Cross sections for excitation of the $n\,^1D$ states of helium by electron impact and polarization of the resulting radiation in Glauber theory

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(Received 17 March 1975)

Recently developed analytic methods, which reduce the Glauber amplitude for charged-particle–neutral-atom collisions to a one-dimensional integral representation involving modified Lommel functions, are used to evaluate the cross sections for the direct excitation of $3\,^1D$ and $4\,^1D$ states of helium by electron impact with incident energies from 40 to 1000 eV. It is shown that the Glauber amplitudes can be written in terms of three generating functions: One of these has already been derived by Thomas and Chan; the detailed reduction of the other two is given in this paper. Comparison is made with the Born, Ochkur, and Woollings-McDowell approximations. The polarization fraction of the 6678-Å helium line emitted in $e^- + \mathrm{He}$ collisions is also calculated in the Glauber approximation. The agreement between theory and experiment is less than satisfactory in the entire energy region.

I. INTRODUCTION

Recently, collisional excitation of the $n\,^1D$ levels of helium has attracted considerable experimental interest since the subsequent radiation lines fall in the visible region of the spectrum and allow total-cross-section measurements. In the last decade, such electron-impact measurements have been made by several groups[1-4] with results which are inconsistant both qualitatively and quantitatively. For example, the results from Ref. 1 are larger than those from Refs. 2 and 3 by ~30–50% in the common energy range of measurements. Moreover, the energy dependences of the experimental cross sections do not agree among themselves. On the theoretical side, calculations have been performed with the Born approximation,[5,6] the Ochkur approximation,[7] and the Woollings-McDowell approximation.[8] But all these calculations give results which are in fairly large discrepancy with experimental values in the entire energy range (40–1000 eV). For example, the total cross section predicted from the Born approximation is smaller than data of Refs. 2 and 3 by ~55% for $3\,^1D$ excitation and by ~26% for $4\,^1D$ excitation even at 1000 eV. The situation is surprising since the Born approximation and the related approximations are expected to be valid in the high-energy region (incident energies ⩾200 eV).

In this paper, we report results obtained from the Glauber approximation (GA),[9] which has recently been applied with partial success to elastic and inelastic scattering of electrons by atomic hydrogen[10-14] and helium[15-19] (the GA is reliable in predicting the magnitude but is incapable of finding the relative phase for the e^--He $1S$-nP excitation amplitude in the intermediate- and high-energy ranges). Furthermore, the present study is interesting in itself since it provides a nontrivial example in which the troublesome $\delta(\vec{q})$ function can and should be removed even for $\vec{q}\neq 0$ (inelastic collision).

We have organized this paper as follows: In Sec. II, we derive the Glauber scattering amplitudes in terms of three generating functions. One of these is given by Thomas and Chan[17]; the detailed derivation of the other two is deferred to an appendix. In Sec. III we derive the expression for the polarization fraction of the 6678-Å helium line in the Glauber approximation; in Sec. IV, we present and discuss the results of numerical calculations of the expressions obtained in Secs. II and III.

II. EXCITATION CROSS SECTIONS

The Glauber scattering amplitudes $F^{(\zeta)}_{n\,^1D,1\,^1S}(\vec{q})$ describing the excitation of the He from the ground state $\Psi_{1\,^1S}(\vec{r}_1,\vec{r}_2)$ to the final state $\Psi_{n\,^1D}(\vec{r}_1,\vec{r}_2)$ by an incident charged particle $Z_i e$ with velocity v_i is given by

$$F^{(\zeta)}_{n\,^1D,\,1\,^1S}(\vec{q})=\frac{iK_i}{2\pi}\int \Psi^*_{n\,^1D}(\vec{r}_1,\vec{r}_2)\Gamma(b;\vec{r}_1,\vec{r}_2)\times\Psi_{1\,^1S}(\vec{r}_1,\vec{r}_2)e^{i\vec{q}\cdot\vec{b}}\,d^2b\,d\vec{r}_1\,d\vec{r}_2\,, \quad (1)$$

where

$$\Gamma(b;\vec{r}_1,\vec{r}_2)=1-(|\vec{b}-\vec{s}_1|/b)^{2i\eta}(|\vec{b}-\vec{s}_2|/b)^{2i\eta} \quad (2)$$

and

$$\eta\equiv -Z_i/v_i \quad (\text{in a.u.}).$$

In Eqs. (1) and (2), $\vec{b}$, $\vec{s}_1$, and $\vec{s}_2$ are the respective projections of the position vectors of the incident particle and the bound electrons ($\vec{r}_1$ and $\vec{r}_2$) onto the plane perpendicular to the direction of the Glauber path integration. The superscript (ζ) represents the $\vec{q}$-dependent coordinate system $C^g(\hat{\xi})$, whose z axis lies along $\hat{\xi}$ and is perpendicular to $\vec{q}$, in which the Glauber amplitudes $F^{(\xi)}_{n\,^1D,\,1\,^1S}(\vec{q},m_L)$ are readily computable. The approximate ground-state wave function chosen (in a.u.) is the one described by Byron and Joachain,[20]

$$\Psi_{1\,^1S}(\vec{r}_1,\vec{r}_2)=(1.6966/\pi)(e^{-1.41r_1}+0.799e^{-2.61r_1})\times(e^{-1.41r_2}+0.799e^{-2.61r_2})\,. \qquad (3)$$

