high-order equations; (ii) for collisional plasmas, one gets greater modifications because the differential equations (2.7) now become integro-differential with the integral operators L_1, L_2, etc., introduced in the Appendix.

(d) One can also study with these methods the propagation of several waves in such plasmas; in this case, one has to fix the relative order of magnitude of the various amplitudes and then define the adequate small parameters. Thus, one can investigate: (i) the coupling of several transverse waves (generation of harmonics and of sums or differences of the initial frequencies[33(a)]); (ii) the effects of collisions on the wave-wave interactions in weakly ionized media (three-wave resonant coupling, coupling coefficients at third-order approximation.[33(a), 37]

Finally, these techniques can be adapted to the study of low-frequency oscillations with $\omega \ll \delta\overline{\nu}$ or $\omega \simeq \delta\overline{\nu}$, for which one has to use the parameters α

and η (defined in Sec. II) instead of α' and η'. As has been seen in previous papers,[21(a)] one can exhibit in this case explicit solutions only if one has $\gamma = \alpha/\delta \ll 1$. The results so obtained allow one to study frequency- or amplitude-modulated fields, for which the modulation frequencies often fulfill the condition $\omega \lesssim \delta\bar{\nu}$; these techniques thus provide a theoretical basis for many previous works.[39-42]

In conclusion, the perturbation methods proposed in this paper allow one to determine, in each particular problem, not only the relative order of magnitude of the various nonlinear effects, but also the time or space scales at which these phenomena occur. The physical interpretation of the results so obtained is very easy, since the expansion parameters are directly linked with the quantities characterizing the state of the system, viz. $\bar{\nu}$, T, δ, ω_p, ω, λ_0, and E_0. Moreover, one can make a kinetic analysis of the role played by the collisions in the nonlinear behavior of the medium; indeed, they occur in the definition of weakly dissipative systems, but they occur also in the general formulas describing the various nonlinear effects through the reduced frequencies $\nu_1'(v)$, $\nu_2'(v), \ldots,$ which are determined by the e-n interaction law. Thus, the involved phenomena closely depend on the nature of this law: particularly, one can establish again the privileged role played by the Maxwellian interaction (in $1/r^5$) and then prove, in the stationary case, that the first odd harmonic of ϵ^2 order cancels and that $F_{(0)}^{(0)}$ is Maxwellian with an electronic temperature determined by the wave intensity.

Let us close this paper by emphasizing two essential features of this method.

(a) The use of Cartesian tensors in $\vec{w}$, as a well-suited basis of expansion, has led to the replacement of the kinetic equation by a system of coupled equations which is in fact equivalent to the system of moment equations.[22,23] Thus these techniques have the same domain of application as the moment equations and present the same limitations as those quoted by Buneman[45]; especially, they cannot be applied without modifications to plasmas where Landau damping must be taken into account. Nevertheless, they may be adapted to the analysis of the influence of collisions on the spectra of weakly instable or turbulent systems, in the spirit of the methods discussed in Refs. 46–48.

(b) By the definition of the small parameters ϵ, these perturbation expansions are generally equivalent to expansions in powers of the electric field, so that one is involved with singular expansions. Thus, the domain of validity and the limitations of these approximations are those of expansions of this kind, whose convergence difficulties have been analyzed in other works.[43]

APPENDIX

In a partially ionized plasma, constituted of electrons (e), ions (i), and neutrals (n), the collisional term relative to the electronic distribution function may be written

$$\left(\frac{\delta f_e}{\delta t}\right)_{\text{coll}} = \left(\frac{\delta f_e}{\delta t}\right)_{e-e} + \left(\frac{\delta f_e}{\delta t}\right)_{e-i} + \left(\frac{\delta f_e}{\delta t}\right)_{e-n}, \tag{A1}$$

in which the three terms of the right-hand side represent, respectively, the effects of e-e, e-i, and e-n collisions. One has now to express the various contributions to these terms when f_e is expanded in the basis of the irreducible Cartesian tensors which have been defined elsewhere.[21-23]

Let us first consider the e-n collisions; as in previous works,[22(a)] $(\delta f_e/\delta t)_{e-n}$ can be split into two terms: (i) one, called perfectly Lorentzian, in which the neutrals are considered at rest and the ratio $\delta(=2m/M)$ is zero; at this approximation, the collisional operator is invariant under rotation group and has spherical tensors as eigenfunctions; (ii) the other, called imperfectly Lorentzian, which is proportional to δ and which acts only on the isotropic part of f_e. Thus, one can write

$$\left(\frac{\delta f_e}{\delta t}\right)_{e-n} = \frac{\delta}{2v^2}\frac{\partial}{\partial v}\left[v^2\nu_1\left(vf^{(0)} + \frac{kT}{m}\frac{\partial f^{(0)}}{\partial v}\right)\right] - \sum_{l>0} \nu_l(v)\vec{\mathrm{f}}^{(l)}, \tag{A2}$$

