Semiclassical electrodynamics of a two-level system

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(Received 21 August 1974)

We analyze the problem of a spin-1/2 system coupled to a longitudinal field B_0 and a transverse field $2B\cos\omega t$. We show that the magnetization is not periodic in general. Time-averaged population inversion is non-negative for the case $\omega = -\gamma B_0$. We discuss the significance of this result and show that eigenvalues of the Floquet Hamiltonian of Shirley's theory must be discontinuous at this point. The solution also exhibits satellite frequencies symmetrically displaced about odd multiples of ω.

I. INTRODUCTION

Consider a spin-$\frac{1}{2}$ system in a static magnetic field B_0 and an oscillating field $2B\cos\omega t$ applied transverse to it. The static field removes degeneracy and the oscillating field induces transitions. Suppose that initially the spin is "pointing down." We wish to calculate the probability $P_{-1/2,1/2}$ that the system has made a transition to the "spin-up" state. This problem is equivalent to that of solving Schrödinger's equation for a two-level atom with a periodic potential. Therefore all results derived in this paper are also applicable to the latter case.

The magnetic moment $\vec{M}$ evolves according to the well-known equation

$$d\vec{M}/dt = \gamma(\vec{M}\times\vec{B}) , \qquad (1.1)$$

where $\vec{M}\equiv(M_x, M_y, M_z)$ and $\vec{B}\equiv(2B\cos\omega t, 0, B_0)$. We define

$$\omega_0 \equiv -\gamma B_0, \quad b\equiv -\tfrac{1}{2}\gamma B, \quad \vec{M}\equiv \tfrac{1}{2}\gamma\hbar\vec{m}.$$

The component m_z of $\vec{m}$ measures the relative population inversion: $m_z = 1$ corresponds to total inversion and $m_z = -1$ means that the atom is in the ground state. It is related to the transition probability $P_{-1/2,1/2}$ in a simple manner:

$$P_{-1/2,1/2} = \tfrac{1}{2}(1+m_z) . \qquad (1.2)$$

Generally the time average of $P_{-1/2,1/2}$ is measured. This average is taken over times much greater than the period of oscillation. The result of this averaging is that all oscillating terms drop out and the attention is focussed on the static component of m_z, a_0 say. Those values of the static field B_0 for which a_0 is maximum determine positions of resonance. The difference $\omega - \omega_0$ is known as the Bloch-Siegert shift[2] and has been calculated theoretically[2-5] and measured experimentally[1] to a high degree of accuracy. Shirley[3] has described an elegant method for calculating a_0 by showing that it is related to the eigenvalues q of an infinite Floquet matrix

$$a_0 = -4(\partial q/\partial\omega_0)^2 . \qquad (1.3)$$

Equations (1.2) and (1.3) show that the time average P_{ave} of the transition probability is maximized when the average occupation of the two states is equally likely and the average induced rates out of each state are equal.[5]

In this paper we shall describe a general method for obtaining a bounded solution[6-8] for the transition probability $P_{-1/2,1/2}$. In particular, we find that when $\omega=\omega_0$, a_0 is non-negative. We shall discuss the physical significance of this result in Sec. IV below and suggest how theories based on (1.3) may have to be modified.

The plan of the paper is as follows: In Sec. II we briefly comment on the perturbation series solution for m_z and compare it with the solution in the rotating-field approximation (RFA), i.e., when the oscillating field $2B\cos\omega t$ is replaced by a rotating field

$$B\cos\omega t\,\hat{i} + B\sin\omega t\,\hat{j} ,$$

where $\hat{i}$ and $\hat{j}$ are unit vectors in the x and y directions, respectively. In Sec. III we describe the general solution for m_z and discuss it in detail for the special case $\omega_0=\omega$ in Sec. IV. Section V summarizes the results.

