PHYSICAL REVIEW A VOLUME 12, NUMBER 1 JULY 1975

Particle approach to the Fresnel coefficients

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(Received 11 March 1975)

The Fresnel reflectivity and transmissivity coefficients for a semi-infinite medium are derived for normal and oblique incidence using a particle approach and a probabilistic condition at the boundary. The method depends in part on the behavior of the momentum of light in ordinary refractive media, and this subject is discussed. The method clarifies why the Schrödinger equation for the comparable problem (potential-step) and Maxwell's equations yield the same coefficients for normal incidence. The use of the phase velocity rather than the group velocity in formulating the probabilistic condition is interpreted from a quantum viewpoint, although the absence of Planck's constant in the analysis suggests one is dealing with a kind of nonclassical but prequantal indeterminacy. An analysis and review of the momentum, inertia, and energy-momentum tensor of light in ordinary refractive media is also given.

I. INTRODUCTION

In a reexamination of the particle approach to refraction, it was remarked that we need the wave theory to obtain the probability of transmission of light into a medium.[1] Now while this is certainly true in general, there are special cases when a particle picture suffices. Thus, the main purpose of this work is to derive the Fresnel reflectivity and transmissivity coefficients at a plane boundary for normal and oblique incidence using the particle picture.

The treatment will be restricted to a homogeneous, isotropic, semi-infinite medium, since the interference effects that would arise from the other face of the medium obviously vanish in this idealized situation. Even with the above restrictions, one of the assumptions we have made is most curious: It involves the introduction at the boundary of a probabilistic condition on the motion at the boundary which is to be contrasted with the "classical" utilization of a limiting condition to provide a deterministic description. The relation to quantum mechanics is discussed.

In Sec. II the momentum of light in an ordinary refractive medium will be reviewed, since this is needed for the formulation of the condition which utilizes the phase velocity rather than the group velocity. The detailed formulation of the condition is given in Sec. III. Effects due to polarization will be taken up in Sec. IV, where oblique incidence is considered. In Sec. V a further analysis of the probabilistic condition is given. In Sec. VI some further remarks are made about the momentum of light in refractive media and its bearing on the derivation of the Fresnel coefficients. Also the quantum significance of the use of the phase velocity in formulating the probabilistic condition is discussed. In an appendix, arguments are given in favor of the Newton-Minkowski momentum rather than the Abraham momentum.

II. PHOTON MOMENTUM IN AN ORDINARY REFRACTIVE MEDIUM, AND THE MOMENTUM RULE

We suppose that light is normally incident on a plane, semi-infinite, homogeneous, isotropic, nonabsorbing medium. (An entirely similar problem in quantum mechanics is the case of a particle incident on an attractive, semi-infinite potential well.) If the photon is transmitted, the magnitude of its momentum p' in the medium is related to the magnitude of its momentum p in free space by the equation

$$p' = np\,, \tag{1}$$

where n is the index of refraction. This equation is readily derived from the Cartesian-Newtonian assumption of a normally directed, short-ranged, attractive force between the medium and the particle, conservation of momentum in the transverse direction, and the definition of n through the law of refraction.[1] In this model, the relation holds, independent of the angle of incidence, and hence it holds for normal incidence as a special case. For simplicity, we consider the case of a boundary between free space and the medium, but more generally the argument below holds if n is only the relative index. There is no restriction that the medium be nondispersive; however, double refraction is excluded.

Although the momentum increase predicted by Eq. (1) is a basic consequence of the attractive force in the classical corpuscular theory of refraction, the result is implicit in the writings of J. J. Thomson[2] on the wave theory of light, and it was also discussed in the electromagnetic theory of light by Goldhammer.[3] In his classic 1905 paper, Poynting[4] gave a more general argument based on the momentum of light in a wave train as discussed by Larmor, and he showed that Eq. (1) follows "without any further appeal to the theory of wave motion." In this remarkable paper, Poynting describes a mechanical-optical experiment he performed with Barlow on unpolarized light in which they verified the momentum increase predicted by Eq. (1) to about 15% accuracy. More explicitly, they verified that there is an attractive force between the light and the medium (glass prisms) as demanded by the Newtonian model. After an extraordinary gap of nearly half a century, one finds more recent and more accurate experiments under different conditions by Jones,[5] and with greater accuracy by Jones and Richards,[6] who verified the increase in pressure on a vane immersed in a fluid medium of index n. Quite recently, this attractive force predicted by Newton was verified in a rather dramatic way with the aid of a laser beam incident on the surface of a liquid by Ashkin and Dziedzic.[7] The paper of Jones and Richards contains a useful bibliography of the earlier classical experimental and theoretical work on the pressure of light. Some later references are to be found in the paper of Ashkin and Dziedzic. Theoretical support for an increase in momentum of a light beam in an optically denser medium is to be found in a footnote in Whittaker[8] which seems to be making an oblique reference to Jones' preliminary results.[5]

