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Asymptotic expansion for the temporal coherence functions of a finite blackbody

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Planckian thermal radiation of temperature T shows a broad continuous spectrum and poor, monotonically decreasing, temporal coherence. We study the effects due to the discreteness of the exact resonances in a finite, cube-shaped cavity (volume L^3) bounded by perfectly reflecting walls. Using the Poisson summation technique, we calculate the almost periodic spatially averaged temporal (t) correlation tensors in terms of the simultaneous asymptotic series in powers of both $1/LT$ and t/L with Riemann and Epstein zeta-function coefficients. The according cross-spectral tensors yield all orders of the expansion of the spectral mode density including the oscillatory terms.

I. INTRODUCTION

The study of blackbody radiation at the turn of the century[1] started a revolution in thought that has dominated physics ever since. The current interest in this oldest subject of modern physics has two main sources. First, quantum optics and the theory of partial coherence study the blackbody as the best known example for chaotic fields.[2-6] Second, size, shape, and proximity effects and their impact on the venerable laws of thermal radiation are studied in connection with the Casimir effect,[7-9] the radiation transfer between closely spaced bodies,[10,11] and the spectral and total energy densities of small cavities[12-17] related to the problem of far-infrared radiation standards as well as to the concept of the density of states and quantum-size effects in the statistical mechanics of noninteracting systems. Both the aspects of coherence and boundaries are combined in the studies of the correlation of thermal radiation in finite cavities[18] and the quantum electrodynamics in the vicinity of material walls.[19-21]

The discreteness of the exact eigenvalue spectrum is waived in the statistical physics of very large systems, where the quasicontinuum approach leading to the analytical density of states is adopted. This approach can be extended to finite, but not too small, systems by describing surface and shape effects in terms of the refined smoothed spectral distribution. The corresponding correction terms for the Planck and Stefan-Boltzmann radiation laws were derived from studying the eigenvalue distribution of the electromagnetic wave equation in finite domains bounded by perfectly reflecting walls.[12-17] The quasicontinuum approach, however, suppresses the fluctuations around the averaged mode density related to the finite distance between adjacent levels.

A systematic study of the oscillatory "fine structure" of the distribution is available for the scalar wave equation[22] related to the quantum-size effects in nonrelativistic perfect gases, e.g., for electrons in metal grains or the nuclear-shell correction. The analogous problem for the more involved electromagnetic wave equation related to the thermal photon gas in small blackbodies is unsolved hitherto, although oscillations have been observed earlier[23-24] in numerically computed spectral densities. The discreteness of the spectrum is of particular interest in time-dependent statistical mechanics. It is responsible for the Poincaré cycles of the time correlation functions, whereas continuous spectra lead to the well-known aperiodic long-time relaxations. Thus, the quasicontinuum approach produces the well-known rapid and monotonic decrease of the temporal coherence $|\gamma(t)|$ of the blackbody radiation,[3-5,25] showing the small coherence length (below $\hbar c/KT$) and the long-time behavior $|\gamma| \propto (Tt)^3$, with T denoting the temperature and t the time. Unfortunately, the improved quasicontinuum approach in terms of refined smoothed spectral densities is meaningful for very short times t only.[18]

In this paper, both these problems—the exact analytical density of states including the oscillatory terms, and the exact analytical temporal coherence tensors for any time t—are solved simultaneously. We consider the case of the thermal radiation field in a cube-shaped cavity of arbitrary size with perfectly reflecting walls and use the Poisson-lattice summation technique and the quantum optical Wiener-Khintchine theorem.[2,6] As we are not interested in proximity effects[19-21] here, we consider the *spatial average* of the temporal correlation tensors, i.e.,

$$\overline{\mathcal{E}}_{\mu\nu}(t) = V^{-1} \int_V d\vec{x}\, \mathcal{E}_{\mu\nu}(\vec{x}, 0; \vec{x}, t), \qquad (1)$$

