Interaction between two coupled oscillators

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The quantum dynamics for two coupled harmonic oscillators is presented. Using the coupled-boson representation, the assumed problem is shown to be isomorphic with a perturbed angular-momentum oscillator. The current operator is obtained and its associated expectation values with respect to the number states, coherent states (Glauber states), and atomic coherent states are given. The eigenvalue spectrum of the current operator is seen to be finite and discrete. An interesting correspondence between this analysis and past work on Josephson tunneling and quantum interference is discussed.

I. INTRODUCTION

In this paper, we consider the quantum-mechanical behavior of two coupled linear oscillators. In the past, the coupled oscillator system has been extensively studied in quantum mechanics; in this analysis, however, the quantum aspects of the system are treated from a less familiar point of view. As such, the discussions are somewhat pedagogical; yet, there is an interesting correspondence between this analysis and recent treatments of quantum optical interference[1] and phase effects in Josephson tunneling.[2]

The analysis, in part, is carried out in the Heisenberg picture. The equations of motion are obtained and are shown to be isomorphic with a perturbed angular-momentum oscillator, an isomorphism which is simply a manifestation of the coupled-boson representation. The current operator (the rate of exchange of excitation number) is obtained in terms of the appropriate angular-momentum operators.

The dynamical behavior of the coupled system is studied using various initial states, namely, number states, coherent states, and angular-momentum "atomic" coherent states. The eigenvalues of the Hamiltonian and the current operator are also discussed.

II. EQUATIONS OF MOTION

The coupled oscillator system under investigation is described by the model Hamiltonian

$$H = \omega_1 N_1 + \omega_2 N_2 + \epsilon (a_1 a_2^\dagger + a_2 a_1^\dagger) , \qquad (1)$$

where $N_i = a_i^\dagger a_i$ $(i = 1,2)$ represents the number operator for each oscillator and the a_i's satisfy the usual boson commutation relations. The coupling parameter ϵ is assumed to be a real, positive constant and the units are chosen so that $\hbar = 1$ throughout. For purposes of our discussion, the above Hamiltonian represents an idealization of two boson systems exchanging excitations via tunneling. A similar model, together with some phase operator assumptions, was adopted by Nieto[2] in order to establish the role of phase in Josephson tunneling phenomena.

In the spirit of a tunneling problem, we seek to calculate the current or rate of exchange of excitation between the two oscillators. In this regard, the current is naturally manifested when the coupled oscillators are treated as an angular-momentum oscillator. The correspondence between two linear oscillators and an angular-momentum oscillator has been discussed by Schwinger[3] and is commonly referred to as the coupled-boson representation.[4] Within this representation, it can easily be shown that

$$L_x = \tfrac{1}{2}(a_1^\dagger a_2 + a_2^\dagger a_1) , \qquad (2a)$$

$$L_y = (1/2i)(a_1^\dagger a_2 - a_2^\dagger a_1) , \qquad (2b)$$

$$L_z = \tfrac{1}{2}(N_1 - N_2) , \qquad (2c)$$

$$L = \tfrac{1}{2}(N_1 + N_2) , \qquad (2d)$$

where the L_{x_i} operators satisfy the usual angular-momentum commutation relations. In terms of the angular-momentum operators, the Hamiltonian in Eq. (1) becomes

$$H = \omega L_z + 2\epsilon L_x + (\omega_1 + \omega_2) L , \qquad (3)$$

with

$$\omega = \omega_1 - \omega_2 , \qquad (4)$$

and the Heisenberg equations of motion become

$$\dot{L}_x = -\omega L_y , \qquad (5a)$$

$$\dot{L}_y = \omega L_x - 2\epsilon L_z , \qquad (5b)$$

$$\dot{L}_z = 2\epsilon L_y , \qquad (5c)$$

$$\dot{L} = 0 . \qquad (5d)$$

We also note that when a constant biasing voltage V_0 is applied across the tunnel junction, an additional term,

$$H^1 = \omega_0 N_2 \equiv \omega_0 (L - L_z) ,$$

with

$$\omega_0 = qV_0 ,$$

appears in the Hamiltonian, thereby changing ω_2 to $\omega_2 + \omega_0$. From these equations of motion, it is clear that we can describe the exchange of excitation between two oscillators in terms of the free precessional motion of a perturbed angular-momentum oscillator.

