Wave packets in an absorptive and strongly dispersive medium

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The evolution of slowly varying wave pulses in strongly dispersive and absorptive media is studied by a recursive method. It is shown that the resulting envelope function may be obtained by including correction terms of arbitrary dispersive and absorptive orders. Different definitions of pulse velocity are discussed in some detail and finally, qualitative examples of a Gaussian pulse in a cold plasma and an atomic medium are given, illustrating the importance of higher-order terms in the velocity expression. Particularly, the packet velocity is shown to be dependent on pulse width besides dispersion.

I. INTRODUCTION

The propagation of signals in absorptive and dispersive media is continually of basic theoretical and practical interest. The conventional method of treating pulse-wave propagation in such media is to use the Fourier representation of the wave packet in terms of its spectral function at some given position.[1] This representation is developed by means of a Taylor expansion of the function $k = k(\omega)$, where k and ω denote wave number and frequency, respectively. However, for higher expansion terms the resulting integrals become difficult, and resort is taken to numerical computations. Another approach to the problem is to express the Fourier integral as a convolution integral,[2] or to calculate the wave field by the saddle-point method (see for example Ref. 3).

The evolution of slowly varying pulses in strongly dispersive, but nonabsorbing media can be studied by a recursive method,[4] i.e., a system of coupled first-order differential equations for successive approximations is derived and solved recursively. In the present paper we will extend this analysis to absorptive media and show that it is possible to obtain correction terms of arbitrary dispersive and absorptive orders for the envelope function, under certain conditions.

It is well known that the velocity of propagation of wave packets through a dispersive but nonabsorbing medium to first approximation is equal to the group velocity and to the velocity of energy propagation. For an absorbing medium, on the other hand, difficulties arise in attempting to determine the packet velocity which has a definite physical meaning.

The first attempt to determine the group velocity was made by Brillouin,[5] who found for a special case, that the packet velocity for weak absorption is

$$v_g = (\partial k_r/\partial\omega)^{-1}, \tag{1.1}$$

where k_r is the real part of the wave number. It is found, however, that v_g as given by (1.1) may be greater than the free-space velocity of light in certain frequency ranges. Normally this is not justified except in certain cases (see Ref. 6).

Terina[7] has investigated the propagation of a quasimonochromatic pulse with Gaussian envelope through a dispersive and strongly absorptive cold plasma. The pulse velocity was defined here as the velocity of propagation of the temporal maximum of the envelope at a given distance.

Another possibility to determine the pulse velocity is to calculate the velocity of propagation of the spatial maximum of the envelope function at a given time.[8] This concept of pulse velocity is discussed in more detail by Suchy.[3]

Thus, it appears from the literature that different definitions and resulting different expressions of the pulse velocity are used without any direct physical justifications. The purpose of this paper is to give a more detailed analysis of these problems. We will investigate the propagation of a slowly varying, particularly Gaussian, pulse in an absorptive and strongly dispersive medium. The temporal and spatial concepts of pulse velocity will be treated and differences between them discussed. We will also show that it is possible to obtain correction terms of higher dispersive and absorptive orders to the pulse velocity. The results are then applied to the case of a cold plasma with collisions, and to an atomic medium.

II. STANDARD ANALYSIS

In one-dimensional propagation of nonstationary waves, the field is defined by a Fourier integral,

$$E(t,x)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\hat{f}(\omega)\exp\{i[\omega t-k(\omega)x]\}\,d\omega\,, \quad (2.1)$$

where $k(\omega)$ is the complex wave number, and $\hat{f}(\omega)$ is the spectral function of the initial wave form $E(t,0)$. Writing

$$E(t,x)=E_0(t,x)\exp\{i[\omega_0 t-k_r(\omega_0)x]\}\,, \quad (2.2)$$

we identify the envelope function $E_0(t,x)$ as

$$E_0(t,x)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\hat{f}(\omega)\exp i\{(\omega-\omega_0)t-[k_r(\omega)-k_r(\omega_0)]x-ik_i(\omega)x\}\,d\omega\,, \quad (2.3)$$

where $k_r(\omega)$ and $k_i(\omega)$ are the real and imaginary parts of the wave number. The analytical evaluation of the Fourier integral (2.3) is not possible for general functions $\hat{f}(\omega)$ and $k(\omega)$. The standard approximation to obtain the main contribution to this integral is to expand $k_r(\omega)$ and $k_i(\omega)$ in a Taylor series around ω_0.