with V denoting the volume of the cavity and where $\mathcal{E}_{\mu\nu}(\vec{x}, 0; \vec{y}, t)$ is the electric field correlation tensor as introduced by Glauber[2] and Mehta and Wolf.[3] The magnetic and mixed tensors $\overline{\mathcal{B}}_{\mu\nu}(t)$ and $\overline{\mathfrak{M}}_{\mu\nu}(t)$ are defined in the same manner. It is easily inferred from Refs. 2 and 3 that the Fourier transform of the trace $\sum_\mu \overline{\mathcal{E}}_{\mu\mu}(t)$ of (1) produces the spectral energy density of the radiation field. We calculate (1) with the aid of the Poisson summation and find the spectral density by inverting the Fourier transform. The above average correlation tensor is of physical relevance in a situation where the thermal emission of a blackbody source is studied by correlation experiments or Fourier-transform spectroscopy. The average temporal coherence tensors are global properties of the cavity radiation field considered as a whole, very much like the spectral energy density responsible for the spectral emissivity. Furthermore, the study of the spatial average is sufficient for the demonstration of the long-time correlation effects typical for a finite system showing a discrete spectrum. We would, however, like to mention that the position-dependent time correlation $\mathcal{E}_{\mu\nu}(\vec{x}, 0; \vec{x}, t)$ is as well a quantity of great physical interest (see, e.g., Refs. 20–22).

In Sec. II we establish the general correlation tensors for the cube of finite edge length L. In Sec. III we calculate the average tensors in terms of both the exact series and the complete asymptotic expansion around $\hbar c/KTL \to \infty$ along with the electric field autocorrelation function $\langle \vec{E}(0)\vec{E}(t)\rangle$. Section IV is devoted to the discussion of the results as well as to the impact on the density of states and the total radiation energy.

II. CORRELATION TENSORS FOR THE CUBE-SHAPED CAVITY

Let us consider the thermal radiation field of temperature T enclosed in a finite domain G of volume V and study the electric, magnetic, and mixed correlation tensors introduced by Glauber[2] and Mehta and Wolf.[3] We assume that G has a discrete spectrum of resonance frequencies ω_l with $0<\omega_1 \leq \omega_2 \leq \cdots$ belonging to the normal modes

$$\vec{u}_l(\vec{x})e^{-i\omega_l t}, \quad \vec{x}\in G, \tag{2}$$

where the $\vec{u}_l = (u_{l,1}, u_{l,2}, u_{l,3})$ are an orthonormal system of solutions of the Helmholtz equation obeying $\operatorname{div}\vec{u}_l = 0$ and some boundary conditions. Using the same units as in Refs. 2 and 4, we expand the electric and magnetic field operators $\vec{E} = \vec{E}^{(+)} + \vec{E}^{(-)}$ and $\vec{B} = \vec{B}^{(+)} + \vec{B}^{(-)}$ in terms of the $\vec{u}_l$. Avoiding the usual approximations valid in the free-space limit only,[2-5] we obtain

$$\begin{aligned}\mathcal{E}_{\mu\nu} &\equiv \operatorname{tr}\{\rho E_\mu^{(-)}(\vec{x}, t_1)E_\nu^{(+)}(\vec{y}, t_2)\} \\ &= (\hbar/2)\sum_l \omega_l F_l(t) u_{l,\mu}^*(\vec{x}) u_{l,\nu}(\vec{y}),\end{aligned} \tag{3}$$

$$\begin{aligned}\mathcal{B}_{\mu\nu} &\equiv \operatorname{tr}\{\rho B_\mu^{(-)}(\vec{x}, t_1)B_\nu^{(+)}(\vec{y}, t_2)\} \\ &= (\hbar c^2/2)\sum_l \omega_l^{-1} F_l(t)[\vec{\nabla}\times\vec{u}_l^*(\vec{x})]_\mu[\vec{\nabla}\times\vec{u}_l(\vec{y})]_\nu,\end{aligned} \tag{4}$$

$$\begin{aligned}\mathfrak{M}_{\mu\nu} &\equiv \operatorname{tr}\{\rho E_\mu^{(-)}(\vec{x}, t_1)B_\nu^{(+)}(\vec{y}, t_2)\} \\ &= (-i\hbar c/2)\sum_l F_l(t)\vec{u}_{l,\mu}^*(\vec{x})[\vec{\nabla}\times\vec{u}_l(\vec{y})]_\nu,\end{aligned} \tag{5}$$