The current operator, which is defined as the net rate of exchange of excitation with time, is given as

$$I \equiv 2\dot{L}_z = 4\epsilon L_y(t) . \tag{6}$$

The equation of motion for $L_y(t)$ is easily obtained by taking the time derivative of Eq. (5b) and then eliminating $\dot{L}_x$ and $\dot{L}_z$ through the use of Eqs. (5a) and (5c). The result is

$$L_y(t) = L_y(0)\cos\bar{\omega}t + (\omega/\bar{\omega}) \times \{L_x(0) - [(\bar{\omega}/\omega)^2 - 1]^{1/2} L_z(0)\} \sin\bar{\omega}t , \tag{7}$$

with

$$\bar{\omega} = [\omega^2 + 4\epsilon^2]^{1/2} . \tag{8}$$

The other components of the perturbed angular-momentum oscillator can easily be found since the Hamiltonian in Eq. (3) is diagonalized by the unitary transformation

$$U(\gamma) = e^{i\gamma L_y(t)} , \tag{9}$$

with

$$\tan\gamma = [(\bar{\omega}/\omega)^2 - 1]^{1/2} . \tag{10}$$

III. EXPECTATION VALUE OF THE CURRENT OPERATOR; COHERENT STATES

A measure of the current passing from one oscillator to the other is given by the expectation value of $I(t)$ with respect to the initial state of the system. The explicit form of the current, obtained by substituting Eq. (7) into Eq. (6), is

$$I(t) = 4\epsilon\{L_y(0)\cos\bar{\omega}t + (\omega/\bar{\omega}) \times [L_x(0) - ((\bar{\omega}/\omega)^2 - 1)^{1/2}] \sin\bar{\omega}t\} . \tag{11}$$

If we assume that each oscillator is initially prepared in a number state, then the expectation value of the current operator in Eq. (11) results in the familiar expression

$$\langle n_1 n_2 | I | n_1 n_2 \rangle = (4\epsilon^2/\bar{\omega})(n_2 - n_1)\sin\bar{\omega}t . \tag{12}$$

If, on the other hand, the system is prepared in an eigenstate of the Hamiltonian,[5] then the expectation value of the current operator is zero.

We now consider the situation where the system is prepared in an exact coherent state

$$|\alpha\rangle = e^{-i\gamma L_y(0)} |\alpha\rangle_0 , \tag{13a}$$

with

$$|\alpha\rangle_0 = |\alpha_1\rangle |\alpha_2\rangle . \tag{13b}$$

Here, $|\alpha_i\rangle$ represents the coherent state for each free oscillator and is expressed as

$$|\alpha_i\rangle = \exp(-\tfrac{1}{2}|\alpha_i|^2) \sum_{n_i=0}^{\infty} \frac{\alpha_i^{n_i}}{(n_i!)^{1/2}} |n_i\rangle ,$$

with

$$\alpha_i = (\bar{N}_i)^{1/2} e^{i\phi_i} ,$$

where $\bar{N}_i = \langle \alpha_i | N_i | \alpha_i \rangle$ is the mean excitation number and ϕ_i is the phase. Since the $|\alpha_i\rangle$ states are right eigenstates of a_i, the expectation value of the current operator with the α state is easily obtained as

$$\langle \alpha | I | \alpha \rangle = 4\epsilon \bar{l} \sin\theta \sin\Phi(t) , \tag{14a}$$

with

$$\sin\theta = 2(\bar{N}_1 \bar{N}_2)^{1/2}/\bar{N} . \tag{14b}$$

Here $\Phi(t) = \bar{\omega}t + \phi$ and $\bar{l} = \frac{1}{2}\bar{N}$, where $\phi = \phi_2 - \phi_1$ and $\bar{N} \equiv \bar{N}_1 + \bar{N}_2$.

