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Mass and charge renormalization for classical processes with a fluctuation-dissipation theorem

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The fluctuation-dissipation theorem (FDT) valid both for canonical systems in equilibrium and for purely irreversible stochastic processes with detailed balance is shown to imply that the renormalized values of all coefficients in the field equation ("mass" and "charges") are simply related to static cumulants of the field. As a consequence, the problem of determining the static behavior of the system (which reduces to quadratures when the FDT holds) is completely separated from the dynamic problem; the perturbation theory for the time dependence of the two-point cumulant has, if properly renormalized, the full statics of the system incorporated exactly; the renormalized perturbation theory (RPT) will, in general, be much more useful than the unrenormalized one since in contrast to the latter it does not necessarily require the system to be nearly Gaussian. If the nonlinearity in the field equation involves only a quartic charge matrix, the renormalized charge matrix is $C^{(4)}/C^{(2)4}$, where $C^{(2)}$ and $C^{(4)}$ are the static cumulants of order 2 and 4, respectively, whereupon the expansion parameter in the RPT becomes $C^{(4)}/C^{(2)2}$; when the static cumulants are known, the smallness of this expansion parameter (which does not necessarily require $C^{(4)}\approx 0$, i.e., near-Gaussian statics) can be checked explicitly. The usefulness and necessity of mass and charge renormalization is demonstrated by a perturbative calculation of the linewidth of the Van der Pol oscillator; although this oscillator is far from Gaussian near and above threshold, the result (below, near, and above threshold) is quite satisfactory already in second and excellent in fourth order; the unrenormalized perturbation expansion, on the other hand, yields nonsense in any finite order except very far below threshold where the oscillator is indeed nearly Gaussian. For many-body systems the static cumulants defining the renormalized mass and charge(s) can, in general, not be evaluated exactly; however, in many cases sufficiently accurate approximate values or experimental information are available to render the RPT practical.

I. INTRODUCTION

The perturbative construction of correlation functions for classical fields[1] is, in general, a somewhat more complex task than for quantum systems in equilibrium, since there normally is no fluctuation-dissipation theorem (FDT) relating the two-point correlation function and the response function describing the linear response of the field to an external source. As a consequence, one has to deal with two independent propagators, two independent self-energies, and several independent vertices.

However, for canonical systems in equilibrium and for purely irreversible random processes with detailed balance, there is a FDT.[2,3] In such cases, the perturbation theory acquires the following three remarkable features. (a) It can be formulated in terms of one propagator and one self-energy.[2] (b) Static problems can be separated from dynamical ones; the renormalized values of all coefficients in the field equation (mass and charges) can be expressed by static cumulants of the field; since the set of these cumulants uniquely determines the stationary probability distribution of the field, the perturbation scheme has the full statics of the system exactly incorporated after mass and charge renormalization; only the time dependence of the propagator remains to be determined; the static cumulants necessary to specify the renormalized mass and charges can be calculated explicitly for few-body problems, since they are known up to quadratures whenever a FDT holds, whereas for many-body systems approximations or experiments many give sufficient information on them. (c) The usefulness of the renormalized perturbation scheme is, in general, much greater than that of the unrenormalized one; while expansions in terms of the bare charge(s) cannot give sensible results in any finite order unless the system is nearly Gaussian, the renormalized expansion has an eventually quite non-Gaussian static behavior built in exactly.

In this paper we illustrate the above statements for a field equation with only a cubic nonlinearity (i.e., a quartic charge). Other nonlinearities can be treated as well. The argument is carried out for purely irreversible processes with detailed balance but can be carried over without change to canonical systems in thermal equilibrium because the FDT is the same in both cases.[2,3] Sections II and III set out the notation, specify the process considered, and briefly set up the basic perturbation scheme.[1] Sections III–VI present mass and charge renormalization. In Sec. VII we give, as the probably simplest nontrivial application, a perturbative treatment of the Van der Pol oscillator. The renormalized expansion for the linewidth gives very satisfying results already in low orders, even near and above threshold where the oscillator is far from Gaussian; in contrast, an expansion in terms of the bare charge is shown to be nonsensical near and above threshold in any finite order.

II. PURELY IRREVERSIBLE RANDOM PROCESSES WITH DETAILED BALANCE

Consider random variables $\psi_1(t)\in(-\infty,+\infty)$, labeled by the discrete or continuous index 1, and let their dynamics be described by the Langevin equation

$$\frac{\partial}{\partial t}\psi_1(t)=U_{12}\psi_2(t)+U_{1234}\psi_2(t)\psi_3(t)\psi_4(t)+f_1(t)\,, \tag{2.1}$$

where repeated indices are to be summed over. Let $f_1(t)$ represent Gaussian white noise,

$$\langle f_1(t_1)f_2(t_2)\rangle=2D_{12}\delta(t_1-t_2)\,, \tag{2.2}$$

and let the diffusion matrix $D_{12}>0$ be independent of the field ψ. Assume that the process (2.1) is purely irreversible, that is, that the drift vector

$$F_1(\psi)=U_{12}\psi_2+U_{1234}\psi_2\psi_3\psi_4 \tag{2.3}$$

transforms like ψ_1 under time reversal,

$$F_1(\tilde\psi)=\epsilon_1F_1(\psi)\ \text{ with }\ \tilde\psi_1=\epsilon_1\psi_1,\ \epsilon_1=\pm1\,, \tag{2.4}$$

when $\psi\to\tilde\psi$ for $t\to-t$ (no summation convention in formulas involving ϵ_1). Finally, assume detailed balance to get the potential conditions[4]

$$\begin{aligned}&D_{12}=\epsilon_1\epsilon_2D_{12}\,,\\&\frac{\partial}{\partial\psi_1}D^{-1}_{23}F_3=\frac{\partial}{\partial\psi_2}D^{-1}_{13}F_3\,.\end{aligned} \tag{2.5}$$