II. PERTURBATION THEORY

It is well known that ordinary perturbation theory, when applied to the problem of a two-level system, gives rise to secular terms,[7] i.e., terms which increase indefinitely with time and therefore apparently violate the unitarity condition on the transition amplitude. From (1.1) one can easily derive the following integral equation for the relative population inversion m_z[9]:

$$m_z(t) = -1 + 16b^2\int_{t_0}^{t}\int_{t_0}^{t_1}\cos\omega(t_1-t_0)\cos\omega(t_2-t_0)\cos\omega_0(t_1-t_2)\,m_z(t_2)\,dt_2\,dt_1 . \qquad (2.1)$$

We have assumed that initially $m_z = -1$, i.e., the atom is in the "spin-down" state. The equation can be formally solved by iteration thus giving a perturbation series. Up to first order in b^2 we get

$$m_z = -1 + \frac{4b^2}{\omega^2-\omega_0^2} + \frac{16b^2\omega_0^2}{(\omega^2-\omega_0^2)^2} - \frac{4b^2}{\omega^2-\omega_0^2}\left(\cos 2\omega t - \frac{2\omega_0}{\omega+\omega_0}\cos(\omega+\omega_0)(t-t_0) + \frac{2\omega_0}{\omega-\omega_0}\cos(\omega-\omega_0)(t-t_0)\right) + \cdots \,. \tag{2.2}$$

Secular terms appear when we go to higher orders. Letting

$$r = 16b^2\omega_0^2/(\omega^2-\omega_0^2)^2 \,,$$

we notice that continuing (2.2) to all orders, the constant term will contain an infinite geometric progression $-(1 - r + r^2 - \cdots)$ which can be readily summed to get

$$\frac{-1}{1+r} \equiv \frac{-(\omega^2-\omega_0^2)^2}{(\omega^2-\omega_0^2)^2+16b^2\omega_0^2} \simeq \frac{-(\omega-\omega_0)^2}{(\omega-\omega_0)^2+4b^2} \,. \tag{2.3}$$

This is precisely the constant term that one obtains on making the rotating-field approximation.

Now let us set $\omega = \omega_0$ in Eq. (2.1) and develop the perturbation series as before. Secular terms appear even to first order. However, we can isolate a series

$$-[1 - (2b)^2(t-t_0)^2/2! + (2b)^4(t-t_0)^4/4! - \cdots]$$

which sums to give

$$-\cos 2b(t-t_0) \,. \tag{2.4}$$

This is again seen to be the solution of the problem in RFA when $\omega = \omega_0$.

The above examples illustrate that the problems of perturbation theory arise merely because one truncates the series after a few terms. It appears plausible that if one could calculate a large number of terms, then judicious rearrangement and summation would lead to the transition probability being expressed in terms of periodic functions alone. However, this procedure, even if successful, would be very tedious. In Sec. III we describe a more systematic method of tackling this problem.

III. BOUNDED SOLUTION FOR TRANSITION PROBABILITY

In this section we shall describe a method which readily gives m_z as a sum of periodic functions. Instead of Eq. (2.1) we shall work with the differential equation satisfied by m_z:

$$\frac{d^5 m_z}{dt^5} + 2[4b^2 + 4b^2\cos 2\omega(t-t_0) + \omega^2 + \omega_0^2]\frac{d^3 m_z}{dt^3} - 56b^2\omega\sin 2\omega(t-t_0)\frac{d^2 m_z}{dt^2}$$
$$+ [8b^2(\omega_0^2 - 15\omega^2)\cos 2\omega(t-t_0) + 8b^2(\omega^2+\omega_0^2) + (\omega^2-\omega_0^2)^2]\frac{dm_z}{dt} - 8b^2\omega(\omega_0^2 - 9\omega^2)\sin 2\omega(t-t_0)\, m_z = 0 \,. \tag{3.1}$$

The initial conditions are

$$m_z(t_0) = -1,\;\; m_z'(t_0) = 0,\;\; m_z''(t_0) = 16b^2,\;\; m_z'''(t_0) = 0$$

and

$$m_z''''(t_0) = -16b^2(16b^2 + 4\omega^2 + \omega_0^2) \,, \tag{3.2}$$

where primes denote derivatives. We assume a solution of the form

$$m_z = \sum_{n=0}^{\infty} a_n \cos n\omega(t-t_0) + \sum_{r=-\infty}^{\infty} c_r \cos(\Delta + r\omega)(t-t_0). \tag{3.3}$$

It is shown below that only even values of n and r need be taken into account and that the assumed solution is indeed the unique solution of the problem. No sine terms occur in (3.3) since all odd-order derivatives vanish initially. In Appendix A we show that Δ is independent of initial conditions.