where the expansion (2.5) has been used with $\vec{\mathrm{v}}/v$ instead of $\vec{w}$. In (A2), the neutral distribution function is assumed Maxwellian at the temperature T, while the $\nu_l(v)$ are given by

$$\nu_l(v) = 2\pi N_n v \int_0^\infty [1 - P_l(\cos\chi)]\, b\, db, \tag{A3}$$

where N_n is the density of the neutrals. Let us remind the reader that these ν_l depend on the e-n interaction law and that they are independent of v_e for a Maxwellian-type interaction (in r^{-5}). Note also that $\nu_1(v)$ is nothing other than the collision frequency relative to the e-n momentum transfer. On the other hand, it is assumed in this paper that the ν_l are approximately (at least for the small values of l) of the same order of magnitude as $\bar{\nu}$. Thus, the reduced quantities $\nu_l'(v)$, defined by

$$\nu_l'(v) = \nu_l(v)/\bar{\nu}, \tag{A4}$$

can be introduced.

Finally, one sees that $(\delta f_e/\delta t)_{e-n}$ includes two terms: (i) one, which describes the relaxation of the various anisotropies $\vec{\mathbf{f}}^{(l)}$ under the effect of collisions,[21(a), 22(a), 34] and whose order of magnitude is $\bar{\nu}$; (ii) the other, which corresponds to the energy exchanges by e-n collisions, and whose order of magnitude is $\delta\bar{\nu}$.

Let us now consider the collisions between charged particles; one describes the Coulomb interactions by using Fokker-Planck terms expressed by means of the Rosenbluth's potentials.[50] Thus, one can write for the collisional term of particles of species j and k:

$$\left(\frac{\delta f_j}{\delta t}\right)_k = 4\pi Y \frac{m_j}{m_k} f_j f_k + \frac{m_k - m_j}{m_k + m_j} \vec{\nabla}\mathcal{H}\cdot\vec{\nabla} f_j + \frac{\vec{\nabla}\vec{\nabla}\mathcal{G} : \vec{\nabla}\vec{\nabla} f_j}{2}, \qquad \text{(A5)}$$

with the usual definitions of the "potentials" $\mathcal{H}(\vec{\mathbf{v}})$ and $\mathcal{G}(\vec{\mathbf{v}})$ and where

$$Y = 4\pi\left(\frac{Z_j Z_k e^2}{4\pi\epsilon_0 m_j}\right)^2 \log\Lambda,$$

in which Λ is the usual large parameter defined by the ratio of the Debye length to the minimum impact parameter b_0.

Shkarofsky has given[51] general formulas for the expansion of the collisional term (A5) in the basis of Cartesian tensors, expressions which are very intricate and nonlinear in $\vec{\mathbf{f}}^{(l)}$. From our point of view, it is sufficient to use the corresponding linearized formulas that one has to apply to e-e and e-i collisions. For these latter ones, it is also assumed that one can neglect the recoil of the ions whose distribution function is then approximately Maxwellian, as well as that of the neutrals. Under these conditions, the electronic kinetic equation is disconnected from the other kinetic equations and may be studied as independent. So, according to Shkarofsky,[23] one can use the following expressions: For the e-i collisions, one introduces an e-i collision frequency defined by

$$\nu_{ei} = \frac{Y I_0^0}{v^3} \simeq \frac{NY}{v^3}, \qquad \text{(A6)}$$

where N is the density of the positive background. One then gets

$$\left(\frac{\delta f_e^{(0)}}{\delta t}\right)_{e-i} = \frac{\delta}{2v^2}\frac{\partial}{\partial v}\left[\nu_{ei} v^2\left(v f_e^{(0)} + \frac{kT}{m}\frac{\partial f_e^{(0)}}{\partial v}\right)\right], \qquad \text{(A7)}$$

$$\left(\frac{\delta\vec{\mathbf{f}}_e^{(1)}}{\partial t}\right)_{e-i} \simeq -\nu_{ei}\vec{\mathbf{f}}_e^{(1)}, \quad \left(\frac{\delta\overleftrightarrow{\mathbf{f}}_e^{(2)}}{\delta t}\right) \simeq -3\nu_{ei}\overleftrightarrow{\mathbf{f}}_e^{(2)}, \qquad \text{(A8)}$$

in which one recognizes formulas similar to (A2), with ν_{ei} instead of $\nu_1(v)$; thus, at this approximation, the effect of e-i collisions can be compared to that of e-n collisions, their respective weight being determined by the order of magnitude of ν_{ei} and ν_1. On the other hand, for the e-e collisions, one gets for the isotropic part $f_e^{(0)}$:

$$\left(\frac{\delta f_e^{(0)}}{\delta t}\right)_{e-e} = \frac{Y}{v^2}\frac{\partial}{\partial v}\left[f_e^{(0)} I_0^0 + \frac{v}{3}(I_2^0 + J_{-1}^0)\frac{\partial f_e^{(0)}}{\partial v}\right], \qquad \text{(A9)}$$