The momentum relation also follows from Minkowski's energy-momentum tensor in electromagnetic media,[9] but it disagrees with the straightforward predictions using Abraham's tensor,[10] or that of similar theories.[11] A useful, recent review up to 1970 is to be found in a work by Brevick.[12] Additional discussion in the relativistic literature is to be found in works by Møller,[13] von Laue,[14] and Pauli.[15]

Equation (1) also underlies the quantum theory of the Čerenkov effect as developed by Ginzburg.[16] A useful bibliography is given by Jelley.[17] In an earlier analysis of the momentum question from the standpoint of quantum theory and special relativity, it was pointed out by March[18] and later by Jordan,[19] that Eq. (1) follows directly from the de Broglie relation and the reduction of wavelength in a refractive medium,

$$\lambda' = \lambda / n . \qquad (2)$$

A cautious recognition that Eq. (1) follows from quantum theory is in Jones and Richards. Recently, an alternate interpretation of the Jones experiment has been given by Burt and Peierls.[20] Their analysis is based on the Abraham viewpoint and that expressed in Landau.[21] Similar viewpoints are to be found in the works of Schockley,[22] Penfield and Haus,[23] Costa de Beauregard,[24] Gordon,[25] Skobel'tsyn,[26] Ginzburg,[27] and Arnaud.[28] A systematic criticism of the Abraham viewpoint is given in the Appendix.

In support of the Newton-Minkowski momentum, we emphasize (as in previous work[1]), that the de Broglie relation for light follows directly from Eqs. (1) and (2), since one has $p'\lambda' = p\lambda =$ constant, under refraction, although additional arguments are needed to establish the universality of the constant. This will be taken up elsewhere since the de Broglie relation does not actually enter the derivation. But we believe it desirable to emphasize the internal consistency of our approach with the de Broglie relation and with complementarity.

III. FURTHER ASSUMPTIONS; THE STATISTICAL CONDITION

Since we have assumed a sharp boundary, the force on the photon at the boundary becomes singular because of the abrupt change in the index of refraction. Now it is possible to give a definition of the force with the aid of a smoothly varying index which yields a unique, finite impulse upon the passage to the limit of a sharp boundary. A typical technique would be to introduce a δ-function attractive force at the boundary. However, such an approach is deterministic: Every particle incident on the medium would be transmitted. On the other hand, we know from experience that one does not have a unique result: Some of the particles will be found in a transmitted mode and the other particles will be found in a reflected mode. In macroscopic language, there is a reflected and a transmitted beam. To handle this peculiar situation, we shall therefore introduce a statistical method. We shall assume that the probability for a photon to be found in a reflected or a transmitted mode can be calculated from an ensemble average. The ensemble consists of a large number of plane, semi-infinite media of index n. There is one ensemble representative for each incident photon. The probability of reflection will be the fraction of the total number of particles that are observed to have been reflected, and likewise for the probability of transmission. Provided the system is linear, which we shall verify, this is a legitimate way to define the coefficients, although in practice one works with a large number of photons incident on one medium and compares intensities in the respective modes. We shall undertake to reexamine the linearity in Sec. VI, where it will be shown that as a result of a remarkable cancellation, which we might call the independence of collisions, the coefficients are the same for one photon as for N photons. Well-known nonlinear behavior is outside the scope of this work, which has the more modest goal of deducing the standard coefficients.