For the n^1D state of He, we adopt Heisenberg's choice,[21] i.e., we consider the screening of the inner on the outer electron as "complete" so we have

$$\Psi_{3\,^1D}(\vec{r}_1,\vec{r}_2)=\frac{1}{\sqrt{2}}\,[\psi_{1s}(2\,|\,\vec{r}_1)\psi_{3d}(1\,|\,\vec{r}_2)+\psi_{1s}(2\,|\,\vec{r}_2)\psi_{3d}(1\,|\,\vec{r}_1)]$$
$$=[8/81(30\pi)^{1/2}][e^{-2r_2}r_1^2e^{-r_1/3}Y_{2m}(\theta_1,\varphi_1)+e^{-2r_1}r_2^2e^{-r_2/3}Y_{2m}(\theta_2,\varphi_2)] \qquad (4)$$

and

$$\Psi_{4\,^1D}(\vec{r}_1,\vec{r}_2)=\frac{1}{\sqrt{2}}\,[\psi_{1s}(2\,|\,\vec{r}_1)\psi_{4d}(1\,|\,\vec{r}_2)+\psi_{1s}(2\,|\,\vec{r}_2)\psi_{4d}(1\,|\,\vec{r}_1)]$$
$$=[1/32(5\pi)^{1/2}][e^{-2r_2}r_1^2(1-r_1/12)e^{-r_1/4}Y_{2m}(\theta_1,\varphi_1)+e^{-2r_1}r_2^2(1-r_2/12)e^{-r_2/4}Y_{2m}(\theta_2,\varphi_2)]\,, \qquad (5)$$

where $Y_{2m}(\theta,\varphi)$ is the standard spherical harmonic.[22]

A. $1\,^1S$-$3\,^1D$ excitation

Substituting Eqs. (3), (4), and Y_{10} into (1), we obtain the transition amplitude to the $m_L=0$ state, $F^{(\xi)}_{3\,^1D,1\,^1S}(\vec{q},m_L=0)$, in terms of two generating functions $I_0(\lambda_1,\lambda_2;q)$ and $I_z(\lambda_1,\lambda_2;q)$,

$$F^{(\xi)}_{3\,^1D,1\,^1S}(\vec{q},m_L=0)=iK_i\frac{2\times1.6966}{81\sqrt{6}}\sum_{n=1}^{4}c(n)\left(3\frac{\partial^2 I_z(\lambda_1,\lambda_2;q)}{\partial\lambda_1\partial\lambda_2}-\frac{\partial^4 I_0(\lambda_1,\lambda_2;q)}{\partial\lambda_1^3\partial\lambda_2}\right)\Bigg|_{\substack{\lambda_1=\lambda_{31}(n)\\ \lambda_2=\lambda_{32}(n)}}$$
$$=h_0(q)\,, \qquad (6)$$

with

$$c(n)=1,\ 0.799,\ 0.799,\ (0.799)^2;\quad \lambda_{31}(n)=1.743,\ 2.943,\ 1.743,\ 2.943;\quad \lambda_{32}(n)=3.41,\ 3.41,\ 4.61,\ 4.61\,.$$

In Eq. (6), $I_0(\lambda_1,\lambda_2;q)$ is defined and given in Eq. (19) of Ref. 17,

$$I_0(\lambda_1,\lambda_2;q)\equiv\frac{1}{\pi^3}\int\frac{e^{-\lambda_1r_1}}{r_1}\frac{e^{-\lambda_2r_2}}{r_2}[1-(|\vec{b}-\vec{s}_1|/b)^{2i\eta}(|\vec{b}-\vec{s}_2|/b)^{2i\eta}]e^{i\vec{q}\cdot\vec{b}}\,d^2b\,d\vec{r}_1\,d\vec{r}_2$$
$$=-2^4(\lambda_1\lambda_2)^{-2}(2i\eta)^2\Gamma(i\eta)\Gamma(1-i\eta)q^{2i\eta-2}[\lambda_1^{-2i\eta}{}_2F_1(1-i\eta,1-i\eta;1;-\lambda_1^2q^{-2})+\lambda_2^{-2i\eta}{}_2F_1(1-i\eta,1-i\eta;1;-\lambda_2^2q^{-2})]$$
$$-2^5(2i\eta)^4\int_0^\infty b^5\,db\,J_0(qb)(i\lambda_1b)^{-2i\eta-2}\mathfrak{L}_{2i\eta-1,0}(i\lambda_1b)(i\lambda_2b)^{-2i\eta-2}\mathfrak{L}_{2i\eta-1,0}(i\lambda_2b)\,, \qquad (7)$$

and $I_z(\lambda_1,\lambda_2;q)$ is defined by

$$I_z(\lambda_1,\lambda_2;q)\equiv\frac{1}{\pi^3}\int\frac{e^{-\lambda_1r_1}}{r_1}\frac{e^{-\lambda_2r_2}}{r_2}z_1^2[1-(|\vec{b}-\vec{s}_1|/b)^{2i\eta}(|\vec{b}-\vec{s}_2|/b)^{2i\eta}]e^{i\vec{q}\cdot\vec{b}}\,d^2b\,d\vec{r}_1\,d\vec{r}_2\,. \qquad (8)$$