The rotating-field approximation is equivalent to truncating the summations (3.3) only after their first terms to get

$$m_z = -(\omega-\omega_0)^2/\Delta^2 - (4b^2/\Delta^2)\cos\Delta(t-t_0) \,, \tag{3.4}$$

where Δ is the so-called "nutation frequency,"

$$\Delta^2 = (\omega - \omega_0)^2 + 4b^2 \,. \tag{3.5}$$

Note that m_z as given by (3.3) is *not periodic*. In general Δ will be an irrational number and each term in the second sum will have a different period. This is in contrast with theories which assume *a priori* that all components of the magnetization are periodic and therefore can be Fourier expanded.[4c] However, one can determine positions of resonances correctly solely because this requires a consideration of the maxima of

a_0 which can be done independently of the nonperiodic part of the solution.[5]

By substituting (3.3) in (3.1) we see that the a's and c's must satisfy certain recurrence relations. The relation satisfied by a_{2n} is[5]

$$\{\omega_0^4 - 2[(4n^2+1)\omega^2 - 4b^2]\omega_0^2 + (4n^2-1)\omega^2[(4n^2-1)\omega^2 - 8b^2]\}a_{2n} - 4b^2\epsilon_n(1-1/2n)[(2n+1)^2\omega^2 - \omega_0^2]a_{2n-2} - 4b^2(1+1/2n)[(2n-1)^2\omega^2 - \omega_0^2]a_{2n+2} = 0 \ , \tag{3.6}$$

where $\epsilon_n = 2$ when $n=1$ and $\epsilon_n = 1$ otherwise. We shall assume that

$$\lim_{n\to\infty} a_n = 0 \text{ and } \lim_{r\to\infty} c_r = 0 \ . \tag{3.7}$$

We can express the ratios $(b^2/\omega^2)(a_0/a_2)$, $(b^2/\omega^2)\times(a_2/a_4)$, etc. as infinite continued fractions whose convergence is fast for most values of b/ω. If we write the recurrence relation (3.6) in the form

$$p_n a_{2n} - q_n a_{2n-2} - r_n a_{2n+2} = 0 \ ,$$

where p_n, q_n, r_n have obvious definitions, we can easily derive

$$q_n \frac{a_{2n-2}}{a_{2n}} = p_n - \cfrac{r_n q_{n+1}}{p_{n+1} - \cfrac{r_{n+1}q_{n+2}}{p_{n+2} - \cdots}} \ . \tag{3.8}$$

By using Eq. (3.8) repeatedly, one can express each a_n in terms of a_0.

Analysis for the second series in (3.3) follows along similar lines. In Appendix A we show that the equation determining Δ^2 has in general at least two distinct roots giving rise to two aperiodic solutions:

$$K_{\text{I}} \equiv \sum_{r=-\infty}^{\infty} c_{2r}\cos[(\Delta_1 + 2r\omega)(t-t_0)] \tag{3.9a}$$

and

$$K_{\text{II}} \equiv \sum_{r=-\infty}^{\infty} c'_{2r}\cos[(\Delta_2 + 2r\omega)(t-t_0)] \tag{3.9b}$$

where $\Delta_1 \neq \Delta_2$. One can again write expressions of the form (3.8) and express each c_{2r} in terms of c_0 and c'_{2r} in terms of c'_0. This leaves us with three undetermined coefficients a_0, c_0, and c'_0.

These will be fixed when we use initial conditions (3.2). Since the proposed solution satisfies the differential equation and is consistent with the initial conditions, it must indeed be the unique solution of the problem.