Then the stationary probability distribution $\overline{P}(\psi)$ of the field is uniquely determined by its gradient in ψ space,[4]

$$\frac{\partial}{\partial\psi_1}\ln\overline{P}=D^{-1}_{12}F_2\,. \tag{2.6}$$

We will be concerned with the stationary two-point correlation function

$$\begin{aligned}C_{12}(t_1-t_2)&=\langle\psi_1(t_1)\psi_2(t_2)\rangle\\&=\int d\psi\,\psi_1[\exp L(t_1-t_2)]\psi_2\overline{P}\end{aligned} \tag{2.7}$$

and the response function

$$\begin{aligned}R_{12}(t_1-t_2)&=\Theta(t_1-t_2)\langle\psi_1(t_1-t_2)\hat\psi_2(0)\rangle\\&=\Theta(t_1-t_2)\int d\psi\,\psi_1\exp[L(t_1-t_2)]\\&\qquad\times\left(-\frac{\partial}{\partial\psi_2}\right)\overline{P}\,,\end{aligned} \tag{2.8}$$

which relates the linear response of the field $\psi_1(t_1)$ to a time-dependent external source. In the above two equations, L is the Fokker-Planck differential operator in the equation of motion for the probability distribution $P(t)$,

$$\dot P(\psi,t)=LP(\psi,t),\quad L\overline{P}=0\,, \tag{2.9}$$

which is equivalent to the Langevin equation (2.1).[1] The correlation functions $C(t)$ and $R(t)$ are related by the fluctuation-dissipation theorem[2,5]

$$\frac{\partial}{\partial t}C_{12}(t)=-R_{13}(t)D_{32}+D_{13}R_{23}(-t)\,. \tag{2.10}$$

While the static value $C(0)$ of $C(t)$ can, in principle, be evaluated as a second moment of $\overline{P}$, the time-dependent $C(t)$ is, in general, not obtainable in terms of quadratures. Therefore, a perturbative method of constructing $c(t)$ is of interest.

III. PERTURBATION THEORY

A perturbation theory for classical processes has been developed by Martin, Siggia, and Rose.[1] An adaption of the general scheme to random processes for which the FDT (2.10) holds has been given by Deker and Haake.[2] In this section we sketch the formalism inasmuch as it is needed for our present considerations.

The variable $\hat\psi$, already defined implicitly for $t=0$ in Eq. (2.8) as $\hat\psi=-\partial/\partial\psi$, is defined for all times by the requirements

$$[\psi_1(t),\hat\psi_2(t)]=\delta_{12},\quad\langle\hat\psi(t)[\cdots]\rangle=0\,. \tag{3.1}$$

It then obeys the field equation

$$-\frac{\partial}{\partial t}\hat\psi_1(t)=U_{21}\hat\psi_2(t)+3U_{2341}\hat\psi_2(t)\psi_3(t)\psi_4(t). \tag{3.2}$$

The field equations (2.1) and (3.2) may be combined to give

$$\begin{aligned}&-\frac{\partial}{\partial t_1}i\sigma(12)\Phi(2)\\&\quad=\gamma(12)\Phi(2)+\frac{1}{3!}\gamma(1234)\Phi(2)\Phi(3)\Phi(4)+\begin{pmatrix}0\\f_1(t_1)\end{pmatrix}\,,\end{aligned} \tag{3.3}$$

where

$$\Phi(1)=\begin{pmatrix}\psi_1(t_1)\\\hat\psi_1(t_1)\end{pmatrix},\quad i\sigma(12)=\delta_{12}\delta(t_1-t_2)\begin{pmatrix}0&1\\-1&0\end{pmatrix}\,. \tag{3.4}$$

The matrices $\gamma(12)$ and $\gamma(1234)$ will be referred to as "bare mass" and "bare charge," respectively; they may be chosen symmetric in all arguments; only those of their elements do not vanish for which one index refers to $\hat\psi$ and all others to

$$\gamma(\hat12)=\gamma(2\hat1)=\delta(t_2-t_1)\gamma_{\hat12},\quad \gamma_{\hat12}=U_{12}\ ,$$
$$\gamma(\hat1234)=\delta(t_2-t_1)\delta(t_3-t_1)\delta(t_4-t_1)\gamma_{\hat1234}\ , \qquad (3.5)$$
$$\gamma_{\hat1234}=3!\,U_{1234}\ .$$

The field cumulants are then defined, with the help of the generating functional

$$S(\Lambda)=\langle[\exp\Lambda(1)\Phi(1)]_+\rangle,\quad \Lambda(1)=\begin{pmatrix}\lambda_1(t_1)\\ \lambda_2(t_2)\end{pmatrix}\ , \qquad (3.6)$$

as

$$G(123\ldots n)=\frac{\delta^n}{\delta\Lambda(1)\delta\Lambda(2)\cdots\delta\Lambda(n)}\ln S(\Lambda)\bigg|_{\Lambda=0}\ . \qquad (3.7)$$