The detailed reduction of Eq. (8) is given in Appendix A, where we show that the first term (independent of η) under the integral in Eq. (8), which leads to a δ function in $\vec{q}$, is exactly canceled by a similar factor stemming from the second term (dependent on η). Therefore the $\delta(\vec{q})$ function is explicitly removed both in $I_0(\lambda_1,\lambda_2;q)$ and $I_z(\lambda_1,\lambda_2;q)$ even for $\vec{q}\neq0$ (inelastic collision). $I_z(\lambda_1,\lambda_2;q)$ is given by Eq. (A19) of Appendix A,

$$I_z(\lambda_1, \lambda_2; q) = -2^5\lambda_1^{-4}\lambda_2^{-2}(2i\eta)^2\Gamma(i\eta)\Gamma(1-i\eta)[(1+i\eta)q^{2i\eta-2}\lambda_1^{-2i\eta}\,{}_2F_1(1-i\eta, 1-i\eta; 1; -\lambda_1^2q^{-2})$$
$$+(1-i\eta)^2q^{2i\eta-4}\lambda_1^{-2i\eta+2}\,{}_2F_1(2-i\eta, 2-i\eta; 2; -\lambda_1^2q^{-2})$$
$$+q^{2i\eta-2}\lambda_2^{-2i\eta}\,{}_2F_1(1-i\eta, 1-i\eta; 1; -\lambda_2^2q^{-2})]$$
$$-2^6\lambda_1^{-4}\lambda_2^{-2}(2i\eta)^4\int_0^\infty b\,db\,J_0(qb)[(1+i\eta)(i\lambda_1 b)^{-2i\eta}\mathfrak{L}_{2i\eta-1,0}(i\lambda_1 b)$$
$$+(1-i\eta)(i\lambda_1 b)^{-2i\eta+1}\mathfrak{L}_{2i\eta-2,1}(i\lambda_1 b)](i\lambda_2 b)^{-2i\eta}\mathfrak{L}_{2i\eta-1,0}(i\lambda_2 b)\,. \tag{9}$$

For excitation to the $m_L=\pm1$ states, one sees that by introducing the cylindrical coordinates for $\vec{r}_1$ and $\vec{r}_2$, $\vec{F}(\vec{q})$ vanishes from Eq. (1) since the integrand under the integral is an odd function of z.

Substituting Eqs. (3), (4), and $Y_{2,\pm2}$ into (1), we obtain the transition amplitudes to $m_L=\pm2$ states, $F^{(\zeta)}_{3\,{}^1D,\,1\,{}^1S}(\vec{q}, m_L=\pm2)$, in terms of the third generating function $I_s(\lambda_1,\lambda_2;q)$,

$$F^{(\zeta)}_{3\,{}^1D,\,1\,{}^1S}(\vec{q}, m_L=\pm2) = e^{\mp i2\phi_q} iK_i\frac{1.6966}{81}\sum_{n=1}^{4}c(n)\left(\frac{\partial^2 I_s(\lambda_1,\lambda_2;q)}{\partial\lambda_1\partial\lambda_2}\right)\Bigg|_{\substack{\lambda_1=\lambda_{31}(n)\\ \lambda_2=\lambda_{32}(n)}} = e^{\mp i2\phi_q}h_2(q)\,, \tag{10}$$

where ϕ_q is the azimuthal angle of $\vec{q}$ in $C^g(\xi)$. In Eq. (10), $I_s(\lambda_1,\lambda_2;q)$ is defined by

$$I_s(\lambda_1, \lambda_2; q) = e^{\pm i2\phi_q}\frac{1}{\pi^3}\int\frac{e^{-\lambda_1 r_1}}{r_1}\frac{e^{-\lambda_2 r_2}}{r_2}s_1^2\left[1-\left(\frac{|\vec{b}-\vec{s}_1|}{b}\right)^{2i\eta}\left(\frac{|\vec{b}-\vec{s}_2|}{b}\right)^{2i\eta}\right]e^{i\vec{q}\cdot\vec{b}}e^{\mp i2\varphi_1}d^2b\,d\vec{r}_1\,d\vec{r}_2\,. \tag{11}$$

The detailed reduction of Eq. (11) is again given in Appendix B, where $I_s(\lambda_1,\lambda_2;q)$ is given by Eq. (B11),

$$I_s(\lambda_1, \lambda_2; q) = 2^5\lambda_1^{-2}\lambda_2^{-2}(2i\eta)^2[\Gamma(i\eta)\Gamma(1-i\eta)](1-i\eta)(2-i\eta)q^{2i\eta-4}\lambda_1^{-2i\eta}$$
$$\times[2\,{}_2F_1(3-i\eta, 1-i\eta; 1; -\lambda_1^2q^{-2}) - 4(1+i\eta)\,{}_2F_1(3-i\eta, 1-i\eta; 2; -\lambda_1^2q^{-2})$$
$$+(1+i\eta)(2+i\eta)\,{}_2F_1(3-i\eta, 1-i\eta; 3; -\lambda_1^2q^{-2})]$$
$$+2^5\lambda_1^{-2}\lambda_2^{-2}(2i\eta)^3\int_0^\infty b^3\,db\,J_2(qb)[2i\eta(i\lambda_1 b)^{-2i\eta}\mathfrak{L}_{2i\eta-1,0}(i\lambda_1 b) - 4(1+i\eta)(i\lambda_1 b)^{-2i\eta-1}\mathfrak{L}_{2i\eta,1}(i\lambda_1 b)$$
$$+2(2+i\eta)(i\lambda_1 b)^{-2i\eta-2}\mathfrak{L}_{2i\eta+1,2}(i\lambda_1 b)](i\lambda_2 b)^{-2i\eta}\mathfrak{L}_{2i\eta-1,0}(i\lambda_2 b)\,. \tag{12}$$