We can calculate all a_{2n} by using the value of a_0 calculated by a completely different approach due to Shirley.[3] In his theory a_0 is given as Eq. (1.3), where q is an eigenvalue of an infinite Floquet matrix. The square of q has a development in terms of $b^2(\omega+\omega_0)$:

$$q^2 = \tfrac{1}{4}(\omega-\omega_0)^2 + \frac{2\omega_0 b^2}{\omega+\omega_0} - \frac{2\omega_0 b^4}{(\omega+\omega_0)^3} + \frac{8\omega_0(\omega^2 - 5\omega\omega_0 - 2\omega_0^2)b^6}{(\omega+\omega_0)^5(9\omega^2-\omega_0^2)} \cdots \ . \tag{3.10}$$

By using (3.8), (1.3), and (3.10) we can find all a_{2n}. We will later show that this method fails when $\omega = \omega_0$.

We have solved the problem without encountering difficulties associated with the perturbation theory. For sufficiently small values of b one may express Δ and various coefficients as a power series in b, which we do in the next section. However, the accuracy will be much improved if one uses expressions of the form (3.8) as they stand, since they are valid for any value of b.

The calculation becomes simpler for the special case $\omega = \omega_0$, since a first integral of Eq. (3.1) can be obtained in this case. We shall treat this in detail in the following section. This will illustrate the application of the method described and will also enable us to make comparison with some previous theories.

IV. THE CASE $\omega = \omega_0$

We set $\omega = \omega_0$ in (3.1) and integrate to obtain

$$\frac{d^4 m_z}{dt^4} + 4[2b^2 + \omega^2 + 2b^2\cos 2\omega(t-t_0)]\frac{d^2 m_z}{dt^2} - 40b^2\omega\sin 2\omega(t-t_0)\frac{dm_z}{dt} - \{32b^2\omega^2\cos 2\omega(t-t_0) - 16b^2\omega^2\}m_z = 0 \ , \tag{4.1}$$

with $m_z(t_0) = -1$, $m'_z(t_0) = 0$, $m''_z(t_0) = 16b^2$, and $m'''_z(t_0) = 0$. We assume a solution

$$m_z = a_0 + \sum_{n=1}^{\infty} a_{2n}\cos 2n\omega(t-t_0) + c_0\cos\Delta(t-t_0) + {\sum_{r=-\infty}^{\infty}}' c_{2r}\cos(\Delta + 2r\omega)(t-t_0) \ .$$

The prime indicates that $r=0$ is to be omitted. The recurrence relation for a_{2n} is given by (3.6) by setting

$\omega=\omega_0$; the corresponding relation for c_{2n} is as follows:

$$[n^2(n^2-1)+2n(2n^2-1)sz+(6n^2-1)s^2z^2-(2n^2-1)s^2+4ns^3(z^3-z)+s^4(z^4-2z^2)]c_{2n} - \tfrac{1}{2}s^2(2n-1+2sz)(n+1+sz)c_{2n-2}-\tfrac{1}{2}s^2(2n+1+2sz)(n-1+sz)c_{2n+2}=0\ , \tag{4.2}$$

where $s\equiv b/\omega$ and $sz\equiv\Delta/2\omega$. It is shown in Appendix B that the frequency Δ can be expressed as a power series in b/ω,

$$\Delta=2b(1-\tfrac{1}{8}b^2/\omega^2-\tfrac{13}{128}b^4/\omega^4-\cdots)\ . \tag{4.3}$$

The first term on the right-hand side corresponds to the RFA solution. Other terms represent corrections of various orders.

If we call the smallest frequency that occurs in (4.1) the "nutation frequency," Δ_n say, we see that this frequency grows as a function of b/ω until it reaches

$$\Delta_n=\omega\ .$$

This occurs when $b/\omega=0.52$ approximately. After that it starts decreasing because now $2\omega-\Delta$ is more appropriately labeled the "nutation frequency."