The second-order cumulant reads explicitly

$$G(12)=\begin{pmatrix}C_{12}(t_1-t_2) & R_{12}(t_1-t_2)\\ R_{21}(t_2-t_1) & 0\end{pmatrix}\ , \qquad (3.8)$$

and obeys the Dyson equation

$$\left(-\frac{\partial}{\partial t}\,i\sigma(12)-\gamma(12)\right)G(23)$$
$$=\delta(13)+\left[2D(12)+\tfrac12\gamma(1245)G(45)+\Sigma(12)\right]G(23)\ , \qquad (3.9)$$

with

$$\delta(12)=\delta_{12}\delta(t_1-t_2)\begin{pmatrix}1&0\\0&1\end{pmatrix}\ ,$$
$$D(12)=\delta(t_1-t_2)D_{12}\begin{pmatrix}0&0\\0&1\end{pmatrix}\ .$$

The Langevin noise source enters this equation only through the first term in the brackets on the right-hand side of (3.9). The next term in the brackets is the Hartree-Fock part of the self-energy, while

$$\Sigma(12)=\begin{pmatrix}0 & \Sigma_{1\hat2}(t_1-t_2)\\ \Sigma_{2\hat1}(t_2-t_1) & \Sigma_{\hat1\hat2}(t_1-t_2)\end{pmatrix} \qquad (3.10)$$

is the collisional part of the self-energy. We shall call Σ "self-energy." It is related to the four-point vertex $\Gamma(1234)$ by[6]

$$\Sigma(12)=\tfrac16\gamma(1345)G(36)G(47)G(58)\Gamma(6782) \qquad (3.11)$$

and

$$\Gamma(1234)=\gamma(1234)+\tfrac12\gamma(1256)G(58)G(69)\Gamma(8934)$$
$$+2\frac{\delta\Sigma(12)}{\delta G(34)}+\frac{\delta\Sigma(12)}{\delta G(56)}\,G(58)G(69)\Gamma(8934). \qquad (3.12)$$

Equations (3.11) and (3.12) can be iterated to yield perturbation expansions for Σ and Γ in terms of the bare charge γ. The first few terms of these expansions are represented in Fig. 2 in terms of the graphical symbols for G, γ, Γ, Σ given in Fig. 1.

Note that the vertex Γ has, in general, four independent nonvanishing components with respect to the spinor indices, since one, two, three, or four of its indices may refer to $\hat\psi$ (see Fig. 3). However, the number of independent components of G, Σ, and Γ in our case reduces by virtue of the FDT (2.10). As for G, the FDT immediately allows us to eliminate the response function R in favor of the pure-ψ cumulant C. By using the Dyson equation (3.9), the FDT for G may be expressed in terms of the self-energy as

$$\frac{\partial}{\partial t}\Sigma_{\hat1\hat2}(t)=-\Sigma_{\hat13}(t)D_{32}+D_{13}\Sigma_{\hat23}(-t)$$
$$=\begin{cases}-\Sigma_{\hat13}(t)D_{32} & \text{for } t>0\\ D_{13}\Sigma_{\hat23}(-t) & \text{for } t<0\ ,\end{cases} \qquad (3.13)$$

whereupon all components of Σ derive from $\Sigma_{\hat1\hat2}(t)$.[7] As a consequence, the Dyson equation (3.9) reduces to a single equation for $C(t)$,

$$\dot C_{13}(t)=\left[\gamma_{\hat12}+\tfrac12\gamma_{\hat1245}C_{45}(0)+\Sigma_{\hat1\hat4}(0)D^{-1}_{42}\right]C_{23}(t)$$
$$-\int_0^t dt'\,\Sigma_{\hat1\hat4}(t-t')D^{-1}_{42}\dot C_{23}(t'),\quad t>0. \qquad (3.14)$$

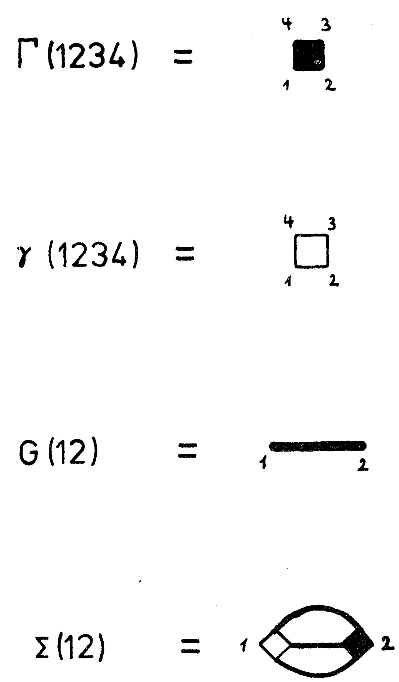

FIG. 1. Graphs for Γ, γ, G, Σ.

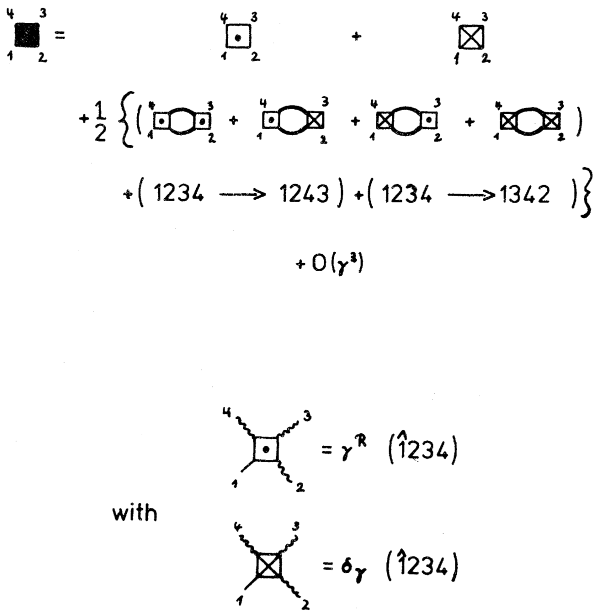

FIG. 2. Perturbation expansions for Γ and Σ in terms of the bare charge.