Let I denote the incident particle flux, let I_R be the flux of reflected particles, and let I_T be the flux of transmitted particles. Under the assumption that the recoil energy transmitted to the media is negligibly small, the energy of each photon E_γ will be conserved independently of whether it is reflected or transmitted. Hence from total energy conservation we have

$$I_R E_\gamma + I_T E_\gamma = I E_\gamma . \tag{3}$$

As is well known, the energy conservation law expressed by this equation can also be deduced in electromagnetic theory (Poynting's theorem). However, Eq. (3) is more general than electromagnetic theory and also it contains more information. Indeed, since E_γ is a common factor, we can deduce from it the conservation of photon flux,

$$I_R + I_T = I . \tag{4}$$

An entirely similar equation also follows from quantum mechanics and the surface integral of the conserved probability current. However, as is well known, an equation such as Eq. (4) does not hold in general for light because of emission and absorption phenomena. Ordinary refraction is a remarkable special case for which the above equation holds. Thus, as is well known, the problem of light normally incident on a plane, refractive medium has features entirely similar to that of a nonrelativistic particle incident on a potential well in quantum mechanics as expressed in Eqs. (1) and (4). Nevertheless, as we shall see below, the quantum of action is not involved in the derivation of the Fresnel coefficients.

In accordance with standard notation, we now introduce the ratios $R = I_R/I$ and $T = I_T/I$, which are taken to define the probabilities of reflection and transmission, respectively. These probabilities satisfy the so-called conservation of probability condition

$$R + T = 1 . \tag{5}$$

Since Eq. (5) (or its equivalent) is fundamental to any derivation, the problem becomes one of finding another condition on R and T. The following considerations are necessary to the formulation of this condition.

From the conservation of over-all momentum and Eq. (1), the momentum transferred to an ensemble member if the photon is transmitted is

$$\Delta P_T = -(n-1)p , \tag{6}$$

and the corresponding kinetic energy taken up by the ensemble member is

$$E_T = (n-1)^2 p^2 / 2M , \tag{7}$$

where M is the mass of the ensemble member. If E_γ is the energy of the incident photon, it is assumed that $E_T \ll E_\gamma$. This is to ensure that the recoil of the medium does not significantly alter the frequency of the photon, a condition which is certainly fulfilled in ordinary refraction. Thus, higher-order corrections to Eq. (7) are neglected. Similarly, if the photon is reflected, the momentum transferred is $\Delta P_R = -2p$, and the kinetic energy taken up by the ensemble member is

$$E_R = 2p^2/M . \tag{8}$$

The rate at which energy is transmitted to the ensemble media for these two modes will be assumed to be proportional to the *phase velocity* of classical Hamiltonian mechanics. We introduce the phase velocity U_R in the reflected mode,

$$U_R = E_\gamma / p , \tag{9}$$

and likewise for the phase velocity in the transmitted mode we have

$$U_T = E_\gamma / np . \tag{10}$$

We write for the average rate of energy delivered to the ensemble member in the reflected mode, the quantity

$$R U_R E_R , \tag{11}$$

and for the average rate of energy delivered in the transmitted mode, the quantity

$$T U_T E_T . \tag{12}$$

Now in the limit of infinite mass of the refracting medium, both E_R and E_T vanish, and hence it is trivially true that

$$R U_R E_R = T U_T E_T . \tag{13}$$

Let us suppose, however, that this relation also holds for M merely very large, so that we have from Eqs. (7), (8), and (10),

$$R \frac{E_\gamma}{p} \frac{2p^2}{M} = T \frac{E_\gamma}{np} \frac{(n-1)^2 p^2}{2M} . \tag{14}$$

This equation yields the following additional relation on the intensities

$$T = \frac{4n}{(n-1)^2} R . \tag{15}$$

This result, in conjunction with the conservation of probability condition [Eq. (5)], yields the standard Fresnel coefficients for normal incidence:

$$R = \frac{(n-1)^2}{(n+1)^2}, \quad T = \frac{4n}{(n+1)^2}. \tag{16}$$

It is a consequence of the above derivation that the Fresnel coefficients hold independently of whether the incident particle is relativistic or nonrelativistic. Also, because of the isotropic assumption, they are independent of polarization. Hence, the Fresnel coefficients are of the same form whether one derives them from Maxwell's equations or from Schrödinger's equation; a result which is not immediately evident from the equations themselves. A further discussion of the statistical condition is given in Secs. V and VI.