Once Δ is known it is an easy matter to calculate coefficients c_{2n} in terms of c_0. We then make use of initial conditions to fix a_0 and c_0. The final result is as follows:

$$a_0=\tfrac{1}{4}s^2(1+\tfrac{3}{2}s^2\cdots)\ , \tag{4.4a}$$

$$a_2=-2a_0\ , \tag{4.4b}$$

$$a_4=-\tfrac{3}{4}s^2(1+\tfrac{7}{12}s^2\cdots)a_0\ , \tag{4.4c}$$

and so on. Also

$$c_0=-(1-\tfrac{3}{4}s^2-\tfrac{1}{2}s^4\cdots)\ , \tag{4.4d}$$

$$c_2=\tfrac{1}{2}s(1+\tfrac{1}{2}s+\tfrac{1}{8}s^2+\tfrac{5}{8}s^3+\cdots)c_0\ , \tag{4.4e}$$

$$c_4=\tfrac{3}{16}s^3(1-\tfrac{5}{6}s\cdots)c_0\ , \tag{4.4f}$$

$$c_{-n}=c_n(s\to-s)\ .$$

Here $s=b/\omega$. For small intensities of the transverse field ($b\simeq0$), we can use the result of the rotating-field approximation, Eq. (2.4), to obtain the transition probability

$$P_{-1/2,\,1/2}=\sin^2 b(t-t_0)\ .$$

This is zero when $b\to0$. For large values of intensity, the concept of averaging becomes meaningful and the time-averaged transition probability P_{ave} is now given by

$$P_{\text{ave}}=\tfrac{1}{2}(1+a_0)\ .$$

It is approximately equal to $\frac{1}{2}$ for $s\ll1$. As one increases the intensity, it grows larger. However, it again equals $\frac{1}{2}$ for a certain value of b/ω, s_0 say, lying between 1.23 and 1.24 and yet again when b/ω is approximately 2.07, and so on.

It seems plausible that P_{ave}, as a function of intensity, goes through successive maxima and minima. A similar conclusion was reached by Reiss[10] who considered a $2s\to1s$ transition in hydrogen, although his curves show only two extremum points. We also notice that a_0 is positive and presumably remains positive until $s=s_0$ when a_0 vanishes. This clearly disagrees with Shirley's result that a_0 is always negative; hence P_{ave} cannot exceed 0.5 [see Eq. (1.3)].[3] It is imperative to resolve the discrepancy between the two theories. This is done by noting that Shirley's original result, Eq. (1.3), will apply at $\omega_0=\omega$ *only if* the derivative of q with respect to ω_0 exists at that point. What one does in practice is obtain an expression for q^2 [Eq. (3.10)] which is differentiable at $\omega=\omega_0$. Extrema of q are found by solving[3,4a] $\partial q^2/\partial\omega_0=0$. It is possible that $\partial q^2/\partial\omega_0$ exists at a certain point while $\partial q/\partial\omega_0$ does not. We shall show that this is actually the case at $\omega_0=\omega$. Since q^2 is continuous at this point it follows that in the neighborhood of $\omega_0=\omega$

$$q=\pm(q^2)^{1/2}\ \text{for}\ \omega_0>\omega \tag{4.5a}$$

and

$$q=\mp(q^2)^{1/2}\ \text{for}\ \omega_0<\omega\ , \tag{4.5b}$$

where either upper or lower signs are to be taken.

We shall give a simple argument to support the above conclusion. From (1.1) it follows that m_x and m_z are coupled through the equation

$$d^2m_x/dt^2+\omega_0^2m_x=4b\omega_0\cos\omega(t-t_0)m_z\ . \tag{4.6}$$

When $\omega_0^2-\omega^2$ is small, the quantity $a_0+\frac{1}{2}a_2$ must be proportional to $\omega_0^2-\omega^2$ for any value of b, since otherwise one can make the coefficient of the $\cos\omega(t-t_0)$ term in m_x arbitrarily large by choosing ω_0 sufficiently close to ω. Here a_0 and a_2 are the same as in (3.1). Assume

$$a_2+2a_0=\gamma(\omega_0^2-\omega^2)\ . \tag{4.7}$$

From (3.8) we get

$$[x^2-2(5-4s^2)x+3(3-8s^2) - 18s^4(25-x)(1-x)/G(x,s)]a_2=4s^2(9-x)a_0\ . \tag{4.8}$$