Of course, the elimination of R is completed explicitly only after replacing each R appearing in the perturbation expansion for $\Sigma_{\hat{1}\hat{2}}$ (Figs. 4 and 5) by invoking the FDT. Equation (3.14) has to be complemented by the initial condition

$$\dot{C}_{12}(0^+) = -D_{12} , \tag{3.15}$$

which also follows from the FDT.

IV. MASS RENORMALIZATION

The initial condition (3.15) allows for the following identification of the bracket in the first term in the Dyson equation (3.14):

$$[\gamma_{\hat{1}2} + \tfrac{1}{2}\gamma_{\hat{1}245}C_{45}(0) + \Sigma_{\hat{1}\hat{4}}(0)D^{-1}_{42}] = -D_{14}C^{-1}_{42}(0) . \tag{4.1}$$

If the perturbation expansion for $\Sigma_{\hat{1}\hat{4}}(0)$ is inserted here, we obtain a perturbative equation for the static cumulant $C_{12}(0)$.[8] It is unnecessary, however, to determine $C_{12}(0)$ from this equation, since we consider $C_{12}(0)$ known as a second

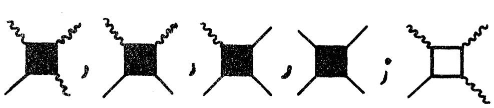

FIG. 3. Nonvanishing spinor components of Γ and γ; straight (wavy) legs represent $\hat{\psi}$- (ψ-) indices.

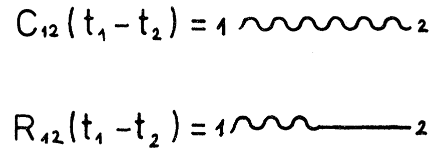

FIG. 4. Graphs for C and R.

moment of the stationary probability distribution $\overline{P}$. We may thus rewrite Eq. (3.14) as

$$\dot{C}_{13}(t) = -D_{14}C^{-1}_{42}(0)C_{23}(t) - \int_0^t dt'\, \Sigma_{\hat{1}\hat{4}}(t-t')D^{-1}_{42}\dot{C}_{23}(t') . \tag{4.2}$$

Since the bare mass $\gamma_{\hat{1}2}$ is now eliminated in favor of the exact $C_{12}(0)$, we have carried out mass renormalization. Note that no counterterms are necessary in the perturbation expansion for $\Sigma_{\hat{1}\hat{2}}$ according to Fig. 5, since this expansion involves the exact propagator $C(t)$ rather than its approximation in zeroth order in the bare charge $\gamma_{\hat{1}234}$. Note also that, while mass renormalization is possible for processes with or without a FDT, it is precisely the FDT which allows one to relate the renormalized mass to the static second-order cumulant in our case.

V. CHARGE RENORMALIZATION

The perturbation expansion for $\Sigma_{\hat{1}\hat{2}}(t)$ displayed in Fig. 5 goes in terms of the bare charge $\gamma_{\hat{1}234}$. We expect the usefulness of this expansion to improve when it is reorganized as a power series in terms of the renormalized charge:

$$\gamma^R(\hat{1}234) = \delta(t_2 - t_1)\delta(t_3 - t_1)\delta(t_4 - t_1)\gamma^R_{\hat{1}234} ,$$

$$\gamma^R_{\hat{1}234} = \int dt_2 \int dt_3 \int dt_4\, \Gamma_{\hat{1}234}(t_2 - t_1, t_3 - t_1, t_4 - t_1)$$

$$= \Gamma_{\hat{1}234}(\omega_i = 0) . \tag{5.1}$$

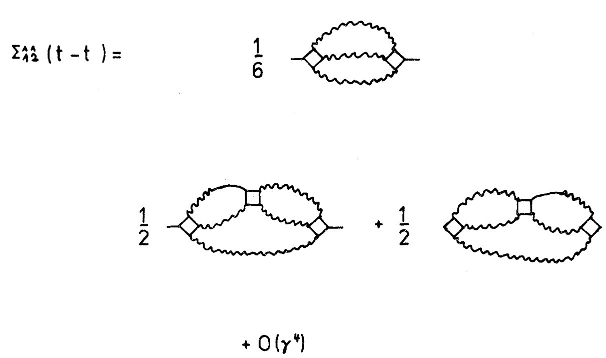

FIG. 5. Perturbation expansion for $\Sigma_{\hat{1}\hat{2}}$, obtained from Fig. 2. by carrying out the summation over the spinor indices.

This reorganization follows standard lines and will be sketched briefly below. The important point, however, is that the renormalized charge γ^R in our case is, due to the FDT, related to the static pure-ψ cumulants of order two and four, as

$$\gamma^R_{\hat{1}234} = D_{15} C_{56}(0)^{-1} C_{27}(0)^{-1} C_{38}(0)^{-1} C_{49}(0)^{-1} C_{6789}(t_i = 0)\ , \tag{5.2}$$

and can thus in principle be evaluated by quadratures from $\overline{P}$. The algebra needed to establish Eq. (5.2) is presented in the Appendix.