Taking into account the two first terms of the expansion of $k_r(\omega)$ and only the first term of the expansion of $k_i(\omega)$, we obtain the first-order expression

$$E_0(t,x)\approx e^{k_i(\omega_0)x}f(t-x/v_g)\,, \quad (2.4)$$

i.e., the initial form of the envelope function is damped by the factor $e^{k_i(\omega_0)x}$ and travels with the velocity $v_g=(\partial k_r/\partial\omega)^{-1}_{\omega_0}$. If we choose the input pulse with a Gaussian envelope,

$$E_0(t,0)=f(t)=A\exp(-bt^2)\,, \quad (2.5)$$

it is possible to evaluate explicitly the second-order expression obtained by including quadratic terms in the Taylor expansion

$$E_0(t,x)=\frac{Ae^{k_i(\omega_0)x}}{[1+2ixb(\partial^2k_r/\partial\omega^2)_{\omega_0}]^{1/2}}\times\exp\left(-\frac{b[t-(\partial k/\partial\omega)_{\omega_0}x]^2}{1+2ixb(\partial^2k_r/\partial\omega^2)_{\omega_0}}\right), \quad (2.6)$$

i.e., the wave packet spreads out.[1]

The above procedure is adequate if the spectral function $\hat{f}(\omega)$ is sharply peaked around ω_0, and the medium is moderately dispersive and absorptive. In the next section we will give an analytical treatment of situations where one or both of these conditions is violated and it becomes necessary to include higher-order terms in the expansion for $k_r(\omega)$ and $k_i(\omega)$.

III. RECURSIVE METHOD

The analysis will be based on the following assumptions: (a) The field of a wave pulse travelling through a linear, homogeneous, dispersive, and absorptive medium is characterized by a set of differential equations.[9,10] (b) The solutions are of the form

$$E(t,x)=E_0(t,x)\exp\{i[\omega_0 t-k_r(\omega_0)x]\}\,, \quad (3.1)$$

where $E_0(t,x)$ is a slowly varying envelope function compared to $\exp\{i[\omega_0 t-k_r(\omega_0)x]\}$. This may be expressed qualitatively as

$$\frac{|\partial E_0(t,x)/\partial t|}{\omega_0|E_0(t,x)|}=\epsilon_t\ll 1\,,\qquad \frac{|\partial E_0(t,x)/\partial x|}{k_r(\omega_0)|E_0(t,x)|}=\epsilon_x\ll 1\,. \quad (3.2)$$

(c) The imaginary part of the wave number is one order of magnitude smaller than the real part of the wave number, i.e.,

$$|k_i(\omega_0)/k_r(\omega_0)|=\epsilon\ll 1\,. \quad (3.3)$$

(d) We will assume ϵ_t, ϵ_x, and ϵ to be of the same order and use only the symbol ϵ. (e) Only a single-mode problem is considered.

The differential equation describing the evolution of the envelope function can be written as

$$P_{\rm op}\left(\frac{\partial}{\partial t},\frac{\partial}{\partial x}\right)E_0(t,x)=\sum_{n=0}^{+\infty}P_n\left(\frac{\partial}{\partial t},\frac{\partial}{\partial x}\right)E_0(t,x)=0\,, \quad (3.4)$$

where

$$P_0=k_0-k_r(\omega_0)\,,$$
$$P_1=-ik_i(\omega_0)+i\left[\left(\frac{\partial k_r}{\partial\omega}\right)_{\omega_0}\frac{\partial}{\partial t}+\frac{\partial}{\partial x}\right]\equiv i\frac{D}{Dx}\,,$$
$$\vdots$$
$$P_n=\frac{1}{i^n(n-1)!}\left(\frac{\partial^{n-1}k_i}{\partial\omega^{n-1}}\right)_{\omega_0}\frac{\partial^{n-1}}{\partial t^{n-1}}-\frac{1}{i^n n!}\left(\frac{\partial^n k_r}{\partial\omega^n}\right)_{\omega_0}\frac{\partial^n}{\partial t^n}\,. \quad (3.5)$$

If we, in agreement with our assumptions, express the envelope function as a Fourier integral (2.3) and expand $k_r(\omega)$ and $k_i(\omega)$ in a Taylor series around ω_0, it is easy to show that Eq. (3.4) is satisfied.