with

$$\mu, \nu = 1, 2, 3, \quad \vec{x}, \vec{y}\in G, \quad t = t_2 - t_1,$$

and

$$F_l(t) \equiv \frac{e^{-i\omega_l t}}{e^{\hbar\omega_l/KT} - 1}, \tag{6}$$

ρ being the canonical density operator, c the vacuum velocity of light, $\hbar$ the Planck constant divided by 2π, and K the Boltzmann constant. The asterisk indicates the complex conjugate. We observe that the above tensors cannot be expected to be spatially homogeneous as in the free-space limit[3-5] and hence do not necessarily depend on the relative distance $(\vec{y} - \vec{x})$ only. For the same reason, the *temporal* correlation tensors (where $\vec{y} = \vec{x}$) as well as the energy densities (where $\vec{y} = \vec{x}$ and $t_2 = t_1$) are expected to depend on the actual position $\vec{x}\in G$.

Let us now consider the cube-shaped cavity with perfectly reflecting walls, where the normal modes are characterized by the indices $(\vec{k};\ \alpha) = (k_1, k_2, k_3; \alpha)$. Here, $\vec{k}$ denotes the wave vector $(\pi/L)\vec{n}$, L being the edge length of the cavity and $\vec{n}$ a lattice vector (n_1, n_2, n_3) with non-negative integers n_i, and α is the polarization index with the possible value 1 or 2. The eigenfrequencies are $\omega(\vec{k}; \alpha) = k/c$, with k denoting $|\vec{k}|$. The μth components of the normal modes read

$$u_{(\vec{k};\alpha),\mu} = (8/L^3)^{1/2} e_\mu(\vec{k}, \alpha)\cos k_\mu x_\mu \prod_\sigma \sin k_\sigma x_\sigma, \tag{7}$$

with $\mu = 1, 2, 3$ and $\sigma\in\{1, 2, 3\}$, $\sigma \neq \mu$, and with the unit polarization vectors $\vec{e}(\vec{k}; \alpha)$ defined as usual.[2-4] From (3) and (7) we calculate

$$\mathcal{E}_{\mu\nu} = 4\hbar c L^{-3}\sum_{\vec{k}} (k^2\delta_{\mu\nu} - k_\mu k_\nu)k^{-1}F(k, t)S_{\mu\nu}^E(\vec{k}; \vec{x}, \vec{y}), \tag{8}$$

with

$$F(k, t) \equiv \frac{e^{-ikct}}{e^{\hbar ck/KT} - 1} \tag{9}$$

and

$$S^E_{\mu\nu} \equiv \cos k_\mu x_\mu \cos k_\nu y_\nu \prod_{\sigma\neq\mu} \sin k_\sigma x_\sigma \prod_{\sigma'\neq\nu} \sin k_{\sigma'} y_{\sigma'} \,. \tag{10}$$

Results similar to (8) hold for $\mathcal{B}_{\mu\nu}$ and $\mathfrak{M}_{\mu\nu}$; one has only to replace the spatial function $S^E_{\mu\nu}$ by

$$S^B_{\mu\nu} \equiv \sin k_\mu x_\mu \sin k_\nu y_\nu \prod_{\sigma\neq\mu} \cos k_\sigma x_\sigma \prod_{\sigma'\neq\nu} \cos k_{\sigma'} y_{\sigma'} \tag{11}$$

and

$$S^M_{\mu\nu} \equiv -i \cos k_\mu x_\mu \sin k_\nu y_\nu \prod_{\sigma\neq\mu} \sin k_\sigma x_\sigma \prod_{\sigma'\neq\nu} \cos k_{\sigma'} y_{\sigma'} \,, \tag{12}$$

respectively.

III. SPATIAL-AVERAGE EXPANSIONS

We now calculate the spatial-average temporal correlation tensors by putting $\vec{y}=\vec{x}$ and integrating over the cube. From (8)–(12) we obtain

$$\bar{\mathcal{E}}_{\mu\nu} = \delta_{\mu\nu}\bar{\mathcal{E}}_{\mu\mu}(t)\,, \quad \bar{\mathcal{B}}_{\mu\nu} = \delta_{\mu\nu}\bar{\mathcal{B}}_{\mu\mu}(t)\,, \quad \bar{\mathfrak{M}}_{\mu\nu} = 0\,. \tag{13}$$