Here $s\equiv b/\omega$, $x\equiv\omega_0^2/\omega^2$, and $G(x,s)$ denotes a continued-fraction expression whose details do not concern us here, except that G does not vanish when $x=1$. The condition $\omega_0 \to \omega$ has now become $x \to 1$. From (4.7) and (4.8) we get

$$(x-1)H(x,s)a_0 = \gamma\omega^2(1-x), \tag{4.9}$$

where $H(x,s) \equiv 18 - 2x - 12s^2 + 36s^4(25-x)/G(x,s)$. Now it is clear from (4.9) that

$$\lim_{\omega_0 \to \omega} a_0 = \frac{\gamma\omega^2}{H(1,s)}. \tag{4.10}$$

On the other hand, when $\omega_0 = \omega$, Eq. (4.10) no longer applies. In this case one must resort to some other means as, for example, to the method used in this section to get (4.4a). The limit of a_0 when $\omega_0 \to \omega$ may indeed be different from the value of a_0 when $\omega_0 = \omega$. Comparison of Shirley's theory with the present one clearly shows that the two quantities *must* be different, therefore Eqs. (4.5) are justified.

The physical significance of the above result is best understood by considering the system in a rotating reference frame. Pegg[8] has shown that the eigenvalues q of the Floquet Hamiltonian of Shirley's theory are the same as the eigenvalues of the system in a frame so that the fields are static. The frequency diagram of a spin-$\frac{1}{2}$ system in a static magnetic field but without any transverse field, viewed in a reference frame rotating with angular frequency ω around the z axis, is shown in Fig. 1. The levels cross at $\omega_0 = \omega$. If we apply a very weak perturbation the picture should not change drastically.

Now suppose we apply a weak oscillating field in the transverse plane. In the rotating frame the magnetic field along the z axis is proportional to $\omega - \omega_0$. For positive values of $\omega - \omega_0$ this field is pointing upwards, and for negative values of $\omega - \omega_0$ it points downwards. The leading term in a power-series expansion of q is also proportional to the magnetic field along the z axis. Pegg has shown that[8]

$$q^2 = \tfrac{1}{4}(\omega - \omega_0)^2 + 2\omega_0 b^2/(\omega + \omega_0) + \cdots .$$

We must take the positive square root when $\omega \gg \omega_0$ and the negative square root when $\omega \ll \omega_0$ [so that $q \simeq \frac{1}{2}(\omega - \omega_0)$ in both cases]. Only then will the field point upwards in one case and downwards in the other.

The frequency diagram when both longitudinal and transverse fields are present is usually drawn as in Fig. 2(a). In light of the above discussion we see that this diagram must be modified as in Fig. 2(b). The similarity of Figs. 1 and 2(b) is now obvious. As the perturbation grows weaker, Fig. 2(b) will steadily change into Fig. 1. In this interpretation, levels still cross at $\omega_0 = \omega$ in the sense that the energy of λ+ at $\omega_0 = \omega + 0$ equals that of

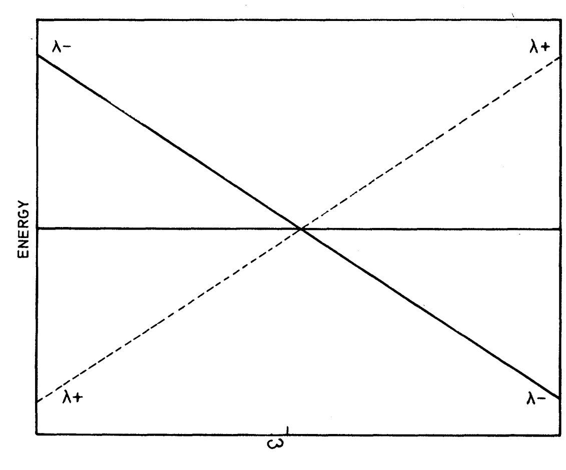

FIG. 1. Eigenvalues of a two-state system as a function of ω_0 in a rotating reference frame when there is no oscillating field.