To solve Eq. (3.4) we will use the perturbation technique, which leads to the solution of a recurrence formula. We introduce the ordering parameter ϵ and write

$$P_{\rm op}\left(\frac{\partial}{\partial t},\frac{\partial}{\partial x}\right)=\sum_{n=0}^{+\infty}\epsilon^n P_n\left(\frac{\partial}{\partial t},\frac{\partial}{\partial x}\right),\qquad E_0(t,x)=\sum_{n=0}^{+\infty}\epsilon^n e_n(t,x)\,. \quad (3.6)$$

The power series in ϵ obtained by inserting Eq. (3.6) into Eq. (3.4) is identically zero if

$$\sum_{m=0}^{n} P_{n-m}\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x}\right) e_m(t, x) = 0, \quad n = 0, 1, \ldots . \tag{3.7}$$

For $n = 0$ we obtain

$$P_0 e_0(t, x) = 0, \tag{3.8}$$

which has nontrivial solutions if $P_0 = 0$. Omitting terms proportional to P_0 which vanish, we can rewrite (3.7) as a recursive system of equations

$$\frac{De_n}{Dx} = i \sum_{m=0}^{n-1} P_{n-m+1}\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x}\right) e_m(t, x), \tag{3.9}$$

which yields

$$\frac{De_0}{Dx} = 0, \quad \frac{De_1}{Dx} = iP_2 e_0,$$
$$\frac{De_2}{Dx} = i(P_3 e_0 + P_1 e_2), \quad \ldots . \tag{3.10}$$

Using (3.5) we find the particular solutions to (3.10) as

$$e_0(t, x) = e^{k_i(\omega_0)x} f\left[t - \left(\frac{\partial k_r}{\partial \omega}\right)_{\omega_0} x\right] = e^{k_i(\omega_0)x} f(\xi),$$

$$e_1(t, x) = ie^{k_i(\omega_0)x} x\left[\frac{1}{2}\left(\frac{\partial^2 k_r}{\partial \omega^2}\right)_{\omega_0} f''(\xi) - \left(\frac{\partial k_i}{\partial \omega}\right)_{\omega_0} f'(\xi)\right],$$

$$e_2(t, x) = e^{k_i(\omega_0)x}\left\{\frac{x}{6}\left[\left(\frac{\partial^3 k_r}{\partial \omega^3}\right)_{\omega_0} f'''(\xi) - 3\left(\frac{\partial^2 k_i}{\partial \omega^2}\right)_{\omega_0} f''(\xi)\right]\right.$$
$$\left. - \frac{x^2}{8}\left[\left(\frac{\partial^2 k_r}{\partial \omega^2}\right)^2 f^{(4)}(\xi) - 4\left(\frac{\partial^2 k_r}{\partial \omega^2}\right)_{\omega_0}\left(\frac{\partial k_i}{\partial \omega}\right)_{\omega_0} f'''(\xi) + 4\left(\frac{\partial^2 k_i}{\partial \omega^2}\right)^2_{\omega_0} f''(\xi)\right]\right\}, \quad \ldots , \tag{3.11}$$

where f is an arbitrary function which satisfies the initial condition $E_0(t, 0) = f(t)$. It is obvious that this process could be continued to give higher-order correction terms to the envelope function.

The most well known and simple form of a slowly varying wave train is that with a Gaussian envelope given by (2.5). The envelope function according to (3.6) and (3.11) then becomes

$$E_0(t, x) \approx A\left\{1 + ixb\left[\left(\frac{\partial^2 k_r}{\partial \omega^2}\right)_{\omega_0}(2b\xi^2 - 1) + 2\left(\frac{\partial k_i}{\partial \omega}\right)_{\omega_0}\xi\right] + \frac{xb}{3}\left[\left(\frac{\partial^3 k_r}{\partial \omega^3}\right)_{\omega_0} 2b\xi(3 - 2b\xi^2) - \left(\frac{\partial^2 k_i}{\partial \omega^2}\right)_{\omega_0} 3(2b\xi^2 - 1)\right]\right.$$
$$\left. - \frac{x^2 b}{2}\left[\left(\frac{\partial^2 k_r}{\partial \omega^2}\right)_{\omega_0} b(3 - 12b\xi^2 + 4b^2\xi^4) - 4b\xi\left(\frac{\partial^2 k_r}{\partial \omega^2}\right)_{\omega_0}\left(\frac{\partial k_i}{\partial \omega}\right)_{\omega_0}(3 - 2b\xi^2) + \left(\frac{\partial k_i}{\partial \omega}\right)^2_{\omega_0} 2(2b\xi^2 - 1)\right]\right\}$$
$$\times \exp[k_i(\omega_0)x - b\xi^2] + \cdots , \tag{3.12}$$