Now we consider the situation where the system is prepared in an exact angular-momentum coherent state

$$|\mu\rangle = e^{-i\gamma L_y(0)} |\mu\rangle_0 . \tag{15}$$

$|\mu\rangle_0$ represents the coherent state for a free angular-momentum oscillator and may be derived by "suitably restricting" the Kronecker product of Glauber states in Eq. (13b) to preclude all terms not satisfying the condition that $n_1 + n_2 = N$, a constant total particle number. In doing so, we obtain a normalized expression for $|\mu\rangle_0$ as

$$|\mu\rangle_0 = \frac{1}{(|\alpha_1|^2 + |\alpha_2|^2)^{N/2}} \times \sum_{S=0}^{N} \left(\frac{N!}{S!(N-S)!}\right)^{1/2} \alpha_1^S \alpha_2^{N-S} |S\rangle_1 |N-S\rangle_2 . \tag{16}$$

Letting $S = l + m$, $N = 2l$ and noting that an angular-momentum state, $|l,m\rangle$, is expressed as a product of the appropriate linear oscillator number states,[3,4] $|\mu\rangle_0$ in Eq. (16) can be rewritten as

$$|\mu\rangle_0 = \frac{1}{(|\alpha_1|^2 + |\alpha_2|^2)^l} \times \sum_{m=-l}^{l} \binom{2l}{l+m}^{1/2} \alpha_1^{l+m} \alpha_2^{l-m} |l, m\rangle . \tag{17}$$

Here, ( ) signifies the binomial coefficient.

The above $|\mu\rangle_0$ state has interesting properties, most of which have already been elucidated by Arecchi *et al.*[6] (see Appendix). In previous discussions,[6,7] the $|\mu\rangle_0$ states have been utilized in describing the collective behavior of N two-level atoms and, as such, have been called "atomic" coherent states.

The expectation value of the current operator with respect to the $|\mu\rangle$ state can easily be obtained (see Appendix) to give

$$\langle\mu|I|\mu\rangle = 4\epsilon l \sin\theta \sin\Phi(t) , \tag{18}$$

where $\sin\theta$ and $\Phi(t)$ are discerned from Eq. (14) and $l=\frac{1}{2}N$.

Lastly, we establish the relationship between the $|\alpha\rangle$ state and the $|\mu\rangle$ state. In this regard, the $|\alpha\rangle_0$ state is written as

$$|\alpha\rangle_0 = \exp[-\tfrac{1}{2}(|\alpha_1|^2+|\alpha_2|^2)] \times \sum_{N=0}^{\infty}\sum_{S=0}^{N}\frac{\alpha_1^S\alpha_2^{N-S}}{[S!(N-S)!]^{1/2}}|S\rangle_1|N-S\rangle_2 .$$

By multiplying and dividing by $(N!/\overline{N}^N)^{1/2}$ inside the sum of over N and then using Eq. (16) to identify $|\mu,N\rangle_0$, the above expression for $|\alpha\rangle_0$ becomes

$$|\alpha\rangle_0 = \sum_{N=0}^{\infty}\frac{e^{-\overline{N}/2}\,\overline{N}^{-N/2}}{(N!)^{1/2}}|\mu,N\rangle_0 , \tag{19}$$

thereby indicating that the $|\alpha\rangle$ state is the Poisson "packet" of a $|\mu\rangle$ state distributed over total particle number N.

IV. SPECTRUM OF THE CURRENT OPERATOR; DISCUSSION

In the Schrodinger picture, the current operator [Eq. (6)] is $I(0)=4\epsilon L_y(0)$. Thus, the eigenstates of $I(0)$ are $|l,m\rangle_y$ with discrete eigenvalues $4\epsilon m$, $-l\leq m\leq l$. The current operator can then be written as

$$I(0) = I_{\max}(L_y(0)/l) , \tag{20}$$

with $I_{\max}=4\epsilon l$. Since $l=\frac{1}{2}N$, it then follows that the current is quantized in intervals of $2/N$. In a model calculation, Nieto[2] has made use of sine and cosine phase operators[8] in the Hamiltonian of Eq. (1) to describe quantized current effects in Josephson tunneling. However, it has been argued[1,9] that the quantum-mechanical definition of such phase operators is not unique. In this regard, we have shown that the use of the coupled boson representation in this model naturally results in a quantized current. Other conclusions by Nieto[2] can also be deduced here.

The coupled boson representation has also been used by Shalom and Zak[1] to construct an ideal model for quantum-mechanical interference.