IV. OBLIQUE INCIDENCE

The Fresnel coefficients for oblique incidence can be derived in a manner similar to that in which we derived them for normal incidence, although the effects of polarization of light requires additional assumptions. We denote the angle of incidence by θ and the angle of refraction by θ'. For the component of momentum transferred to the medium in the normal direction we have

$$\Delta P_T = -(p'\cos\theta' - p\cos\theta), \tag{17}$$

when the photon is transmitted. When the photon is reflected we have

$$\Delta P_R = 2p\cos\theta. \tag{18}$$

For the phase velocity of the reflected beam in the direction normal to the surface, we have

$$U_R = E_\gamma / p\cos\theta. \tag{19}$$

Likewise, the phase velocity of the transmitted beam in the direction of the normal is

$$U_T = E_\gamma / p'\cos\theta'. \tag{20}$$

Therefore, upon generalizing the equations for normal incidence, for the component polarized perpendicular to the plane of incidence, we have

$$R_\perp + T_\perp = 1, \tag{21}$$

and also

$$R_\perp \frac{E_\gamma}{p\cos\theta}\frac{(2p\cos\theta)^2}{2M} = T_\perp \frac{E_\gamma}{p'\cos\theta'} \times \frac{(p'\cos\theta' - p\cos\theta)^2}{2M}. \tag{22}$$

After some simplification, and the use of the Newtonian mechanical form of the Snell-Descartes law (transverse-momentum conservation condition)

$$p'\sin\theta' = p\sin\theta, \tag{23}$$

we find

$$\frac{T_\perp}{R_\perp} = \frac{\sin 2\theta \sin 2\theta'}{\sin^2(\theta-\theta')}. \tag{24}$$

Hence, upon substitution in Eq. (21), we obtain the standard Fresnel coefficients for $T_\perp$ and $R_\perp$:

$$T_\perp = \frac{\sin 2\theta \sin 2\theta'}{\sin^2(\theta+\theta')}, \quad R_\perp = \frac{\sin^2(\theta-\theta')}{\sin^2(\theta+\theta')}. \tag{25}$$

In order to obtain the transmission and reflection coefficients for the radiation polarized parallel to the plane of incidence, we use an equation entirely analogous to Eq. (22), but in which the polarization vectors $\vec{e}$, $\vec{e}'$ have been introduced to describe the polarization of the reflected and transmitted beams, respectively. We write the following generalization of Eq. (22), which allows for both perpendicular and parallel polarization:

$$R\frac{E_\gamma}{p\cos\theta}\frac{(2p\cos\theta)^2}{2M} = T\frac{E_\gamma}{p'\cos\theta'}\frac{(p'\cos\theta' - p\cos\theta)^2}{2M} \times (\vec{e}\cdot\vec{e}')^2. \tag{26}$$

For the case that $\vec{e}$ and $\vec{e}'$ are both perpendicular to the plane of incidence, the factor $(\vec{e}\cdot\vec{e}')$ is unity and Eq. (26) reduces to Eq. (22). However, for the case of the polarization vector parallel to the plane of incidence, one finds

$$(\vec{e}\cdot\vec{e}')^2 = \cos^2(\theta+\theta'). \tag{27}$$

This extra factor yields zero reflection at the Brewster angle. More generally, one has from Eq. (26),

$$\frac{R_\parallel}{T_\parallel} = \frac{\sin^2(\theta-\theta')}{\sin 2\theta \sin 2\theta'}\cos^2(\theta+\theta'), \tag{28}$$

which in conjunction with the conservation of probability condition yields the standard Fresnel coefficients for the parallel component,

$$T_\parallel = \frac{\sin 2\theta \sin 2\theta'}{\sin^2(\theta+\theta')\cos^2(\theta-\theta')}, \quad R_\parallel = \cot^2(\theta+\theta')\tan^2(\theta-\theta'). \tag{29}$$

It should be pointed out that one can treat the comparable case of light emerging from a semi-infinite medium into free space by the above methods. For simplicity, let us consider the radiation to be polarized perpendicular to the plane of incidence so that Eq. (22) is the appropriate one. As one approaches the critical angle, the angle of refraction θ' tends to $\pi/2$, and hence $\cos\theta' \to 0$; consequently the denominator in Eq. (22) vanishes. However, one can multiply both sides of the equation by $p'\cos\theta'$ and then pass to the limit. One finds $T_\parallel = 0$, which agrees with the limit obtained from the Fresnel relations and experience.