B. $1\,{}^1S$-$4\,{}^1D$ excitation

Substituting Eqs. (3), (5), and Y_{2m} into Eq. (1), we obtain the corresponding transition amplitudes for the $1\,{}^1S$-$4\,{}^1D$ excitation,

$$F^{(\zeta)}_{4\,{}^1D,\,1\,{}^1S}(\vec{q}, m_L=0) = iK_i\frac{1.6966}{4\times32}\sum_{n=1}^{4}c(n)\left[\left(1+\frac{1}{12}\frac{\partial}{\partial\lambda_1}\right)\left(3\frac{\partial^2 I_z}{\partial\lambda_1\partial\lambda_2} - \frac{\partial^4 I_0}{\partial\lambda_1^3\partial\lambda_2}\right)\right]\Bigg|_{\substack{\lambda_1=\lambda_{41}(n)\\ \lambda_2=\lambda_{42}(n)}}\,, \tag{13}$$

$$F^{(\zeta)}_{4\,{}^1D,\,1\,{}^1S}(\vec{q}, m_L=\pm1) = 0\,, \tag{14}$$

and

$$F^{(\zeta)}_{4\,{}^1D,\,1\,{}^1S}(\vec{q}, m_L=\pm2) = e^{\mp i2\phi_q} iK_i\frac{1.6966}{4\times32}\sqrt{\tfrac{3}{2}}\sum_{n=1}^{4}c(n)\left[\left(1+\frac{1}{12}\frac{\partial}{\partial\lambda_1}\right)\frac{\partial^2 I_s}{\partial\lambda_1\partial\lambda_2}\right]\Bigg|_{\substack{\lambda_1=\lambda_{41}(n)\\ \lambda_2=\lambda_{42}(n)}}\,, \tag{15}$$

with

$$\lambda_{41}(n) = 1.66,\ 2.86,\ 1.66,\ 2.86\,,$$
$$\lambda_{42}(n) = 3.41,\ 3.41,\ 4.61,\ 4.61\,.$$

In Eqs. (7), (9), and (12), Γ, J, ${}_2F_1$, and $\mathfrak{L}_{\mu,\nu}$ are the usual gamma, Bessel, hypergeometric, and modified Lommel function,[17] respectively. From Eq. (A10) of Ref. 17,

$$\frac{d}{dx}\mathfrak{L}_{\mu,\nu}(ix) = i(\mu+\nu-1)\mathfrak{L}_{\mu-1,\nu-1}(ix) - \frac{\nu}{x}\mathfrak{L}_{\mu,\nu}(ix)\,, \tag{16}$$

and[23]

$$\frac{d}{dx}\,{}_2F_1(a,b,c;x) = \frac{ab}{c}\,{}_2F_1(a+1,\ b+1;\ c+1; x)\,, \tag{17}$$

one can easily obtain the expressions of various derivatives of I_0, I_z, and I_s and hence the Glauber amplitudes $F^{(\zeta)}_{3\,^1D,1\,^1S}(\vec{q}, m_L)$ and $F^{(\zeta)}_{4\,^1D,1\,^1S}(\vec{q}, m_L)$. We would like to point out that we need to calculate explicitly only two of the modified Lommel functions via Eq. (A6) of Ref. 17 [for example, $(ix)^{-2i\eta}\times\mathfrak{L}_{2i\eta-1,0}(ix)$ and $(ix)^{-2i\eta+1}\mathfrak{L}_{2i\eta-2,1}(ix)$]; all others in Eqs. (6), (10), (13), and (15) may be obtained from these two using the recurrence relations.[17] The procedures for numerical computation of Eqs. (6), (10), (13), and (15) are the same as those of $I_0(\lambda_1, \lambda_2; q)$, and are described in detail in Ref. 17.

III. POLARIZATION OF THE 6678-Å HELIUM LINE

For the 6678-Å line ($3\,^1D$-$2\,^1P$) emitted by the helium atom following electron excitation to the $3\,^1D$ state, the polarization fraction

$$P=\frac{I_\parallel - I_\perp}{I_\parallel + I_\perp}, \tag{18}$$

according to the theory of Percival and Seaton,[24] is given by

$$P(E_i)=\frac{3(Q_0+Q_1-2Q_2)}{5Q_0+9Q_1+6Q_2}. \tag{19}$$

In Eq. (18), $I_\parallel$ and $I_\perp$ are the intensities, observed at 90° to the incident-electron-beam direction, of the respective 6678-Å line having electric vectors parallel and perpendicular to the incident-electron-beam direction. In Eq. (19), E_i is the incident-electron energy, the quantities Q_m, $m=0$, 1, and 2, are the total cross sections for exciting the helium atom from the ground state to the $3D_m$ sublevels. It was pointed out by Gerjuoy, Thomas, and Sheorey (GTS)[14] that the total cross sections Q_m, which appear in Eq. (19) for $P(E_i)$, are computed from the Glauber scattering amplitudes $F^{(i)}_{3\,^1D,1\,^1S}(\vec{q}, m_L)$, quantized along the direction $\vec{K}_i$ of the incident electron [which we denote by the superscript (i)]. The connection between these two sets of Glauber amplitudes, according to the theory of GTS, is found by the following transformation:

$$F^{(i)}_{3\,^1D,1\,^1S}(\vec{q}, m_L)=\sum_{m_L'} D^{(2)}_{m_L m_L'}(\alpha,\beta,\gamma)\times F^{(\zeta)}_{3\,^1D,1\,^1S}(\vec{q}, m_L'). \tag{20}$$