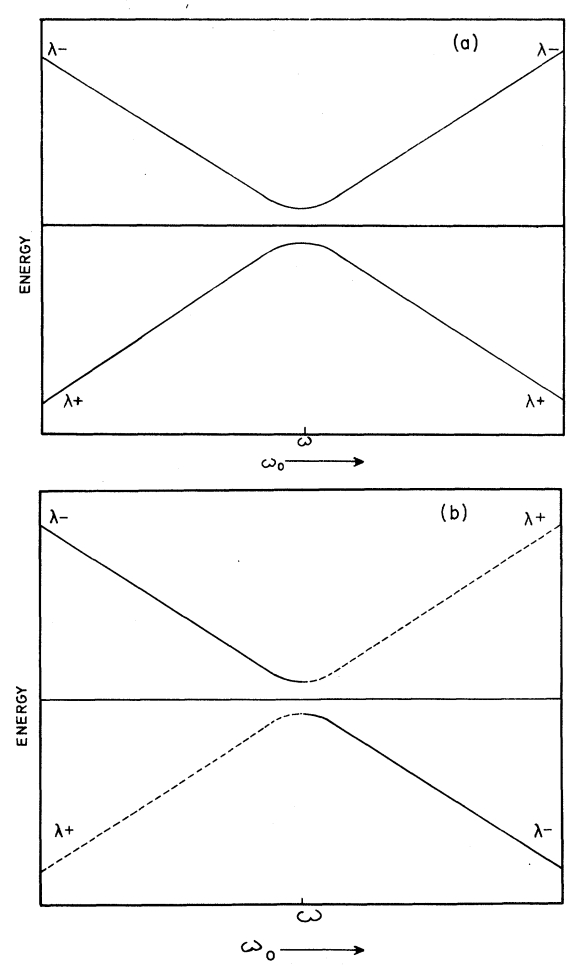

FIG. 2. (a) An incorrect view of eigenvalues when an oscillating field is applied. (b) Correct interpretation of the eigenvalue plot.

$\lambda-$ at $\omega_0=\omega-0$. It appears that eigenstates $|\lambda+\rangle$ and $|\lambda-\rangle$ of the Floquet Hamiltonian become degenerate near $\omega_0=\omega$, and as the static magnetic field is swept through this point the system makes a transition without any transfer of energy.

V. CONCLUSIONS

We have shown that the time-averaged transition probability is discontinuous at the point $\omega_0=\omega$. We have not explored the possibility that it may be discontinuous at other points as well. One can use an argument like the one leading to (4.10) to show that the limit of a_{2n} when $\omega_0 \to (2n+1)\omega$ may be different from the value of a_{2n} when $\omega_0=(2n+1)\omega$. An understanding of this phenomenon is essential for a consistent formulation of the theory. In the present analysis the width of the transition at $\omega_0=\omega$ is exactly zero, thus making it undetectable in any experiment. It is possible that, when due account is taken of relaxation processes, this transition may be broadened, giving rise to a peak in the observed magnetic resonance curve at $\omega_0=\omega$. Since this peak has not been observed so far, we must conclude that the broadening due to relaxation, if it exists, must be very small.

Our solution clearly exhibits satellites, i.e., m_x will contain terms oscillating not only at ω, 3ω, 5ω, etc., but also at frequencies symmetrically displaced around them at $\omega\pm\Delta$, $3\omega\pm\Delta$, etc. The shape and relative intensities of the components at the driving frequency ω (i.e., the Rayleigh line) and its satellites are of special interest. Stroud[11a] and Gush and Gush[11b] have predicted the ratio of the heights of the central peak and its satellites to be 2:1 whereas Mollow[11c] and Newstein[11d] have obtained a 3:1 ratio. Here we shall make a qualitative estimate of the relative intensities of the satellites only. Denote amplitudes of $\cos(\omega+\Delta)(t-t_0)$ and $\cos(\omega-\Delta)(t-t_0)$ by $s+$ and $s-$ respectively. From (3.3) and (4.6) it follows that