Just as in the wave treatment of metallic reflection and evanescent states, so too in the method developed here, one can formally extend the index of refraction into the complex plane and likewise the momentum rule. On the other hand, the experiments of Poynting, Jones, and Ashkin verified the momentum rule only for real n with $n > 1$, and it would of course be mathematically and physically desirable to have a mechanical verification for cases with n complex (e.g., a plasma) with the real part of $n < 1$.

V. INDEPENDENCE OF COLLISIONS

It will be observed that it is basic to the above ensemble approach that the collisions of the particles with the physical medium may be treated as independent. However, it is not clear why the form of the energy transferred to the physical medium should be the same for one particle as for a "rain" of particles. Indeed, on the basis of the above derivation, it might at first appear that the Fresnel coefficients would hold only for very low intensities, in conflict with experience. Actually, we have found that due to the structure of the coefficients themselves a very remarkable simplification takes place which justifies the independent collision result.

The total momentum ΔP transferred to the medium for N photons normally incident on the physical medium is

$$\Delta P = [R2p - T(n-1)p]N, \tag{30}$$

where the direction of the incident beam has been taken to be positive. Hence the energy transferred to the physical medium of mass M is given by

$$\frac{\Delta P^2}{2M} = \frac{[R2p - T(n-1)p]^2}{2M} N^2 . \tag{31}$$

Upon introducing the Fresnel values for R and T, we can obtain

$$\frac{\Delta P^2}{2M} = \frac{2p^2}{M} RN^2 . \tag{32}$$

Since RN is the number of reflected photons, the energy transferred per reflected photon is $(2p^2/M)N$ and has the same form as our treatment based on individual collisions, except for a multiplicative factor N associated with the number of photons. One can also reduce Eq. (31) to an expression which involves the transmissivity. One finds

$$\frac{\Delta P^2}{2M} = \frac{(n-1)^2}{n} \frac{p^2}{2M} TN^2 , \tag{33}$$

where TN/n is proportional to the phase velocity flux of the transmitted photons and $N(n-1)^2 p^2/2M$ represents the energy transferred to the medium by the transmitted photons and has the same form as for the individual photon case enhanced by the factor N. Upon equating Eqs. (32) and (33), we obtain the equivalent of Eq. (16), which confirms the consistency of the method based on individual collisions. An interesting relation between T and R results if instead of using the Fresnel values we require that Eq. (31) reduce to Eq. (32). Upon solving for T we obtain

$$T = \frac{2}{n-1} R^{1/2}(R^{1/2} + 1) . \tag{34}$$

Similarly, if we require Eq. (31) to reduce to Eq. (33), we obtain

$$R = \frac{n-1}{2} T^{1/2}\left(T^{1/2} - \frac{1}{n^{1/2}}\right) . \tag{35}$$

From Eqs. (34) and (35) the Fresnel values of R and T can be deduced. It is rather curious to see appear in Eqs. (34) and (35) the square roots of probability, that is, the terms $R^{1/2}$ and $T^{1/2}$. It is therefore conceivable that a deeper study of the independence of collisions assumption may help to clarify the probability amplitude concept in quantum mechanics. Another use for Eqs. (34) and (35) would be to see how well they hold in the case of nonideal boundary conditions.

VI. CONCLUDING REMARKS

Let us first consider the bearing that the derivation of the Fresnel coefficients has on the momentum rule. It will be noted that we would still obtain the same coefficients for normal incidence if in the probabilistic energy transfer equation we assumed that the momentum of light had diminished in the optically denser medium, instead of increasing, since the right-hand side of Eq. (14) is invariant under $n \to 1/n$. However, this would lead to an inconsistency with the standard identification of the phase velocity with the wave velocity and the fact that the latter is known to diminish in the optically denser medium by the Huygens' construction. For oblique incidence, the derivation of the coefficients uses the conservation of transverse momentum and this obviously requires $p'/p = n$. Further discussion is given in the Appendix.