and for the first anisotropy $\vec{\mathbf{f}}_e^{(1)}$:

$$\left(\frac{\delta\vec{\mathbf{f}}_e^{(1)}}{\delta t}\right)_{e-e} = \frac{Y}{3v}\frac{\partial^2\vec{\mathbf{f}}_e^{(1)}}{\partial v^2}(I_2^0 + J_{-1}^0) + \frac{Y}{3v^3}\vec{\mathbf{f}}_e^{(1)}(-3I_0^0 + I_2^0 - 2J_{-1}^0) + \frac{Y}{3v^2}\frac{\partial\vec{\mathbf{f}}_e^{(1)}}{\partial v}(3I_0^0 - I_2^0 + 2J_{-1}^0)$$
$$+ 8\pi Y\vec{\mathbf{f}}_e^{(1)} f_e^{(0)} + \frac{Y}{5v}\frac{\partial^2 f_e^{(0)}}{\partial v^2}(\vec{\mathbf{I}}_3^1 + \vec{\mathbf{J}}_{-2}^1) + \frac{Y}{15v^2}\frac{\partial f_e^{(0)}}{\partial v}(-3\vec{\mathbf{I}}_3^1 + 2\vec{\mathbf{J}}_{-2}^1 + 5\vec{\mathbf{I}}_1^1), \qquad \text{(A10)}$$

as well as similar expressions for the higher anisotropies; in (A9) and (A10), the I and J are the integrals introduced by Allis.[52] Finally, by (A2), (A7)–(A10) and by introducing the reduced variables (2.1), the whole collision term (A1) may be written for the first anisotropies:

$$\left(\frac{\delta F_e^{(0)}}{\delta t}\right)_{\text{coll}} = \frac{\delta\bar{\nu}}{2w^2}\frac{\partial}{\partial w}\left[(\nu_1')_t w^2\left(wF_e^{(0)} + \frac{1}{3}\frac{\partial F_e^{(0)}}{\partial w}\right)\right] + \frac{NY}{\bar{v}^3 w^2}\frac{\partial}{\partial w}\left(F_e^{(0)} I_0'^0 + \frac{w}{3}(I_2'^0 + J_{-1}'^0)\frac{\partial F_e^{(0)}}{\partial w}\right), \qquad \text{(A11)}$$

$$\left(\frac{\delta\vec{\mathbf{F}}_e^{(1)}}{\delta t}\right)_{\text{coll}} = -\bar{\nu}(\nu_1')_t\vec{\mathbf{F}}_e^{(1)} + \frac{NY}{\bar{v}^3}L_1(\vec{\mathbf{F}}_e^{(1)}),$$
$$\left(\frac{\delta\overleftrightarrow{\mathbf{F}}_e^{(2)}}{\delta t}\right)_{\text{coll}} = -\bar{\nu}(\nu_2')_t\overleftrightarrow{\mathbf{F}}_e^{(2)} + \frac{NY}{\bar{v}^3}L_2(\overleftrightarrow{\mathbf{F}}_e^{(2)}), \qquad \text{(A12)}$$

where L_1, L_2, ... are integrodifferential linear operators depending on $F_e^{(0)}$ and defined by (A10), ..., and where

$$(\nu_1')_t = \nu_1' + \nu_{ei}' = \nu_1' + \frac{NY}{\bar{v}^3\bar{\nu}}\frac{1}{w^3},$$
$$(\nu_2')_t = \nu_2' + 3\nu_{ei}' = \nu_2' + \frac{3NY}{\bar{v}^3\bar{\nu}}\frac{1}{w^3}; \qquad \text{(A13)}$$

let us note that in (A11), the primed integrals $\overleftrightarrow{\mathbf{I}}'$ or $\overleftrightarrow{\mathbf{J}}'$ are equal to $(1/N)\overleftrightarrow{\mathbf{I}}$ or $(1/N)\overleftrightarrow{\mathbf{J}}$.

It is thus seen that the relative effect of Coulomb collisions with respect to e-n collisions is characterized by the dimensionless parameter $NY/\bar{v}^3\bar{\nu}$. If this ratio is equal to (or greater than) unity, the Coulomb terms will be dominant in (A11) on account of the presence of the factor δ in the e-n term, while one gets integrodifferential operators for the anisotropies $\vec{F}^{(1)}$, $\bar{\bar{F}}^{(2)}$, ... But if one has $NY/\bar{v}^3\bar{\nu} \ll 1$, the Coulomb terms in (A12) may be neglected (at least in the first approximations); thus, one can put $(\nu_1')_t \simeq \nu_1'$ and $(\nu_2')_t \simeq \nu_2'$ and one thus gets the collisional expressions given in (2.8). In this case, the contributions of Coulomb collisions occur only in the equation relative to the isotropic part of the electronic distribution function and one then has to compare $NY/\bar{v}^3\bar{\nu}$ with δ.

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