where $\xi = t - (\partial k_r/\partial \omega)_{\omega_0} x$. To compare this with the expression (2.6) obtained from the integral representation we use the relation

$$\frac{\partial^2}{\partial u^2}\left[\frac{1}{\sqrt{z}} \exp\left(-\frac{u^2}{4z}\right)\right] = \frac{\partial}{\partial z}\left[\frac{1}{\sqrt{z}} \exp\left(-\frac{u^2}{4z}\right)\right], \tag{3.13}$$

and rewrite (2.6) as

$$E_0(t, x) = A\left\{\exp\left[\frac{ix}{2}\left(\frac{\partial^2 k_r}{\partial \omega^2}\right)_{\omega_0}\frac{\partial^2}{\partial \xi^2} - ix\left(\frac{\partial k_i}{\partial \omega}\right)_{\omega_0}\frac{\partial}{\partial \xi}\right]\right\}$$
$$\times \exp[k_i(\omega_0)x - b\xi^2]. \tag{3.14}$$

For sufficiently small values of the quantities $|x(\partial^2 k_r/\partial \omega^2)_{\omega_0}|$ and $|x(\partial k_i/\partial \omega)_{\omega_0}|$ we can expand the exponential operator in (3.14) and obtain a result in accordance to (3.12). Thus, we note that the recursive method and the standard analysis give equivalent results within the degree of approximation. However, the expression for the envelope function resulting from the recursive method is obtained for a general pulse form and can include correction terms of arbitrary dispersive and absorptive orders. Finally, we also note that the parameter ϵ can be identified with characteristic physical quantities as follows: From (3.11) we obtain

$$\epsilon \sim \left|\frac{e_1}{e_0}\right|$$
$$\sim x\,\frac{\left|\frac{1}{2}(\partial^2 k_r/\partial \omega^2)_{\omega_0} f''(\xi)\right| + \left|(\partial k_i/\partial \omega)_{\omega_0} f'(\xi)\right|}{f(\xi)}$$
$$\sim x\left[\left|\frac{(\partial^2 k_r/\partial \omega^2)_{\omega_0}}{\sigma_0^2}\right| + \left|\frac{(\partial k_i/\partial \omega)_{\omega_0}}{\sigma_0}\right|\right], \tag{3.15}$$

where σ_0^2 is the original width of the packet ($\sigma_0^2 = 1/b$ for a Gaussian pulse). This implies that the asymptotic solution (3.6) is valid for

$$x < x_{\mathrm{crit}} \sim \frac{\sigma_0^2}{|(\partial^2 k_r/\partial\omega^2)_{\omega_0}| + |\sigma_0(\partial k_i/\partial\omega)_{\omega_0}|} . \quad (3.16)$$

IV. THE VELOCITIES OF A WAVE PACKET

We will now discuss the basic characteristics and nature of the concepts of the pulse velocity. It is obvious that the envelope function $|E_0(t,x)|$ may have two different types of maxima: a temporal maximum at a given distance x, and a spatial maximum at a given time t. Consequently we can define two types of pulse velocity: the "temporal" and "spatial" pulse velocity, respectively. Our analysis is based on the results derived from the recursive method in Sec. III. For convenience we will use the notations

$$\left(\frac{\partial^{n+1}k_r}{\partial\omega^{n+1}}\right)_{\omega_0} = \frac{1}{v_g^{(n)}},$$

$$\left(\frac{\partial^{n+1}k_i}{\partial\omega^{n+1}}\right)_{\omega_0} = \frac{1}{a^{(n)}}, \quad n = 0, 1, 2, \ldots \quad (4.1)$$

$$\xi = t - x/v_g .$$

A. "Temporal" pulse velocity

A number of authors[5,7,11] define the pulse velocity as the velocity of propagation of the temporal maximum of the envelope function $|E_0(t,x)|$ at a given distance x according to the formula