In order to generate the perturbation expansion of $\Sigma_{\hat{1}\hat{2}}$ in powers of γ^R, we insert the identity

$$\gamma(1234) = \gamma^R(1234) + \delta\gamma(1234) \tag{5.3}$$

in the expansion for Γ in Fig. 2. The resulting series is displayed, for the sake of completeness, in Fig. 6. The correction $\delta\gamma$ is then determined order by order in γ^R by requiring Eq. (5.1) (which is represented graphically in Fig. 7) to hold in each order. The resulting series for the bare charge γ and the vertex Γ are shown in Fig. 8. When these series are inserted in the self-energy, we obtain Fig. 9, or, with the help of the FDT,

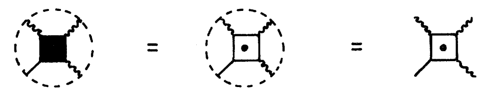

FIG. 7. Definition (5.1) of the renormalized charge; the circle around a vertex (and, later, a vertex part) means taking the $\omega = 0$ Fourier component and multiplication with $\delta(t_1 - t_2)\delta(t_3 - t_1)\delta(t_4 - t_1)$ if one of the four free indices refers to $\hat{\psi}$, and zero if two or more $\hat{\psi}$-indices appear.

$$\Sigma_{\hat{1}\hat{2}}(t) = \tfrac{1}{6}\gamma^R_{\hat{1}345} C_{36}(t) C_{47}(t) C_{58}(t) \gamma^R_{678\hat{2}}$$
$$- \tfrac{1}{2}\int_0^t dt'\, \gamma^R_{\hat{1}345} C_{36}(t-t') C_{47}(t-t') \gamma^R_{678\hat{9}} D^{-1}_{9,10} \frac{\partial}{\partial t'} C_{10,11}(t') C_{8,12}(t') C_{5,13}(t) \gamma^R_{11,12,13,\hat{2}}$$
$$- \tfrac{1}{4}\gamma^R_{\hat{1}345} C_{36}(0) C_{47}(0) \gamma^R_{678\hat{9}} D^{-1}_{9,10} C_{10,11}(t) C_{8,12}(t)\, C_{5,13}(t) \gamma^R_{11,12,13,\hat{2}} + O(\gamma^{R4})\ . \tag{5.4}$$

VI. DISCUSSION OF THE RENORMALIZED EXPANSION

Since we consider $C^{(2)}(0)$ known, there remains

$$c(t) = C^{(2)}(t) C^{(2)}(0)^{-1} \tag{6.1}$$

to be determined by solving the Dyson equation (4.2) with the self-energy (5.4). By inserting (6.1), we easily see that the renormalized expansion for Σ involves $\gamma^R C^{(2)}(0)^2 = C^{(4)}(0)/C^{(2)}(0)^2$ as expansion parameter(s). Inasmuch as the static fourth-order cumulant $C^{(4)}$ is also known, one can, in principle, check whether this expansion parameter is small enough for the expansion to be expected useful.

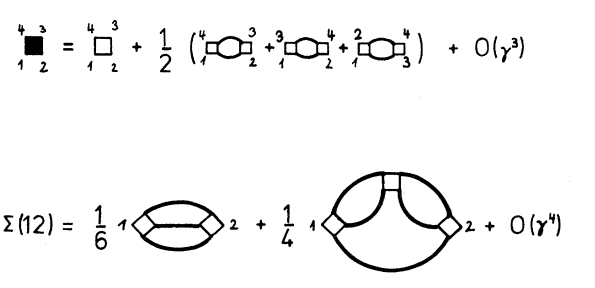

FIG. 6. Perturbation expansion for Γ from Fig. 2, with Eq. (5.3) inserted.

Note that the smallness of $C^{(4)}(0)/C^{(2)}(0)^2$ not necessarily requires the system to be near-Gaussian.

The renormalized expansion has the static cumulants $C^{(2)}(0)$ and $C^{(4)}(0)$ built in exactly. In a sense, even the full probability distribution $\overline{P}$, i.e. the whole statics of the system, is built in exactly, since our field equation involving only $\gamma^{(2)}$ and $\gamma^{(4)}$, the stationary distribution $\overline{P}$ is of the form $\exp(A\psi^2 + B\psi^4)$; the coefficients A and B are uniquely fixed in terms of $C^{(2)}(0)$ and $C^{(4)}(0)$. More generally, for processes with field equations involving $\gamma^{(n)}$ ($n = 2, 3, 4, \ldots, N$), all of these "charges" can be renormalized and the $\gamma^{(n)R}$ be expressed in terms of the static cumulants $C^{(n)}(0)$ ($n = 2, 3, \ldots, N$) if the FDT (2.10) holds; the set of these cumulants again defines the stationary probability distribution uniquely, so that the perturbation scheme for $c(t) = C^{(2)}(t) C^{(2)}(0)^{-1}$ has the exact static behavior of the system incorporated.

Of course, for many-body systems the quadratures giving $\overline{P}$, $C^{(2)}(0)$, $C^{(4)}(0)$, etc., cannot generally be carried out exactly. But an approximate evaluation of γ^R may still be possible or sufficient experimental information available; at least, the present considerations serve to separate static from dynamic problems for processes with the FDT (2.10).