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APPENDIX

Although we have constructed the $|\mu\rangle_0$ state from a suitably restricted $|\alpha\rangle_0$ state, we could have equivalently arrived at such a state by binomially distributing N indistinguishable particles into two energy states, one with phase ϕ_1 and occupation probability p, the other with phase ϕ_2 and occupation probability q, such that $p+q=1$. The binomial "packet" representing this system would then be

$$|p,q,\phi\rangle = e^{iN\phi_2}\sum_{n=0}^{N}\binom{N}{N-n}^{1/2}(\sqrt{p})^n(\sqrt{q})^{N-n} \times e^{in\phi}|n\rangle_1|N-n\rangle_2 , \tag{A1}$$

where $\phi=\phi_1-\phi_2$, and ( ) symbolizes the binomial coefficient. (The phase factor, $e^{iN\phi_2}$ in Eq. (A1) is hereafter suppressed.) In addition, the $|p,q,\phi\rangle$ state is normalized since

$$\langle p,q,\phi|p,q,\phi\rangle = \sum_{n=0}^{N}\binom{N}{N-n}p^nq^{N-n} = (p+q)^N = 1 . \tag{A2}$$

If we identify p and q as

$$p = |\alpha_1|^2/(|\alpha_1|^2+|\alpha_2|^2) \equiv \overline{N}_1/\overline{N} \tag{A3a}$$

and

$$q = |\alpha_2|^2/(|\alpha_1|^2+|\alpha_2|^2) \equiv \overline{N}_2/\overline{N}, \tag{A3b}$$

then $|p,q,\phi\rangle$ in Eq. (A1) is equivalent to the $|\mu\rangle_0$ state of Eq. (16).

For purposes of calculating expectation values, let

$$x=\sqrt{p}\,e^{i\phi/2},\quad y=\sqrt{q}\,e^{-i\phi/2} , \tag{A4}$$

so that Eq. (A1) becomes

$$|p,q,\phi\rangle = e^{i(N/2)\phi}\sum_{n=0}^{N}\binom{N}{N-n}^{1/2}x^ny^{N-n}|n\rangle_1|N-n\rangle_2 . \tag{A5}$$

In using the properties of a_i and $a_i^\dagger$ we can easily see that

$$L_+|p,q,\phi\rangle = a_1^\dagger a_2|p,q,\phi\rangle = y\frac{\partial}{\partial x}|p,q,\phi\rangle\ , \qquad \text{(A6)}$$

$$L_-|p,q,\phi\rangle = a_2^\dagger a_1|p,q,\phi\rangle = x\frac{\partial}{\partial y}|p,q,\phi\rangle\ , \qquad \text{(A6a)}$$

$$L_+L_-|p,q,\phi\rangle = a_1^\dagger a_2 a_2^\dagger a_1|p,q,\phi\rangle = x\frac{\partial}{\partial y}\,y\frac{\partial}{\partial x}|p,q,\phi\rangle\ , \qquad \text{(A7)}$$

$$L_-L_+|p,q,\phi\rangle = a_2^\dagger a_1 a_1^\dagger a_2|p,q,\phi\rangle = y\frac{\partial}{\partial x}\,x\frac{\partial}{\partial y}|p,q,\phi\rangle\ , \qquad \text{(A7a)}$$

$$N_1|p,q,\phi\rangle = a_1^\dagger a_1|p,q,\phi\rangle = x\frac{\partial}{\partial x}|p,q,\phi\rangle\ , \qquad \text{(A8)}$$

$$N_2|p,q,\phi\rangle = a_2^\dagger a_2|p,q,\phi\rangle = y\frac{\partial}{\partial y}|p,q,\phi\rangle\ . \qquad \text{(A9)}$$

The relations in Eqs. (A6)–(A9) permit us to calculate any diagonal or nondiagonal matrix elements with respect to $|p,q,\phi\rangle$ very easily as well as to examine some of the salient features of the $|p,q,\phi\rangle$ state.

As an illustration of the ease with which expectation values can be calculated, we consider the evaluation of $\langle p,q,\phi|L_+|p,q,\phi\rangle$. From Eq. (A6), we see that

$$\langle p,q,\phi|L_+|p,q,\phi\rangle = \langle p,q,\phi|\,y\frac{\partial}{\partial x}\,|p,q,\phi\rangle \equiv \sum_{n=0}^{N}\binom{N}{N-n}\left(\frac{y}{x}\right)n(|x|^2)^n(|y|^2)^{N-n}\ .$$

Using $x=\sqrt{p}\,e^{i\phi/2}$ and $y=\sqrt{q}\,e^{-i\phi/2}$ from Eq. (A4), we see that

$$\langle p,q,\phi|L_+|p,q,\phi\rangle = e^{-i\phi}\sqrt{pq}\,\frac{\partial}{\partial p}\sum_{n=0}^{N}\binom{N}{N-n}p^n q^{N-n}\ ,$$

which reduces to

$$\langle p,q,\phi|L_+|p,q,\phi\rangle = e^{-i\phi}\sqrt{pq}\,\frac{\partial}{\partial p}(p+q)^N = e^{-i\phi}N\sqrt{pq}\ . \qquad \text{(A10)}$$