$$v_t = x/t_M , \quad (4.2)$$

where t_M is the time of the temporal maximum of the envelope and is determined from

$$\left.\frac{\partial|E_0(t,x)|}{\partial t}\right|_{t=t_M} = 0 . \quad (4.3)$$

Using the results of the recursive method for the Gaussian pulse Eq. (3.12) together with (4.2) and (4.3), we obtain for $|E_0(t,x)| \approx |e_0(t,x)|$ the first-order expression $v_t^{(1)} = v_g$, and for $|E_0(t,x)| \approx [|e_0(t,x)+e_2(t,x)|^2 + |e_1(t,x)|^2]^{1/2}$ the velocity becomes to second order

$$\frac{1}{v_t^{(2)}} = \frac{1}{v_g} + \frac{2bx/v_g' a + b/v_g''}{1 + 4bx/a' + 2bx^2(1/a^2 - 3b^2/v_g'^2)} \approx \frac{1}{v_g} + \frac{2bx}{v_g' a} , \quad (4.4)$$

where we have assumed that all terms including ξ^n for $n \geq 2$ are negligible near the maximum. The second expression in (4.4) is obtained for $1/v_g'' \to 0$, $1/a' \to 0$, and by means of the condition (3.16).

It appears from (4.4) that the "temporal" pulse velocity in an absorptive and dispersive medium is not constant. As the distance x increases, the change in the fine-structure frequency (i.e., the spectrum of the pulse) causes a variation in its velocity x/t_M.[11] In fact, each spectral interval $(\omega, \omega + d\omega)$ moves at its own velocity which differs in each interval $(x, x+dx)$. Thus, we can define the differential velocity for each point x according to

$$v_{td} = \left(\frac{dt_M}{dx}\right)^{-1} . \quad (4.5)$$

However, the pulse-velocity definition most appropriate in connection with the experiments is the differential velocity averaged over a distance from the origin $x = 0$ up to the point x,

$$v_t = \frac{x}{\int_0^x [dx'/v_{td}(x')]} = \frac{x}{t_M(x)} \quad (4.6)$$

which is equal to (4.2).

B. "Spatial" pulse velocity

The concept of "spatial" pulse velocity was suggested by Barsukov and Ginzburg,[8] and later discussed in more detail by Suchy.[3] According to the definition

$$v_{xd} = \frac{dx_M}{dt} , \quad (4.7)$$

where x_M is the distance travelled by the spatial maximum of the envelope at a given time t and which is determined from

$$\left.\frac{\partial|E_0(t,x)|}{\partial x}\right|_{x=x_M} = 0 , \quad (4.8)$$

they found the expression

$$v_{xd} = \frac{v_g}{1 - v_g^2/a^2} , \quad (4.9)$$

which is valid for a medium with moderate absorption. This result does not depend on the initial pulse form.

We will now derive the velocity expression for a Gaussian pulse using the results given by the recursive method. Thus, from Eq. (3.12) together with Eqs. (4.7) and (4.8) we get in the first-order approximation $v_{xd}^{(1)} = v_g$, and to second order

$$v_{xd}^{(2)} \approx v_g \frac{1 - v_g^2[1/a^2 - b/v_g'^2 - 2k_i(\omega_0)v_g/v_g' a]}{[1 - v_g^2(1/a^2 - b/v_g'^2 + 4bt/v_g' a)]^2} , \quad (4.10)$$

where we have assumed that $1/v_g'' \to 0$, $1/a' \to 0$ and neglected all terms including ξ^n and x^n for $n \geq 2$ according to the condition (3.16). If we assume in (4.10) that $1/v_g' \to 0$, we obtain directly the expression (4.9). We conclude that the "spatial" velocity

(4.10) depends on the parameters of the medium, the propagation time and the width of the initial pulse form. The relation (4.12) defines the differential velocity of the spatial maximum of the envelope. We can also introduce the averaged "spatial" velocity

$$v_x = \int_0^t v_{xd}(t')\,dt' \Big/ t = [x_M(t) - x_M(0)]/t\ , \qquad (4.11)$$

which for the Gaussian pulse is

$$v_x^{(2)} = \frac{v_g}{1 - v_g^2/a^2 + b v_g^2/v_g'^2} \times \frac{1 - v_g^2[1/a^2 - b/v_g'^2 - 2k_i(\omega_0)v_g/v_g'a]}{1 - v_g^2(1/a^2 - b/v_g'^2 + 4bt/v_g'a)}\ . \qquad (4.12)$$