Accounting for the cube symmetry and rearranging the summations we are led to

$$\bar{\mathcal{E}}_{\mu\mu}(t) = \bar{\mathcal{B}}_{\mu\mu}(t) = \Gamma(t)\,, \quad \mu = 1, 2, 3, \tag{14}$$

with the temporal coherence function

$$\Gamma(t) = (\hbar c/3L^3)\left(\sum_{n_1,n_2,n_3=1}^{\infty} kF(k,t) + \frac{3}{2}\sum_{n_1,n_2=1}^{\infty} \tilde{k}F(\tilde{k},t)\right), \tag{15}$$

with F as defined by Eq. (9), $k=(\pi/L)(n_1^2+n_2^2+n_3^2)^{1/2}$, and $\tilde{k}=(\pi/L)(n_1^2+n_2^2)^{1/2}$. In order to evaluate (15), we symmetrize the summations and obtain

$$\Gamma(t) = (\hbar c/24L^3)\sum_{n_1,n_2,n_3=-\infty}^{+\infty} kF(k) - (\hbar c/8L^3)\sum_{n_1=-\infty}^{+\infty} (\pi n_1/L)F(\pi n_1/L) + KT/12L^3\,, \tag{16}$$

where we define $kF(k)$ for $k=0$ by analytic continuation, $kF(k)\to KT/\hbar c$ for $k\to 0$. We now transform the two sums appearing in (16) using the Poisson summation formula. Introducing the appropriate spherical coordinates and integrating over the angle φ, the first sum is expressed as

$$(2\pi^2/L)\sum_{\nu_1,\nu_2,\nu_3=-\infty}^{+\infty} \int_0^\infty dr\, F(\pi r/L) r^3 \times\left(\int_0^\pi d\theta\, e^{-2\pi i\nu r\cos\theta}\sin\theta\right), \tag{17}$$

with ν denoting $|\vec{\nu}|$, $\vec{\nu}=(\nu_1,\nu_2,\nu_3)$ with integer ν_i, and a similar expression is found for the second sum in (16). Integrating over θ and invoking the Fourier integral representation[26] of the generalized Riemann zeta functions

$$\zeta(s,z) = \sum_{n=0}^{\infty} (n+z)^{-s}\,,$$

we find that the term corresponding to $\nu=0$ in (17) is related to $\zeta(4,z)$, whereas the terms corresponding to $\nu\neq 0$ can be written in terms $\zeta(3,z)$. The second sum is related to $\zeta(2,z)$ in a similar manner. Introducing the reduced parameters

$$\tau \equiv KTt/\hbar\,, \quad \beta \equiv \hbar c/KTL\,, \tag{18}$$

we finally obtain

$$L^4\Gamma(\tau,\beta) = \frac{\hbar c}{\pi^2\beta^4}\zeta(4,1+i\tau) - \frac{\hbar c}{4\pi\beta^2}\zeta(2,1+i\tau) + \frac{\hbar c}{12\beta} - i\frac{\hbar c}{12\pi^2\beta^3}\sum_{\nu_1,\nu_2,\nu_3=-\infty}^{+\infty}{}' \nu^{-1}[\zeta(3,1+i(\tau-2\nu/\beta)) - \zeta(3,1+i(\tau+2\nu/\beta))] - \frac{\hbar c}{8\pi\beta^2}\sum_{m=-\infty}^{+\infty}{}' [\zeta(2,1+i(\tau-2m/\beta)) + \zeta(2,1+i(\tau+2m/\beta))]\,, \tag{19}$$

where the primes indicate that the terms corresponding to $\nu=0$ and $m=0$, respectively, are omitted in the summation.

We now replace the two sums appearing in the above rigorous representation of the temporal coherence function by an *asymptotic* series. Using $\zeta(s,1+z)=\zeta(s,z)-z^{-s}$ and the asymptotic expansions of $\zeta(s,z)$ for $|z|\to\infty$, which we calculate from relations involving the polygamma functions,[27,28] we obtain

$$-i\,\frac{\hbar c}{8\pi}\sum_{n=0}^{\infty}\tau^{2n+1}\times\sum_{m=n}^{\infty}(-1)^{m-n}2^{-2m}B_{2(m-n)}\binom{2m+1}{2n+1}q_m\beta^{2m}-\frac{\hbar c}{16\pi}\sum_{n=0}^{\infty}2^{-2n}(2n+1)q_n(\tau\beta)^{2n}\,, \tag{20}$$