All expectation values with respect to $|p,q,\phi\rangle$ can be calculated in the same fashion to obtain

$$\langle L_+\rangle = N\sqrt{pq}\,e^{-i\phi} \equiv (N/\overline{N})\alpha_1^*\alpha_2\ , \qquad \text{(A11a)}$$

$$\langle L_+^2\rangle = N(N-1)pq\,e^{-2i\phi} \equiv [N(N-1)/\overline{N}^2](\alpha_1^*\alpha_2)^2, \qquad \text{(A11b)}$$

$$\langle L_-L_+\rangle = Nq + N(N-1)pq\ , \qquad \text{(A11c)}$$

$$\langle L_-\rangle = N\sqrt{pq}\,e^{i\phi} = (N/\overline{N})\alpha_1\alpha_2^*\ , \qquad \text{(A11d)}$$

$$\langle L_-^2\rangle = N(N-1)pq\,e^{2i\phi} = [N(N-1)/\overline{N}^2](\alpha_1\alpha_2^*)^2\ , \qquad \text{(A11e)}$$

$$\langle L_+L_-\rangle = Np + N(N-1)pq\ . \qquad \text{(A11f)}$$

In the case of nondiagonal elements, we consider the evaluation of $\langle p',q',\phi'|L_+|p,q,\phi\rangle$.

From Eq. (A6), we see that

$$\langle p',q',\phi'|L_+|p,q,\phi\rangle = y\frac{\partial}{\partial x}\langle p',q',\phi'|p,q,\phi\rangle \equiv e^{i(N/2)(\phi-\phi')}yx'\,N\{xx'^* + yy'^*\}^N\ .$$

Using x, y from Eq. (A4) then gives the results in terms of p, q, and ϕ.

The nature of the $|p,q,\phi\rangle$ state is best appreciated by examining the symmetrical form of Eq. (A1), which is

$$|p,q,\phi\rangle = \sum_{m=-l}^{l}\binom{2l}{l+m}^{1/2}(\sqrt{p})^{l+m}(\sqrt{q})^{l-m} \times e^{i(l+m)\phi}|l,m\rangle\ . \qquad \text{(A12)}$$

Hereafter, the phase factor $e^{il\phi}$ is dropped.

The $e^{+im\phi}$ term is Eq. (A12), when "operating" on $|l,m\rangle$, can be re-expressed as $e^{+iL_z\phi}$, thereby reducing $|p,q,\phi\rangle$ to

$$|p,q,\phi\rangle = e^{-i\phi L_z}|p,q,\phi=0\rangle\ . \qquad \text{(A13)}$$

In addition, if we let

$$q=\sin^2\tfrac{1}{2}\theta,\quad p=\cos^2\tfrac{1}{2}\theta\ , \qquad \text{(A14)}$$

so that $|p,q,\phi=0\rangle$ [Eq. (A12) with $\phi=0$] becomes

$$|p,q,\phi=0\rangle = \sum_{m=-l}^{l}\binom{2l}{l+m}^{1/2}(\sin\tfrac{1}{2}\theta)^{l-m} \times(\cos\tfrac{1}{2}\theta)^{l+m}|l,m\rangle\ ; \qquad \text{(A15)}$$

it then follows from angular-momentum theory[10] that Eq. (A15) reduces to

$$|p,q,\phi=0\rangle = e^{-i\theta L_y}|l,l\rangle\ , \qquad \text{(A16)}$$

$|l,l\rangle$ being the highest eigenstate of L_z.

Thus $|p,q,\phi\rangle$ in Eq. (A13) can be expressed as

$$|p,q,\phi\rangle = e^{+iL_z\phi}e^{-iL_y\theta}|l,l\rangle\ , \qquad \text{(A17)}$$

which shows that $|p,q,\phi\rangle$ is an $|l,l\rangle$ eigenstate of $\vec{L}\cdot\hat{n}$, where

$$\hat{n} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)\ . \qquad \text{(A18)}$$

We, therefore, have established the connection between the $|p,q,\phi\rangle$ state and the atomic coherent states of Arecchi *et al.*[6]

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