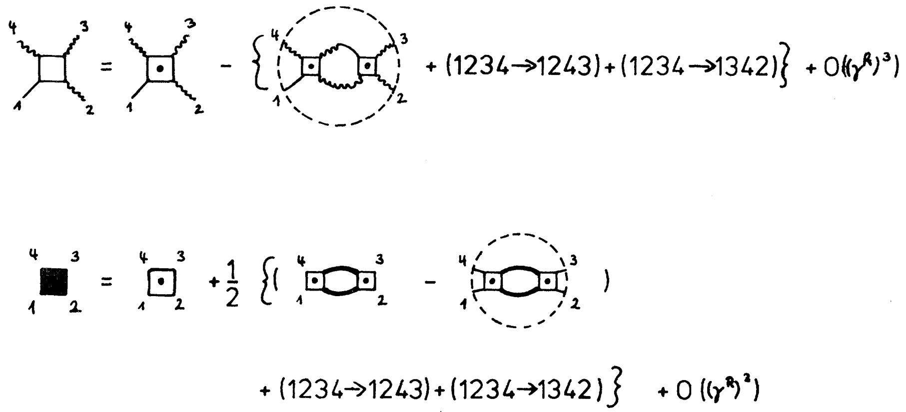

FIG. 8. Expansions of the bare charge and the vertex in powers of the renormalized charge.

The renormalization program presented here makes use of no feature of the process in question other than that the FDT (2.10) should hold. Since this theorem is valid for canonical systems in thermal equilibrium (with a different meaning of the matrix D_{12}),[2,3] all of our considerations are valid for such systems as well.

VII. THE VAN DER POL OSCILLATOR

The two-dimensional Van der Pol oscillator is known as a model for a single-mode laser operated near threshold and has been studied in some detail.[9] Its behavior is described by the Langevin equation

$$\dot{b} = ab - |b|^2 b + f \;, \tag{7.1}$$

where b is the complex oscillator amplitude, f a Gaussian-white-noise source,

$$\langle f(t) f(t')\rangle = \langle f^*(t) f^*(t)\rangle = 0 \;,$$
$$\langle f(t) f^*(t')\rangle = 4\delta(t-t') \;, \tag{7.2}$$

and $a \in (-\infty, +\infty)$ the so-called "pump parameter;" for $a \to -\infty$ (no pumping), the nonlinearity in Eq. (7.1) is negligible, whereas for $a > 0$ (above threshold), the nonlinearity is essential for preventing the oscillator amplitude from blowing up. The stationary probability distribution reads

$$\bar{P}(b, b^*) = N \exp[-\tfrac{1}{4}(bb^* - a)^2] \;. \tag{7.3}$$

This is approximately Gaussian far below threshold ($a \to -\infty$), but definitely not Gaussian near and above threshold. We therefore expect any perturbation theory in powers of the bare charge to fail except for $a \to -\infty$.[10] It is only the mass- and charge-renormalized perturbation scheme which we can hope to be useful near and above threshold.

In order to test the usefulness of the renormalized perturbation scheme, we note

$$D_{12} = 2\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma_{\hat{b} b b b^*} = \gamma_{\hat{b}^* b^* b^* b} = -2, \tag{7.4}$$

$$C_{12}(t) = C^{(2)}(0)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} c(t) \;,$$

and find, from Eqs. (4.2) and (5.4), to third order in γ^R,

$$C^{(2)}(0)\dot{c}(t) = -2c(t) - x^2(1 - \tfrac{5}{2}x)\int_0^t dt'\, c(t-t')^3 \dot{c}(t')$$
$$+ \tfrac{5}{2}x^3 \int_0^t dt' \int_0^t dt''\, c(t-t'')^2 \left(\frac{\partial}{\partial t''} c(t''-t')^2\right) c(t-t')\dot{c}(t'), \tag{7.5}$$

with the expansion parameter

$$x = x(a) = C^{(4)}(0)/C^{(2)}(0)^2 \;. \tag{7.6}$$

Equation (7.5) should be solved numerically. We have, instead, contented ourselves with a Markovian approximation for the linewidth λ,

$$\lambda^{-1} = \int_0^\infty dt\, c(t) \;. \tag{7.7}$$

The Markovian approximation consists in replacing $\int_0^t dt'\, \Sigma_{\hat{b}\hat{b}}(t-t')\mathring{c}(t')$ by $\int_0^\infty dt'\, \Sigma_{\hat{b}\hat{b}}(t')\dot{c}(t)$; this immediately leads to (including the fourth-order contribution)

$$C^{(2)}(0)\lambda = 2 - \tfrac{1}{3}x^2 + \tfrac{5}{18}x^3 - \tfrac{269}{540}x^4 + O(x^5) \;. \tag{7.8}$$

By invoking Risken's results[9] for $C^{(2)}(0)$ and $C^{(4)}(0)$, we find that Eq. (7.8) produces the right asymptotic dependence of λ on a both far below and (!) far above threshold,

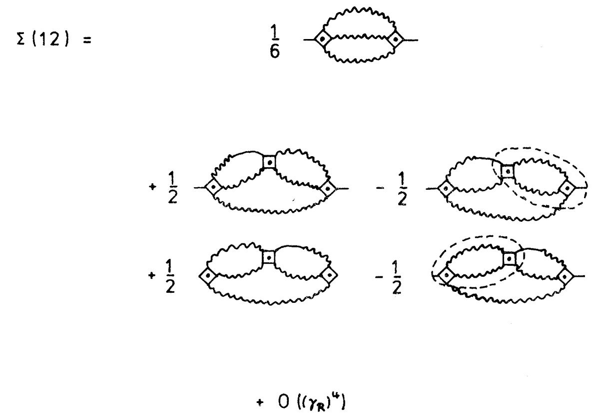

FIG. 9. Charge-renormalized expansion of $\Sigma(12)$.