with

$$q_m=\zeta(2m+2)-(m+1)\zeta_E(2m+4,3)/6\pi\,,$$

where the ordinary Riemann zeta function

$$\zeta(s)=\sum_{n=1}^{\infty}n^{-s}\,,$$

the Epstein zeta function

$$\zeta_E(s,p)=\sum_{\nu_1,\ldots,\nu_p=-\infty}^{+\infty}{}'\,|\vec{\nu}|^{-s}\,,\quad \vec{\nu}=(\nu_1,\nu_2,\ldots,\nu_p),$$

and the Bernoulli numbers B_{2k} are involved. Together with the first three terms of (19), the above asymptotic series are a rigorous representation of $\Gamma(\tau,\beta)$ in the limit $\beta\to 0$, for $\tau\beta<2$. For finite, but not too large β and $\tau\beta$, a finite number of leading terms of the expansion provide a useful approximation of $\Gamma(\tau,\beta)$, say $\beta\lesssim 1$ and $\tau\beta\lesssim 1$ or

$$TL\gtrsim\hbar c/K,\quad L\gtrsim ct. \tag{21}$$

Under these conditions the leading terms of (20) read

$$-i\,\frac{\hbar c}{8\pi}\{q_0\tau[1+O(\beta^2)]+O(\tau^3\beta^2)\}-\frac{\hbar c}{16\pi}\{q_0+O(\tau^2\beta^2)\}. \tag{22}$$

In particular, (19) and (20) enable us to establish an expansion for the electric field autocorrelation function[7]

$$C_E(t)\propto V^{-1}\int_V d\vec{x}\,\mathrm{tr}\{\rho\vec{E}(\vec{x},0)\vec{E}(\vec{x},t)\}$$

of the cube. Up to a normalization constant, we obtain

$$C_E(\tau,\beta)\propto\mathrm{Re}\,\zeta(4,1+i\tau)-\frac{\pi}{4}\beta^2\,\mathrm{Re}\,\zeta(2,1+i\tau)+\frac{\pi^2}{12}\beta^3-\frac{\pi}{16}q_0\beta^4-\frac{\pi}{16}\sum_{n=1}^{\infty}2^{-2n}(2n+1)q_n(\tau\beta)^{2n}. \tag{23}$$

For intermediate times $t\gtrsim 3\hbar/KT$, but still obeying $t\lesssim L/c$, we can use the expansions of $\mathrm{Re}\,\zeta(s,1+i\tau)$ for $\tau\to\infty$.[28] Adopting a new normalization we thus find the approximation

$$C_E(t,L,T)\propto-\frac{1}{2}\left(\frac{ct}{L}\right)^{-4}-\frac{\pi}{8}\left(\frac{ct}{L}\right)^{-2}+\frac{\pi^2}{12}\frac{KTL}{\hbar c}-\frac{\pi}{16}q_0-\frac{3\pi}{64}q_1\left(\frac{KTL}{\hbar c}\right)^4\left[\left(\frac{ct}{L}\right)^2+O\left(\left(\frac{ct}{L}\right)^4\right)\right]. \tag{24}$$