In Eq. (20), $D^{(2)}_{m_L m_L'}$ is the usual representation of R_g [$R_g\equiv R_g(\alpha,\beta,\gamma)$] on the space spanned by eigenvectors of L^2 with angular momentum quantum number $L=2$. The representation $D^{(2)}_{m'm}(\alpha,\beta,\gamma)$ is related to the matrix $d^{(2)}_{m'm}(\beta)$ by[25]

$$D^{(2)}_{m'm}(\alpha,\beta,\gamma)=e^{im'\gamma}d^{(2)}_{m'm}(\beta)e^{im\alpha}, \tag{21}$$

where the Euler angles[14] $\alpha=\phi_q$, $\beta=(\theta_q-\pi/2)$ and $\gamma=-\phi_q$ [θ_q and ϕ_q are the angular coordinates of $\vec{q}$ in $C(\vec{K}_i)$]. Using Eq. (4.1.15) of Ref. 22, one can easily find the matrix $d^{(2)}_{m'm}(\beta)$. Substituting Eqs. (6), (10), (21), and $d^{(2)}_{m'm}(\beta)$ into (20), we find that the $1\,^1S$-$3\,^1D$ Glauber amplitudes, quantized along $\vec{K}_i$, are

$$\begin{aligned}
F^{(i)}_{3\,^1D,1\,^1S}(\vec{q}, m_L=0)&=\tfrac{1}{2}(3\sin^2\theta_q-1)h_0+\sqrt{\tfrac{3}{2}}\cos^2\theta_q h_2,\\
F^{(i)}_{3\,^1D,1\,^1S}(\vec{q}, m_L=\pm1)&=e^{\mp i\phi_q}(-\sqrt{\tfrac{3}{2}}\sin\theta_q\cos\theta_q h_0+\sin\theta_q\cos\theta_q h_2),\\
F^{(i)}_{3\,^1D,1\,^1S}(\vec{q}, m_L=\pm2)&=e^{\mp i2\phi_q}[\tfrac{1}{2}\sqrt{\tfrac{3}{2}}\cos^2\theta_q h_0+\tfrac{1}{2}(1+\sin^2\theta_q)h_2],
\end{aligned} \tag{22}$$

where h_0 and h_2 are defined in Eqs. (6) and (10). From Eq. (22), one immediately sees

$$\begin{aligned}
\left(\frac{d\sigma}{d\Omega}\right)^{(i)}_{1\,^1S-3\,^1D}&=|F^{(i)}_{3\,^1D,1\,^1S}(\vec{q}, m_L=0)|^2+2|F^{(i)}_{3\,^1D,1\,^1S}(\vec{q}, m_L=1)|^2+2|F^{(i)}_{3\,^1D,1\,^1S}(\vec{q}, m_L=2)|^2\\
&=|h_0|^2+2|h_2|^2\\
&=|F^{(\zeta)}_{3\,^1D,1\,^1S}(\vec{q}, m_L=0)|^2+2|F^{(\zeta)}_{3\,^1D,1\,^1S}(\vec{q}, m_L=2)|^2\\
&=\left(\frac{d\sigma}{d\Omega}\right)^{(\zeta)}_{1\,^1S-3\,^1D}.
\end{aligned} \tag{23}$$

Equation (23) indicates that the Glauber cross sections are independent of whether the quantization axis is chosen along $\vec{K}_i$ or along an axis $\perp\vec{q}$.[14] The cross sections Q_{m_L} are constructed from the corresponding amplitudes $F^{(i)}_{3\,^1D,1\,^1S}(\vec{q}, m_L)$ in the usual way, and hence the polarization fraction is easily found.

IV. RESULTS AND DISCUSSION

We have calculated the differential cross sections $d\sigma/d\Omega$ for excitation to $3\,^1D$ and $4\,^1D$ states by means of Eqs. (6), (10), (13), and (15) and the expressions for various derivatives of the generating functions for various incident-electron energies, as a function of scattering angles. The differential cross sections for the $3\,^1D$ and $4\,^1D$ excitation are shown in Figs. 1 and 2, respectively, and need not be discussed in detail. For comparison, we also present in Figs. 1 and 2 the results from the Born approximation[5,6] and from the Woollings-McDowell approximation.[8]

We have also integrated the differential cross sections and therefore obtained the corresponding total cross sections as a function of the incident-electron energy. The total-cross-section results for $3\,^1D$ excitation are shown in Fig. 3 (see also Table I), where they are compared with the results obtained from the other theoretical models, and experimental results of St. John *et al.*[1] and Moustafa Moussa *et al.*[2] We note from Fig. 3 that the Glauber predictions lie between the Born and Ochkur data; for energies greater than 150 eV, the Glauber values are very close to those from the Born approximation. But all the theoretical predictions lie below experimental data even at energies greater than 200 eV (for example, $\sigma_G < \sigma_{\rm expt}$ by ~36% at 1000 eV), where we expect all these high-energy approximations should be valid. The total cross sections for $4\,^1D$ excitation are shown in Fig. 4 and the patterns follow that of the $3\,^1D$ excitation; for energies greater than 150 eV, the Glauber values become slightly larger than