$$s+=\frac{2b\omega_0(c_0+c_2)}{\omega_0^2-(\omega+\Delta)^2}$$

and

$$s-=\frac{2b\omega_0(c_0+c_{-2})}{\omega_0^2-(\omega-\Delta)^2}\,.$$

In general $c_2, c_{-2} \ll c_0$ and $\Delta \ll \omega$. Now magnitudes of $s+$ and $s-$ will be approximately equal when the detuning $|\omega_0-\omega|$ is either much smaller *or* much larger than Δ. Only when

$$|\omega_0-\omega|\simeq\Delta$$

will $|s+|$ be appreciably different from $|s-|$. Therefore, barring the last-mentioned possibility, intensities of the two satellites will be approximately equal. This conclusion seems to be in agreement with recent experiments.[12]

ACKNOWLEDGMENTS

The author is grateful to the Imperial Chemical Industries for financial support and to Professor R. K. Bullough for encouragement and helpful discussions.

APPENDIX A: CALCULATION OF Δ

Assume that the recurrence relation satisfied by the coefficients c_{2r} is

$$\alpha_r c_{2r}+\beta_r c_{2r-2}+\gamma_r c_{2r+2}=0\,. \tag{A1}$$

α_r, β_r, γ_r are functions of b, ω, ω_0 and Δ. For $r=0$ we get

$$\alpha_0 c_0+\beta_0 c_{-2}+\gamma_0 c_2=0\,. \tag{A2}$$

Also we can obtain expressions of the form (3.8) for c_0/c_2 and c_0/c_{-2}. Suppose

$$c_0/c_2=k_1 \text{ and } c_0/c_{-2}=k_2\,,$$

where k_1 and k_2 denote infinite continued fractions. Substituting in (A2) we see that Δ must be chosen so as to satisfy the equation

$$\alpha_0+\beta_0/k_2+\gamma_0/k_1=0\,. \tag{A3}$$

Thus Δ is independent of initial conditions.

To a first approximation one may ignore terms involving c_2 and c_{-2} in (A2) and determine Δ from the condition

$$\alpha_0(\omega,\omega_0,b,\Delta)=0 \tag{A4}$$

which reduces to

$$\Delta[\Delta^4-2(4b^2+\omega^2+\omega_0^2)\Delta^2+8b^2(\omega^2+\omega_0^2)+(\omega^2-\omega_0^2)^2]=0$$

which gives

$$\Delta=0$$

or

$$\Delta^2=\omega^2+\omega_0^2+4b^2\pm(\omega^2\omega_0^2+4b^4)^{1/2}$$
$$\simeq(\omega\mp\omega_0)^2+4b^2 \tag{A5}$$

if $b^2 \ll \omega^2, \omega_0^2$. The root $\Delta=0$ corresponds to the first sum in (3.3) and roots (A5) give rise to two independent solutions which are required so that (3.3) is a unique solution of the problem.

APPENDIX B: DERIVATION OF (4.3)

For $n=0$, Eq. (4.2) reduces to

$$(1-z^2+s^2z^4-2s^2z^2)c_0+\tfrac{1}{2}(1-2sz)(1+sz)c_{-2}+\tfrac{1}{2}(1+2sz)(1-sz)c_2=0\,. \tag{B1}$$

Also to order s^4

$$c_2 = \frac{\frac{1}{2}s(1+2sz)(2+sz)c_0}{2sz+5s^2z^2-s^2+4s^3(z^3-z)+s^4(z^4-2z^2)} \tag{B2}$$

and

$$c_{-2} = c_2(s \to -s)\,. \tag{B3}$$

Substitute (B2) and (B3) in (B1) and solve by iteration. We get

$$z = 1 - \tfrac{1}{8}s^2 - \tfrac{13}{128}s^4 - \cdots .$$

The series can be easily extended by including higher-order terms in (B2) and (B3).

It can also be verified that (B1) is satisfied when $sz = \pm 1$. This gives $\Delta = \pm 2\omega$, which only corresponds to the first sum in Eq. (4.2).

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