It appears that the concepts of "temporal" and "spatial" velocity give different results. The basic problem is now to define which of the velocities has a physical meaning and is a measurable quantity. The literature gives different answers on this question. Suchy[3] is of the opinion that the "temporal" velocity is not a velocity in a physical meaning despite its correct dimensions. This conclusion is based on the fact that the "spatial" velocity given by (4.9) is greater than the "temporal" velocity $v_t = v_g$. Furthermore, only the "spatial" velocity includes the influence of absorption, while the "temporal" velocity remains the same as in the nonabsorbing case.

We will note here that these arguments are unacceptable if we include correction terms of higher dispersive and absorptive orders for the envelope function. Then, the "temporal" and "spatial" velocities become functions of x and t, respectively, and the curves of temporal and spatial maxima in an x-t plane are no longer straight lines. Generally, it is seen that the temporal and spatial maxima for an absorbing medium are situated on two different curves and they always differ from the corresponding curves in the nonabsorbing case. The character of the pulse velocity is related rather to the choice of the initial pulse form. If the initial pulse form is $E_0(0, x)$ and we write the dispersion relation as $\omega = \omega(k)$, it is correct to calculate the "spatial" pulse velocity, which is a function of time. However, if the inital pulse is $E_0(t, x)$ and the dispersion relation is written as $k = k(\omega)$, it is natural to work with the "temporal" pulse velocity, which is a function of distance.

One of the decisive properties in this context is the measurement of the pulse velocity. For the experimentalist it is easier to measure the time at which the maximum of the pulse moves from position x_0 to position x_1, than to determine the distance at which the spatial maximum arrives at a given time point t. Therefore, he will be more interested in the "temporal" velocity of propagation of the pulse. However, it should be mentioned that there are situations when it is natural to use the "spatial" velocity. We may think of an astronomical source giving rise to a pulse at $t = 0$ within the source $|x| < L$ and the signal is received at a certain distance from the source as a function of time.

Many of the measurements of the pulse velocity in resonant systems have been made by placing the material under study in a mode-locked laser cavity and measuring the change in the pulse repetition rate. Using this technique, Faxvog *et al.*[12] found that the pulse velocity in a Ne absorption cell placed inside a self-locked He-Ne laser exceeds c, the free space velocity of light, by about 3 parts in 10^4. The measured velocity is in fact the "temporal" pulse velocity. Measurement of the pulse repetition frequency at the output of a self-locked laser with a fixed cavity length corresponds to a measurement of the time required for the maximum of the pulse to travel a given length of propagation path. A similar technique was used by Casperson and Yariv[13] to determine the "temporal" pulse velocity in a high-gain 3.51-μm xenon laser. The observed velocity was less than the vacuum speed of light by as much as a factor of 2.5. We notice, however, that the experiments in Refs. 12 and 13 consider only resonant pulse propagation for which the correction terms to the group velocity, see (4.4), are equal to zero. An interesting experiment to measure the "temporal" pulse velocity in dilute rubidium vapor was made by Grischkowsky.[14] In contrast to the earlier intracavity experiments, the Rb-vapor cell was here completely passive and located outside the laser. However, this experiment deals with the case far out in the resonance wings where the dispersion of the damping is small and the higher-order correction terms to the group velocity are negligible. No measurements of the pulse velocity are known to the authors which verify the expression (4.4), i.e., take into account correction terms including effects of the pulse width. However, such correction terms should be of interest, for example, in measurement of the propagation of whistler packets in a collisional plasma, like in Ref. 15, where the correction to the group velocity can be up to 5% for the extreme values of the cited parameters.