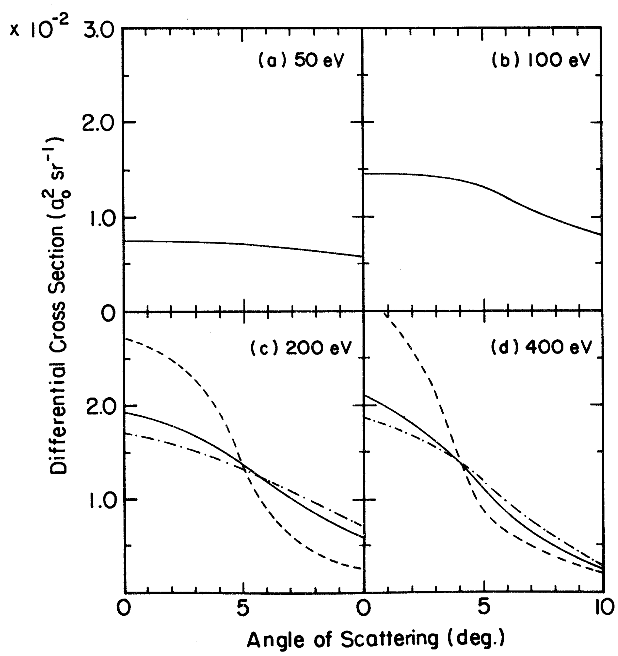

FIG. 1. Differential cross sections for $3\,^1D$ excitation of helium by electrons at (a) 50, (b) 100, (c) 200, and (d) 400 eV. Solid curve, Glauber approximation; dot-dashed curve, Born approximation; dashed curve, Woollings-McDowell approximation.

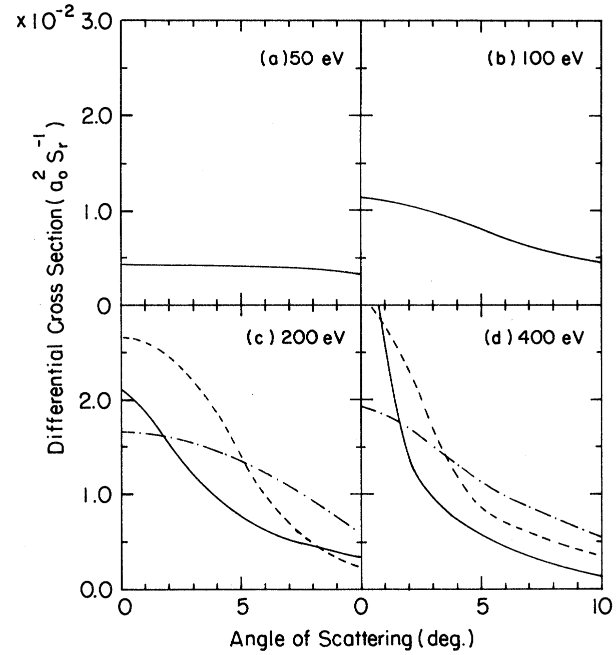

FIG. 2. Differential cross sections for $4\,^1D$ excitation of helium by electrons at (a) 50, (b) 100, (c) 200, and (d) 400 eV. Solid curve, Glauber approximation; dot-dashed curve, Born approximation; dashed curve, Woollings-McDowell approximation.

those obtained from the other models.

We have calculated the polarization fraction $P(E_i)$ at incident energies from 50 to 1000 eV for the 6678-Å helium line via Eqs. (19) and (22). The results are shown in Fig. 5. Although the shape of the Glauber curve resembles the experimental values of Moustafa Moussa *et al.*, we see that the agreement with experiment is also poor.

The poor agreement between the theory and the experimental data is rather disappointing. However, the experimental situation is far from clear. As pointed out by Moustafa Moussa *et al.*,[2] the strong rise of the $\sigma_T E_i$ values of Ref. 1 for $3\,^1D$ and $4\,^1D$ excitation above 200 eV is in contradiction with the Bethe equation. The measurements of van Raan *et al.*[3] (not shown in Fig. 4) on the $4\,^1D$ excitation lie below the Ref. 2 values at $E_i \geq 200$ eV. Therefore, the energy dependences of the total cross sections do not agree among themselves. Furthermore, when the proton-impact data for $4\,^1D$ excitation of helium is scaled to the same electron velocity, one finds that all the theoretical results lie between the experimental values of Thomas and Bent[26] and Hasselkamp *et al.*,[27] which differ among themselves by 60% at 200 eV. On the theoretical side, Eq. (3) used in this paper for the helium ground state is probably reliable (we have calculated the $3\,^1D$ excitation total cross section of helium by electron impact at 1000 eV using the

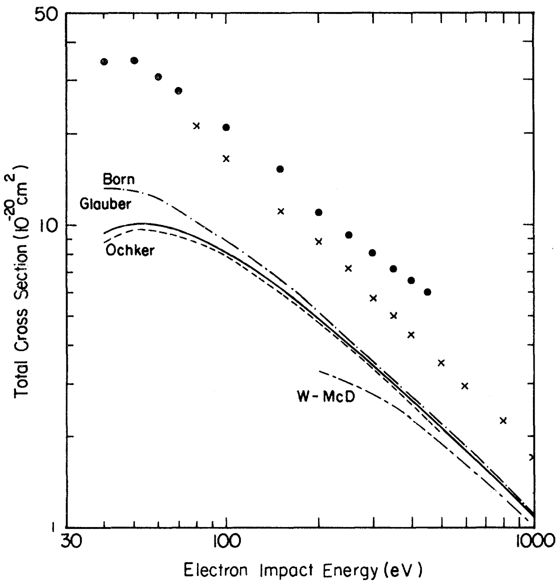

FIG. 3. Total cross sections for $1\,^1S$-$3\,^1D$ excitation of helium by electron impact. Solid curve, this work (GA); dot-dashed curve, Born approximation; dashed curve, Ochkur approximation; solid-dashed, Woollings-McDowell approximation; dots, St. John *et al.*; crosses, Moustafa Moussa *et al.*