Perhaps even more fundamental in our derivation of the Fresnel coefficients is the utilization of the phase velocity rather than the particle velocity. This is somewhat surprising from a purely classical particle viewpoint even if one allows a probabilistic condition instead of a deterministic one at the boundary. Nevertheless, as we have seen in Sec. V, the equation follows from the form of the Fresnel coefficients and the equation expressing the independence of the collisions. A possible explanation (suggested by quantum mechanics) is that the energy transfer equation is not associated with a physical flow of energy but a statistical flow associated with each particle. Indeed, in the determination of R and T we have imagined an arrangement in which individual photons could be incident on widely separated samples at random intervals of time. As a consequence, one cannot regard the equation as describing a true physical current of particles. Rather, it is appropriate to interpret the equation as a probabilistic statement about the interaction of each individual photon (or quantum) with the medium. Also by introducing this condition which does not involve Planck's constant, we have introduced a kind of prequantum indeterminacy that leaves the motion of individual particles undetermined. It should be emphasized that the condition can of course be deduced from Schrödinger's equation and Maxwell's equation to within multiplicative factors such as M^{-1}, the reciprocal of the mass of the medium, which vanishes in the case of a semi-infinite medium. The method has the advantage over strictly classical treatments of scattering in that it gives a reflected beam as well as a transmitted one. The approach should be of value in further clarifying questions about the foundations of quantum mechanics as well as fundamental questions about the interaction of light and matter.

ACKNOWLEDGMENTS

The author acknowledges assistance from a National Science Foundation college departmental research grant during the summer of 1969 when this work was commenced. He wishes to thank Dr. G. Alberi and Dr. I. Kostadinov for useful references to the literature and discussions at the International Centre for Theoretical Physics in Trieste during 1973–74 where the author spent his sabbatical leave; the hospitality there and in Sistiana is warmly acknowledged.

APPENDIX

Several additional arguments in favor of the Minkowski momentum are presented here. The Abraham momentum emerges as a kind of "total momentum" for light plus medium, but not for light itself.

1. Dynamics of internal reflection

Consider oblique incidence with the radiation incident on the boundary at angles greater than the critical angle. This case is not considered, for example, by Burt and Peierls or by Gordon. How can the total internal reflection that results be understood unless the momentum of light is greater in the medium with the greater index? From the standpoint of the classical Newtonian picture, the behavior of the momentum at the boundary can be obtained from the consideration of a block sliding up a finite inclined plane or step at an angle. If the initial angle of incidence is too large, the component of momentum up the plane will not be sufficient to take the block over the top or out of the potential well and one has a turning point. In this familiar elementary analogy, the block experiences no transverse change in its momentum but an attractive impulse down the plane, which is consistent with the behavior of the Minkowski momentum.

From a dynamical standpoint, the treatment of the photon may be based in the ray approximation on the following Hamiltonian, for negligible dispersion:

$$H=\frac{c}{n(z)}(p_x^2+p_y^2+p_z^2)^{1/2}, \tag{A1}$$

where we have set $n=n(z)$ for a medium oriented so that the boundary is in the x-y plane. The advantage of working in the region of total internal reflection is that (neglecting wave-mechanical penetration) the problem can be treated classically, the Fresnel coefficients are $T=0$, $R=1$, and one can legitimately work with $n(z)$ slowly varying and then pass to the limit of a step function. However, one is led to a totally absurd result if in place of (A1), one attempts to relate the photon's Hamiltonian to the Abraham momentum, since $\partial H/\partial p=v$, and for the Abraham energy-momentum relation one would have $v>c$ even with negligible dispersion, which is of course in conflict with experience. Hence the Minkowski momentum is the "canonical momentum." It is therefore the momentum to be used in quantizing the radiation field in a background refractive medium.[27]