C. Applications

1. Cold plasma

We will apply the expression (4.4) to calculate the "temporal" velocity of propagation of a Gaus-

sian pulse in a homogeneous, isotropic, cold plasma with collisions. We assume for simplicity that the dispersion relation can be written as

$$k(\omega)\approx\frac{\omega}{c}\left(1-\frac{\omega_p^2}{2\omega(\omega-i\nu_{\rm eff})}\right), \tag{4.13}$$

where $\omega_p=(N_e^2/\epsilon_0 m)^{1/2}$ is the plasma frequency, and $\nu_{\rm eff}$ is the effective collision frequency. Under the condition $\omega^2\gg\nu_{\rm eff}^2$ (high-frequency case), the velocity obtained from (4.4) is

$$v_t^{(2)}\approx\frac{c}{1+\omega_p^2/2\omega_0^2-2(x/c)b(\omega_p^2\nu_{\rm eff}/\omega_0^6)} \tag{4.14}$$

which according to (3.16) is valid for

$$x<x_{\rm crit}\sim\frac{c\sigma_0^2\omega_0^3}{\omega_p^2(1+\sigma_0\nu_{\rm eff})}, \tag{4.15}$$

where σ_0 is the original pulse width ($\sigma_0^2=1/b$). Using for example the parameters of the ionospheric E layer: $\nu_{\rm eff}=2\pi\times10^5$/sec, $\omega_0=6\pi\times10^6$/sec, $\omega_p/\omega_0\approx1$, and $\sigma_0=4$ μsec, we can calculate from (4.14), $v_t^{(2)}/c\approx1/(1.5-5\times10^{-6}x)$ where $x<x_{\rm crit}\sim26\times10^3$ m.

2. *Atomic medium*

The dispersion relation for an atomic medium is

$$k(\omega)\approx\frac{\omega}{c}\left(1-\frac{\omega_p^2}{4\omega(\omega-\Omega-i/T_2)}\right),\quad|\omega_pT_2|\ll1, \tag{4.16}$$

where Ω is the resonance frequency, and T_2 is the relaxation time. If we introduce $\eta=(\omega_0-\Omega)T_2$, and $\alpha=x|\omega_p^2|T_2^3/4c\sigma_0^2$, the pulse velocity can be written from (4.4) and (4.16) as

$$\frac{1}{v_t^{(2)}}=\frac{1}{c}\mp\frac{|\omega_p^2|T_2^2}{4c(1+\eta^2)^3}(1-\eta^4)\left(1\mp\frac{\alpha}{\alpha_0}\right),$$

$$\alpha_0=\frac{(1-\eta^4)(1+\eta^2)^2}{8\eta^2(3-\eta^2)}, \tag{4.17}$$

$$\alpha_{\rm crit}=\frac{(1+\eta^2)^3}{2|\eta|[|3-\eta^2|+\sigma_0(1+\eta^2)/T_2]},$$

where $-$ [$+$] corresponds to the case of a noninverted ($\omega_p^2>0$) [inverted ($\omega_p^2<0$)] medium.

TABLE I. Conditions when $v_t^{(2)}>c$.

	$0<\|\eta\|<1$	$1<\|\eta\|<\sqrt{3}$	$\|\eta\|>\sqrt{3}$
$\omega_p^2>0$	$\alpha<\alpha_0$	···	$\alpha>\alpha_0$
$\omega_p^2<0$	···	$\alpha>\alpha_0$	$\alpha>0$

We conclude that the pulse velocity will exceed the vacuum velocity c for η and α values given by Table I. Particularly, when $|\eta|=1$ the small correction term proportional to α will determine if the pulse velocity is larger or less than the vacuum velocity.

Finally it should be noted that when the pulse velocity exceeds the free-space velocity of light (i.e., when the velocity of the maximum exceeds the velocity of the pulse front), steepening effects eventually invalidate the assumption of slowly varying amplitudes. A qualitative condition is that the propagation path should be less than the pulse width divided by $(1/c-1/v_t)$.

V. CONCLUSIONS

We have introduced an analytical method to study the evolution of slowly varying pulses in strongly dispersive and absorptive media. This method, contrary to the standard analysis, gives the possibility by simple calculations, to obtain correction terms of arbitrary dispersive and absorptive orders for the envelope function. The results obtained from the recursive method are valid only for a propagation path limited by a critical value.

Different concepts of pulse velocity in dispersive and absorptive media have been discussed. It was shown, using the recursive method, that it is possible to obtain velocity expressions containing correction terms of higher dispersive and absorptive order. The "temporal" pulse velocity has a definite physical meaning and it will be the quantity which is most easy to determine in an experiment. The application of our results to an atomic medium shows that the pulse velocity can be greater or less than the free-space velocity of light, depending on the length of propagation path and the shift between the pulse center frequency and the atomic line-center frequency.

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