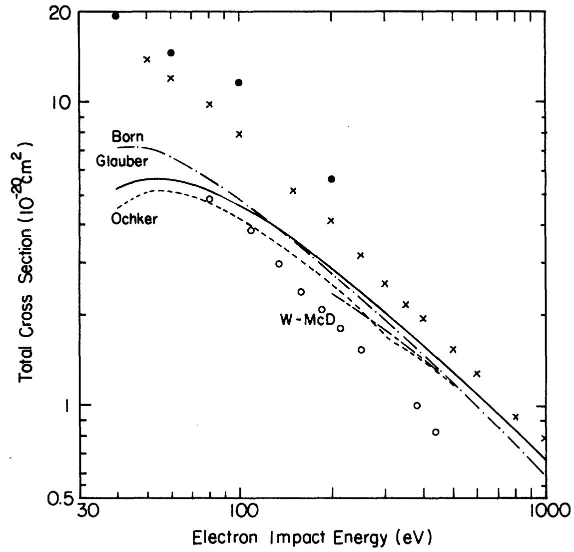

FIG. 4. Total cross sections for $1\,^1S$-$4\,^1D$ excitation of helium by electron impact. Solid curve, this work (GA); dot-dashed curve, Born approximation; dashed curve, Ochkur approximation; solid-dashed curve, Woollings-McDowell approximation; dots, St. John *et al.*; crosses, Moustafa Moussa *et al.*; closed circles, Thomas and Bent (Ref. 26).

Hartree-Fock wave function of Löwdin,[28] the result is about 7% smaller than the value in Table I); but the simple Heisenberg choice, i.e., Eqs. (4) and (5), probably does not adequately represent the shape of the excited He electron, and a better wave function such as a Hartree-Fock wave function or a variationally determined wave function may improve the theoretical results. However, as shown by Byron and Joachain[29] that the GA gives only two

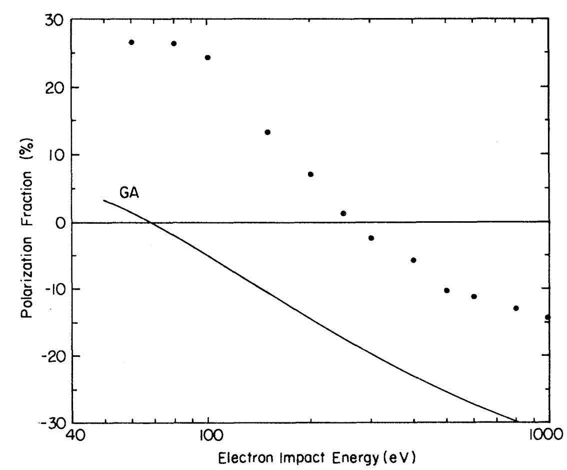

FIG. 5. Polarization fraction of the 6678-Å helium line excited by electron impact. Solid curve, this work (GA); dots, Moustafa Moussa *et al.*

of the four relevant correction terms to the Born approximation, the poor agreement between the theory (GA, BA,...) and the experimental finding may not be surprising. We therefore conclude that the agreement between experiment and theory in this area is less than satisfactory and further experimental and theoretical investigation seems desirable.

After completion of this paper, related works of Bransden and Issa,[30] and Flannery[31] have been brought to our attention.

APPENDIX A: REDUCTION OF THE GENERATING FUNCTION $I_z(\lambda_1,\lambda_2;q)$

The generating function $I_z(\lambda_1, \lambda_2; q)$ is defined by Eq. (8),

$$I_z(\lambda_1,\lambda_2;q) \equiv \frac{1}{\pi^3}\int \frac{e^{-\lambda_1 r_1}}{r_1}\frac{e^{-\lambda_2 r_2}}{r_2} z_1^2 \times\left[1-\left(\frac{|\vec{b}-\vec{s}_1|}{b}\right)^{2i\eta}\left(\frac{|\vec{b}-\vec{s}_2|}{b}\right)^{2i\eta}\right] \times e^{i\vec{q}\cdot\vec{b}}\, d^2b\, d\vec{r}_1\, d\vec{r}_2 . \qquad \text{(A1)}$$

We shall show that the first term (independent of η) under the integral in Eq. (A1), which leads to $\delta(\vec{q})$, is exactly canceled by a similar factor stemming from the second term (dependent on η) in the amplitude integral. Therefore, the $\delta(\vec{q})$ is explicitly removed. By introducing cylindrical coordinates for $\vec{r}_1$ and $\vec{r}_2$ and employing the standard formulas[32]

TABLE I. Total cross sections (10^{-20} cm^2) for $3\,^1D$ and $4\,^1D$ excitation of helium by electron impact (eV).