2. Photon inertia in a medium

An argument that is used in favor of the Abraham momentum is based on the recoil of the medium and the requirement that the center of mass of a free system at rest must not move. The idea for the case of a nonrefractive medium originates with Einstein, but it has been applied to this problem by Costa de Beauregard[24] and Burt and Peierls,[20] although Skobel'tsyn[26] mentions earlier Russian work along these lines. For simplicity, we consider a nondispersive medium. If m is the mass of the photon emitted by the substance imbedded in the refractive material, L is the length through which the photon travels between emission and reabsorption, T is the travel time which equals

$L/(c/n)$, and M is the mass of the medium, one finds with $m \ll M$ for the recoil momentum p_γ of the photon

$$p_\gamma = m(L/T) = m(c/n)\,. \tag{A2}$$

Both groups of authors assume without discussion that in the medium, $m(\text{photon}) = h\nu/c^2$, in accordance with special relativity, and of course the Abraham momentum follows. However, $h\nu/c^2$ is *not* the correct expression for the mass of a photon in an ordinary refractive medium. As we discussed in great detail elsewhere,[1] and, as may be readily inferred from the above Hamiltonian (A1), the correct expression for the mass of a photon in an ordinary, nondispersive medium of index n is

$$m = n^2 h\nu/c^2\,. \tag{A3}$$

Despite its appearance, (A3) does not represent any violation of special relativity or the principle of equivalence. For although the bound photon has greater inertia and would weigh more than in vacuum, the medium would weigh less, and since both photon and medium are bound together, the total weight and hence inertia is unchanged and is given by $h\nu/c^2$. We have given an application of (A3) to the derivation of the Schwarzschild line element in a novel way.[1] For historical completeness, we note that Preston[29] seems to have been the first to hint at a variable mass for the corpuscle in the older Newtonian theory as a solution to the disagreement with the Foucault experiment. This disagreement of course does not effect our approach based on using the Newtonian momentum, since in the text we worked with the *phase velocity* E/p' and, of course, the phase velocity in Newtonian theory diminishes in the optically denser medium in the same way that the wave velocity does. Subsequent work after Preston (whose footnote seems to have been ignored) was carried out by March,[18] using special relativistic considerations. Later in the present form (A3), the result seems to have been first explicitly obtained by Michels and Patterson,[30] albeit by unsatisfactory arguments which were afterwards improved,[31] and which appeared about the same time as our own independent work, which was stimulated by the problem of Huygens' principle in general relativity,[32] and the Hamiltonian method for treating rays in radio propagation in the ionosphere.[33]

For completeness, we note that in the case of an ordinary dispersive medium, one obtains the mass from $p = mv$, and $dE = v\,dp$; or from

$$m = \frac{1}{2}\frac{dp^2}{dE}\,, \tag{A4}$$

and hence with the above Hamiltonian, but with $n = n(E)$ we have

$$m = \frac{En^2}{c^2}\left(1 + \frac{d\ln n}{d\ln E}\right). \tag{A5}$$

The fact that the Fresnel coefficients do not contain the derivative of n, which also shows up in the group velocity is an indication that the Fresnel coefficients depend only on the phase velocity.

3. Symmetry of energy-momentum tensor

The third question that must be dealt with is the fact that the Minkowski energy-momentum tensor $T_M^{\mu\nu}$ is not symmetric in contrast with the Abraham tensor $T_A^{\mu\nu}$. If we choose a frame at rest in the medium, both tensors agree as to energy density, energy flux, momentum flux or stress, but disagree with respect to momentum density. We have for a nonmagnetic medium with $\epsilon = n^2$, $\mu = 1$,

$$\begin{aligned} \vec{G}_A &= \vec{E}\times\vec{H}/c = (1/c^2)\vec{S}\,, \\ \vec{G}_M &= \vec{D}\times\vec{B}/c = n^2\vec{E}\times\vec{H}/c = (n^2/c^2)\vec{S} \end{aligned} \tag{A6}$$

and of course the lack of symmetry is that $c\vec{G}_M \neq \vec{S}/c$, whereas $c\vec{G}_A = \vec{S}/c$, where S is the Poynting vector, or $T_M^{i0} \neq T_M^{0i}$. However, we regard this departure from symmetry as needed to ensure conservation of energy when the matter tensor of the medium $K^{\mu\nu}$ is taken into account. Since proponents of the Abraham tensor agree that the Minkowski momentum is needed to explain the various experiments, such as the Jones experiment, they propose (cf. Gordon) to write