IV. DISCUSSION AND THE SPECTRAL DENSITY

Let us now discuss the main results, (13) and (14), (19)–(21), and (23) and (24), and compare them with the thermodynamic or free-space limit[3-5] $L\to\infty$ and previous partial results obtained by heuristic methods.[18] In the free-space limit, the temporal ($\vec{x}=\vec{y}$) electric and magnetic correlation tensors are diagonal and the mixed tensor vanishes. These results are reproduced for finite cubes [see (13)] provided that the spatial averaging (1) is adopted. Furthermore, the six diagonal components $\mathcal{E}_{\mu\mu}(t)$ and $\mathcal{B}_{\mu\mu}(t)$ are identical in the thermodynamic limit. This result is confirmed here [see (14)] by virtue of the cubical symmetry, but it would not hold, e.g., for a cuboidal cavity showing different edge lengths, $L_1\neq L_2\neq L_3$. The well-known result for the infinite-space temporal coherence function appears as the first term of (19). All the following terms are corrections due to the finite size of the cavity and become obsolete in the limit $L\to\infty$. Obviously the $\bar{\mathcal{E}}_{\mu\mu}$ and $\bar{\mathcal{B}}_{\mu\mu}$ show *no first-order correction* proportional to L^2. This result, too, is a consequence of the symmetry of the problem. We observe that the relevant physical parameters for the complete expansions (19) and (20) are LT and t/L. For some arbitrary finite LT, $\Gamma(t)$ does not vanish in the limit $t\to\infty$; i.e., the finite size of the system produces a *long-time memory* of the temporal coherence. We observe that the real parts of the higher-order corrections appearing in (19) and (20) are smaller and less complicated than the imaginary parts. This remark applies as well to the long-time expansions of $\zeta(4,1+i\tau)$ and $\zeta(2,1+i\tau)$. Consequently, the relatively simple expansions (23) and (24) for $C_E\propto\mathrm{Re}\,\Gamma$ can be established. The first three terms in (23) (and only these) can be conjectured from a heuristic integration of the appropriate smoothed mode density,[18] leaving the range of validity open to *a posteriori* trials. Here, however, we were able to establish the conditions (21) in a rigorous manner, because we know the asymptotic expansion (20) to all orders in both LT and t/L.

With respect to the phase problem[29,30] related to the above temporal coherence function $\Gamma(t)$ we mention that not only $\zeta(4, 1+i\tau)$, but also $\zeta(2, 1+i\tau)$ has no zeros on the axis $\text{Im}\tau=0$ or in the lower half plane $\text{Im}\tau<0$, because the corresponding path integrals over $\Phi=\arg\Gamma$ vanish.

The rigorous expression (19) allows us to calculate the spatial average of *the cross-spectral tensors* $W_{\mu\nu}(\vec{x}, \vec{y}, \omega)$, with $\omega = kc$ introduced by Mehta and Wolf[6]; e.g., for the electric field we have

$$\overline{W}^{(e)}_{\mu\nu}(\omega) = V^{-1}\int_V d\vec{x}\, W^{(e)}_{\mu\nu}(\vec{x},\vec{x},\omega) = \int_{-\infty}^{+\infty} dt\, e^{i\omega t}\,\overline{\mathcal{E}}_{\mu\nu}(t)$$
$$= \delta_{\mu\nu} W(\omega) = \delta_{\mu\nu}\int_{-\infty}^{+\infty} dt\, e^{i\omega t}\Gamma(t), \qquad (25)$$

where the diagonal component $W(\omega)$ is proportional to the spectral energy density of the radiation field. Thus, by taking the Fourier transform of (19) and omitting the thermal-energy factor, we derive a rigorous and complete expression for the *spectral mode density* $D(k)$ of the cube-shaped cavity,

$$D(k) = \frac{L^3k^2}{\pi^2}\left(1 + \sum_{\nu_1,\nu_2,\nu_3=-\infty}^{+\infty}{}' \frac{\sin 2\nu kL}{2\nu kL}\right) - \frac{3L}{2\pi}\left(1 + \sum_{m=-\infty}^{+\infty}{}' \cos 2mkL\right) + \tfrac{1}{2}\delta(k), \qquad (26)$$

where ν and the primes have the same meaning as above [see (17) and (19)]. Here, L^3k^2/π^2, $-3L/2\pi$, and $\frac{1}{2}\delta(k)$ are known as the volume, edge, and corner terms[15,17] and were previously interpreted as the leading terms of an asymptotic expansion of the smoothed mode density $\overline{D}(k)$ around $Lk\to\infty$. The subsequent terms of (26), unknown hitherto, describe the oscillatory behavior of $D(k)$ around $\overline{D}(k)$. The calculation is sketched in the Appendix.

The complete expansion (20) enables us to rederive a previous result[12] for the *total energy density* in terms of $U=\lim_{t\to 0}\{6\Gamma(t)\}$ with a precise remainder,

$$L^4U = \frac{\pi^2K^4}{15(\hbar c)^3}(TL)^4 - \frac{\pi K^2}{4\hbar c}(TL)^2 + \frac{K}{2}TL - \frac{3\hbar c}{8\pi}q_0 + O((TL)^{-M}), \qquad (27)$$

with arbitrarily large $M>0$, where

$$q_0 = \tfrac{1}{6}[\pi^2 - \pi^{-1}\zeta_E(4,3)]$$

with $\zeta_E(4,3) = 16.532\,315\,96\cdots$.[9] We observe that an incorrect value for $\zeta_E(4,3)$ was used in Refs. 12 and 16. We mention that the fourth term in the above expansion is assumed to describe the *Casimir effect* and was independently obtained by Lukosz.[9] The result (27) can be verified as well by integrating $\hbar ckD(k)(e^{\hbar ck/KT}-1)^{-1}$ using (26).