$$\lambda(a\to-\infty)=|a|+\frac{4}{|a|}+O\left(\frac{1}{|a|^3}\right)\,,$$
$$\lambda(a\to+\infty)\propto\frac{1}{a}\,. \tag{7.9}$$

For a more detailed comparison, we have plotted the expansion parameter $x(a)$ in Fig. 10 and the linewidth $\lambda(a)$ obtained in first ($\Sigma=0$), second, third, and fourth order in x in Fig. 11. While these plots demonstrate a surprising success of the renormalized perturbation scheme, Fig. 12 illustrates the failure of a perturbation calculation with renormalized mass but unrenormalized charge; we have not plotted any result of the fully unrenormalized perturbation calculation because the latter gives nonsense except for the uninteresting case $a\to-\infty$.

We conclude that the renormalized perturbation scheme, although possibly being only semiconvergent, leads to a quantitatively good description of the dynamics even in situations where the system is definitely non-Gaussian.

APPENDIX

In order to prove Eq. (5.2), we provide ourselves with the FDT for the fourth-order cumulant,

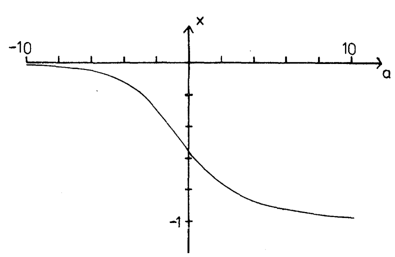

FIG. 10. Expansion parameter $x(a)$.

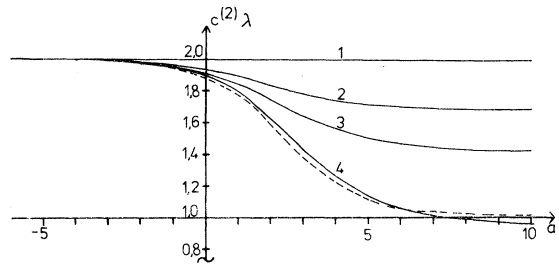

FIG. 11. Linewidth of the Van der Pol oscillator calculated in first (curve 1) through fourth (curve 4) order in mass- and charge-renormalized perturbation theory; the dashed curve gives the exact result (Refs. 9 and 11).

$$\frac{\partial}{\partial t_4}\,G_{123\underline{4}}(t_2,t_3,t_4)=G_{123\hat{5}}(t_2,t_3,t_4)D_{54}\,,$$
$$\text{if } t_4<0,\,t_2,\,t_3, \tag{A1}$$

where for the purposes of this appendix only we introduce the following convention for the indices of G and Γ: an underlined index refers to ψ, an index with a hat refers to $\hat{\psi}$, and indices without either specification may refer to either ψ or $\hat{\psi}$. The proof of (A1) is the same as that of the FDT (2.10) for $G(12)$.[2,5]

Next, we prove the following identity for the static pure-ψ cumulant

$$G_{\underline{1}\underline{2}\underline{3}\underline{4}}(0,0,0)=\int\int\int_{-\infty}^{0} dt_2\,dt_3\,dt_4\,G_{\underline{1}\hat{5}\hat{6}\hat{7}}(t_2,t_3,t_4)\times D_{52}D_{63}D_{74}\,. \tag{A2}$$

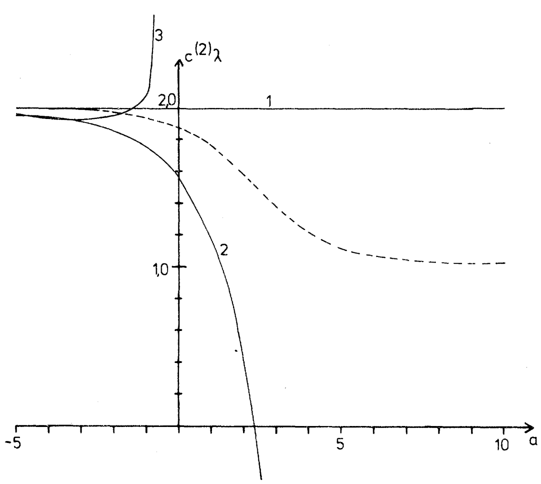

FIG. 12. Linewidth of the Van der Pol oscillator calculated in first (curve 1) through third (curve 3) order in mass- but not charge-renormalized perturbation theory; the dashed curve gives the exact result (Refs. 9 and 11).

After splitting the three-dimensional integration range on the right-hand side of (A2) into six parts such that $t_i \leq t_j \leq t_k$ in each part, we can use the FDT (A1) and carry out the integration over the smallest time; then the right-hand side (A2) reads

$$\left(\int_{-\infty}^{0} dt_2 \int_{-\infty}^{t_2} dt_3\, G_{\underline{1}\hat{5}\hat{6}\hat{4}}(t_2,t_3,t_3) D_{52} D_{63} + \int_{-\infty}^{0} dt_2 \int_{-\infty}^{t_2} dt_4\, G_{\underline{1}\hat{5}\underline{3}\hat{7}}(t_2,t_4,t_4) D_{52} D_{74}\right)$$

$$+\left(\int_{-\infty}^{0} dt_3 \int_{-\infty}^{t_3} dt_2\, G_{\underline{1}\hat{5}\hat{6}\underline{4}}(t_2,t_3,t_2) D_{52} D_{63} + \int_{-\infty}^{0} dt_3 \int_{-\infty}^{t_3} dt_4\, G_{\underline{1}\underline{2}\hat{6}\hat{7}}(t_4,t_3,t_4) D_{63} D_{74}\right)$$

$$+\left(\int_{-\infty}^{0} dt_4 \int_{-\infty}^{t_4} dt_2\, G_{\underline{1}\hat{5}\underline{3}\hat{7}}(t_2,t_2,t_4) D_{52} D_{74} + \int_{-\infty}^{0} dt_4 \int_{-\infty}^{t_4} dt_3\, G_{\underline{1}\underline{2}\hat{6}\hat{7}}(t_3,t_3,t_4) D_{63} D_{74}\right). \tag{A3}$$