$$P_M = P_A + P_K\,, \tag{A7}$$

where P_A is the Abraham momentum and this is supposed to be the momentum of the photon. To it they add P_K, the momentum of the medium which is in the same direction. However, since $T_A^{\mu\nu}$ is symmetric, and since the total energy-momentum tensor must be symmetric, the energy tensor of the medium $K^{\mu\nu}$ would have to be symmetric. But since in this picture one needs a momentum density $(n^2-1)\vec{E}\times\vec{H}/c$, the corresponding energy flux that would have to be carried by the medium would be

$$K^{0i} \Longleftrightarrow \vec{S}_{\text{med}} = (n^2-1)c\vec{E}\times\vec{H}\,. \tag{A8}$$

However in the case of negligible frequency shift, this is not possible because the energy flux is measurable and given by the Poynting vector. In the limit of an infinitely massive medium, there is no energy transferred to the medium, and more generally we can always choose M so that T^{0i} is much smaller than the value in (A8). The vanishing of T^{0i} for an analogous case[34] has been pointed out by Balazs.[35] The problem was later discussed by Haus.[36] Thus, one must abandon (A7). On the other hand, if $\vec{G}_M$ is the momentum density of the light, and $T_M^{\mu\nu}$ is the energy-momentum tensor one can now postulate that the matter tensor for the medium also has an asymmetric part due to the interaction with the electromagnetic field

$$K^{i0} \Longleftrightarrow \vec{G}_{\rm med} = -(n^2-1)\vec{E}\times\vec{H}/c\,, \qquad K^{0i} \Longleftrightarrow \vec{S}_{\rm med} = 0\,. \tag{A9}$$

One can then combine the two asymmetric tensors to form the symmetric tensor

$$T_A^{\mu\nu} = T_M^{\mu\nu} + K^{\mu\nu}\,, \tag{A10}$$

with $T_M^{[\mu\nu]} + K^{[\mu\nu]} = 0$, where $T^{[\mu\nu]}$ means asymmetric part. In this view the medium carries momentum in a direction opposite to which the photon is travelling. Furthermore, one should regard this momentum as having been generated during the boundary interaction. So one has in the present case

$$P_A = P_M - |P_K|\,. \tag{A11}$$

Although the change between (A7) and (A11) is very slight mathematically, it provides a simple way to understand the experimental results which demonstrate an attractive force between the light and the boundary of the medium.

For the case of a dielectric-magnetic medium $\epsilon = \mu = 1$, we point out that the Minkowski momentum density satisfies $G_M = \epsilon^{1/2}\mu^{1/2}W/c$, where W is the energy density as one would expect from the de Broglie relation and from the wave equation dispersion relation $\omega^2 = c^2k^2/\epsilon\mu$. Nevertheless for the case $\epsilon = \mu$, there is a suppression of the reflected wave, as discussed in Sommerfeld[34] and Balazs,[35] which might seem to disagree with our Eq. (14). However, we have found that we can obtain the correct expression for the coefficients while retaining the Minkowski momentum under the assumption that the momentum transferred to the medium as used in Eq. (6) becomes

$$\Delta P = -(\epsilon^{1/2}\mu^{1/2}p - \mu p)\,; \tag{A12}$$

this would imply a momentum enhancement of the photon by the magnetic interaction with the electrons at the boundary prior to the interaction with the lattice. A similar assumption is needed for the momentum transfer used for reflection. Thus in Eqs. (8) and (9) one must also set $p \to \mu p$. The over-all mathematical effect in Eq. (14) is the same as if we had set $p \to p$, $p' \to (\epsilon^{1/2}/\mu^{1/2})p$, but the physical effects are different. In the latter case, one would have a "mixed theory," in which the effective momentum density would be $\vec{G} = \vec{D}\times\vec{H}/c$, but this is far from settled and we definitely need the Minkowski momentum to maintain the Snell-Descartes law, Eq. (23).

We note some very recent work dealing with the Fresnel coefficients and the "suppression of reflection" by Agudin and Platzeck[37] from the complementary wave approach. We also call attention to other references[38-44] that have bearing on the momentum rule.

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