The above investigation is related to further interesting problems. We mention the comparison of the asymptotic expansions with computational results for very small LT directly achievable from the sum (15) and the related Poincaré cycles, the calculation of the visibility $|\Gamma(t)/\Gamma(0)|$, the generalization to less symmetric domains, e.g., the cuboid, and the comparison with different approaches made for the infinite-slab and half-space geometries.[19–21] An according study is in progress.

APPENDIX

Having used the units of Glauber's paper,[2] the cross-spectral tensor (25) differs from the one originally defined by Mehta and Wolf[6] by the factor 4π. It is shown in the first paper of Ref. 6 that

$$\sum_\mu W^{(e)}_{\mu\mu}(\vec{x},\vec{x},\nu) \qquad (A1)$$

represents the contribution from the frequency range ν, $\nu+d\nu$ to the expectation value of the electric energy at the position $\vec{x}$. Integration of (A1) over the cube volume thus yields the spectral electric energy of the whole cavity in that frequency range. Accounting for the isotropy, the spectral magnetic energy, and for $\omega=2\pi\nu$, the total spectral energy density $u(\omega)d\omega$ is obtained as

$$u(\omega) = 6\int_{-\infty}^{+\infty} dt\, e^{i\omega t}\,\Gamma(t), \qquad (A2)$$

with $\Gamma(t)$ as given by (19). The Fourier transform of (19) is achieved by transforming the Riemann zeta series term by term using

$$\sum_{n=1}^{\infty}\int_{-\infty}^{+\infty} d\tau\, e^{-i\hbar\omega\tau/KT}[n+i(\tau\pm 2m/\beta)]^{-l} = \frac{2\pi(-1)^l}{(l-1)!}\left(\frac{\hbar\omega}{KT}\right)^{l-1}\frac{e^{\pm 2im\omega L/c}}{e^{\hbar\omega/KT}-1} \qquad (A3)$$

for $l=2,3$ and leads to

$$u(\omega) = \frac{\hbar\omega}{e^{\hbar\omega/KT}-1}\left(\frac{\omega^2}{\pi^2c^3}\sum_{\nu_1,\nu_2,\nu_3=-\infty}^{+\infty}\frac{\sin(2|\vec{\nu}|\omega L/c)}{2|\vec{\nu}|\omega L/c} - \frac{3}{2\pi cL^2}\sum_{m=-\infty}^{+\infty}\cos(2m\omega L/c)\right) + \tfrac{1}{2}(KT/L^3)\delta(\omega). \qquad (A4)$$

Multiplying by $(e^{\hbar\omega/KT}-1)/\hbar\omega$ and realizing that $\delta(\omega)(e^{\hbar\omega/KT}-1)/\hbar\omega=\delta(\omega)/KT$ we finally obtain the density of states (26).

By applying the Poisson summation, (26) can be rewritten as

$$D(k)=\frac{L^2k}{2\pi}\sum_{\mu_1,\mu_2,\mu_3=-\infty}^{+\infty}\frac{1}{|\vec{\mu}|}\ln\left|\frac{\pi|\vec{\mu}|+kL}{\pi|\vec{\mu}|-kL}\right| -\frac{3L}{3}\sum_{n=-\infty}^{+\infty}\delta(kL-n\pi)+\tfrac{1}{2}\delta(k). \tag{A5}$$

We observe that (26) can as well be obtained directly from the definition[14, 17]

$$D(k)=\tfrac{1}{4}\sum_{n_1,n_2,n_3=-\infty}^{+\infty}\delta(k-|\vec{n}|\pi/L) -\tfrac{3}{4}\sum_{m=-\infty}^{+\infty}\delta(k-m\pi/L)+\tfrac{1}{2}\delta(k) \tag{A6}$$

through a Poisson summation.

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