Since the fourth-order cumulant (in contrast to the second order one) makes no jump at $t_i = t_j$, we may again invoke the FDT (A1) to rewrite (A3) as

$$\int_{-\infty}^{0} dt_2 \int_{-\infty}^{t_2} dt_3 \frac{d}{dt_3}\left[G_{\underline{1}\hat{5}\underline{3}\underline{4}}(t_2,t_3,t_3) D_{52} + G_{\underline{1}\underline{2}\hat{6}\underline{4}}(t_3,t_2,t_3) D_{63} + G_{\underline{1}\underline{2}\underline{3}\hat{7}}(t_3,t_3,t_2) D_{74}\right]$$
$$= \int_{-\infty}^{0} dt_2 \left[G_{\underline{1}\hat{5}\underline{3}\underline{4}}(t_2,t_3,t_3) D_{52} + G_{\underline{1}\underline{2}\hat{6}\underline{4}}(t_3,t_2,t_3) D_{63} + G_{\underline{1}\underline{2}\underline{3}\hat{7}}(t_3,t_3,t_2) D_{74}\right]. \tag{A4}$$

By repeating the argument once more, we recover the left-hand side of (A2).

We now relate the right-hand side in (A2) to the vertex Γ by

$$G(1234) = G(11')G(22')G(33')G(44')\Gamma(1'2'3'4') \tag{A5}$$

or, more specifically,

$$G(\underline{1}\,\hat{2}\hat{3}\hat{4}) = R(11')\Gamma(1'2'3'4')R(2'2)R(3'3)R(4'4)\,. \tag{A6}$$

By expressing the R's in (A6) in terms of $\dot{C}$'s with the help of the FDT (2.10), and inserting the result in (A2), we obtain

$$G_{\underline{1}\underline{2}\underline{3}\underline{4}}(0,0,0)$$
$$= \iiint_{-\infty}^{0} dt_2\, dt_3\, dt_4 \int_{-\infty}^{0} dt_1' \int_{t_2}^{\infty} dt_2' \int_{t_3}^{\infty} dt_3' \int_{t_4}^{\infty} dt_4' (-1)^3 \frac{\partial^3}{\partial t_2' \partial t_3' \partial t_4'} \dot{C}_{51}(t_i) D_{51'}^{-1} \Gamma_{\hat{1}'\underline{2}'\underline{3}'\underline{4}'}(t_2'-t_1', t_3'-t_1', t_4'-t_1')$$
$$\times C_{2'2}(t_2'-t_2) C_{3'3}(t_3'-t_3) C_{4'4}(t_4'-t_4)$$
$$= \int_{-\infty}^{0} dt_1' \iiint_{-\infty}^{0} dt_2'\, dt_3'\, dt_4'\, \dot{C}_{51}(t_1') D_{51'}^{-1} \Gamma_{\hat{1}'\underline{2}'\underline{3}'\underline{4}'}(t_2', t_3', t_4') C_{2'2}(0) C_{3'3}(0) C_{4'4}(0)$$
$$= C_{15}(0) D_{51'}^{-1} \gamma^{R}_{\hat{1}'\underline{2}'\underline{3}'\underline{4}'} C_{2'2}(0) C_{3'3}(0) C_{4'4}(0)\,. \tag{A7}$$

This establishes Eq. (5.2).

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[1]P. C. Martin, E. D. Siggia, and H. A. Rose, Phys. Rev. A 8, 423 (1973).

[2]U. Deker and F. Haake, Phys. Rev. A 11, 2043 (1975).

[3]It is easy to see and will be shown in a separate publication that the validity of the FDT in question [see Eq. (2.10)] is not restricted to the two classes of processes mentioned explicitly. Especially, it also holds for processes with a drift vector part of which is canonical and the irreversible part of which is of the form of a Ginzburg-Landau damping.

[4]R. Graham, *Springer Tracts in Modern Physics* (Springer, Berlin, 1973), Vol. 66.
[5]G. S. Agarwal, Z. Phys. 252, 25 (1972).
[6]C. de Dominicis and P. C. Martin, J. Math. Phys. 5, 14 (1964).
[7]It is shown in Ref. 1 that the FDT (3.13) is preserved in each order of perturbation theory.
[8]It can be shown by using the FDT that the expansion of $\Sigma_{\hat{1}\hat{2}}(t)$ involves the static cumulant $C^{(2)}$ only and not the time-dependent one.
[9]H. Risken, Fortschr. Phys. 16, 261 (1968).
[10]The completely unrenormalized perturbation expansion yields the asymptotic expansion for $C^{(2)}(0)$ around $a^{-1}=0$, in terms of a^{-1}, implied by (7.3); see U. Deker, Diplomarbeit (Stuttgart, 1974) (unpublished).
[11]K. Seybold, H. Risken, and H. D. Vollmer (private communication).