Excited state	Energy (eV) 40	50	80	100	200	300	400	500	600	800	1000
$3\,^1D$	9.325	10.006	9.045	8.046	4.890	3.450	2.657	2.157	1.814	1.376	1.107
$4\,^1D$	5.253	5.705	5.235	4.691	2.893	2.060	1.596	1.302	1.099	0.837	0.075

for K_ν and J_ν,

$$\int_{-\infty}^{+\infty} dz\, \frac{e^{-\lambda(s^2+z^2)^{1/2}}}{(s^2+z^2)^{1/2}} = 2K_0(\lambda s)\,, \tag{A2}$$

$$\int_{-\infty}^{+\infty} z^2\, dz\, \frac{e^{-\lambda(s^2+z^2)^{1/2}}}{(s^2+z^2)^{1/2}} = \frac{2s}{\lambda} K_1(\lambda s)\,, \tag{A3}$$

and

$$\int_0^{2\pi} d\varphi\, e^{iqb\cos\varphi} = 2\pi J_0(qb)\,, \tag{A4}$$

and then changing variables ($s_1 \to s_1 b$, $s_2 \to s_2 b$), we find that $I_z(\lambda_1, \lambda_2; q)$ can be written as

$$\begin{aligned} I_z(\lambda_1,\lambda_2;q) = \frac{2}{\pi^2}\int_0^\infty b\,db\,J_0(qb)\bigg((2\pi)^2\frac{2}{\lambda_1}\int_0^\infty (bs_1)^2K_1(\lambda_1 b s_1)d(bs_1)\,2\int_0^\infty (bs_2)K_0(\lambda_2 b s_2)d(bs_2) \\ -\frac{2}{\lambda_1}\int_0^\infty (bs_1)^2K_1(\lambda_1 b s_1)d(bs_1)\int_0^{2\pi} d\varphi_1\,(1+s_1^2-2s_1\cos\varphi_1)^{i\eta} \\ \times 2\int_0^\infty (bs_2)K_0(\lambda_2 b s_2)d(bs_2)\int_0^{2\pi} d\varphi_2(1+s_2^2-2s_2\cos\varphi_2)^{i\eta}\bigg)\,. \end{aligned} \tag{A5}$$

We now utilize the result[33] that

$$\int_0^\infty sK_0(\lambda b s)\,ds = (\lambda b)^{-2} \tag{A6}$$

and

$$\int_0^\infty s^2K_1(\lambda b s)\,ds = 2(\lambda b)^{-3}\,. \tag{A7}$$

We obtain

$$I_z(\lambda_1,\lambda_2;q) = 2^5\int_0^\infty b^7\,db\,J_0(qb)[2(\lambda_1 b)^{-4}(\lambda_2 b)^{-2} - M_1(\lambda_2 b)M_2(\lambda_1 b)]\,, \tag{A8}$$

where

$$M_1(x) \equiv \int_0^\infty s\,ds\,K_0(xs)\,\frac{1}{2\pi}\int_0^{2\pi} d\varphi\,(1+s^2-2s\cos\varphi)^{i\eta} \tag{A9}$$

and

$$M_2(x) \equiv \frac{1}{x}\int_0^\infty s^2\,ds\,K_1(xs) \times \frac{1}{2\pi}\int_0^{2\pi} d\varphi\,(1+s^2-2s\cos\varphi)^{i\eta}\,. \tag{A10}$$

Equation (A9) for $M_1(x)$ was derived by Thomas and Chan[17] [Eq. (14b) of Ref. 17] and was given in terms of the modified Lommel functions,[17]

$$M_1(x) = x^{-2}[1-(2i\eta)^2(ix)^{-2i\eta}\mathfrak{L}_{2i\eta-1,\,0}(ix)]\,. \tag{A11}$$

In deriving Eq. (A10) for $M_2(x)$, we introduce the integral representation of Thomas and Gerjuoy[13] [Eq. (A6) of Ref. 13] to replace the integral over φ in Eq. (A10) by an equivalent integral involving Bessel functions; namely,

$$M_2(x) = -2^{2i\eta}\frac{\Gamma(1+i\eta)}{\Gamma(1-i\eta)} \times\frac{1}{x}\int_0^\infty s^2\,ds\,K_1(xs)\int_0^\infty dt\,t^{-2i\eta}\frac{d}{dt}[J_0(t)J_0(st)]\,. \tag{A12}$$

The integral over s is simply[34]

$$\int_0^\infty s^2J_0(ts)K_1(xs)\,ds = 2x/(t^2+x^2)^2\,. \tag{A13}$$

Hence,

$$M_2(x) = -2^{2i\eta}\frac{\Gamma(1+i\eta)}{\Gamma(1-i\eta)}2\int_0^\infty dt\,t^{-2i\eta}\frac{\partial}{\partial t}\left(\frac{J_0(t)}{(t^2+x^2)^2}\right)\,. \tag{A14}$$

Since

$$\frac{\partial}{\partial t}\left(\frac{2J_0(t)}{(t^2+x^2)^2}\right) = \frac{1}{x}\frac{\partial}{\partial x}\frac{J_1(t)}{t^2+x^2} + \frac{1}{x^3}\frac{\partial}{\partial x}\frac{tJ_0(t)}{t^2+x^2} - \frac{1}{x^2}\frac{\partial^2}{\partial x^2}\frac{tJ_0(t)}{t^2+x^2}\,, \tag{A15}$$

one has

$$M_2(x) = -2^{2i\eta}\frac{\Gamma(1+i\eta)}{\Gamma(1-i\eta)}\left[\frac{1}{x}\frac{\partial}{\partial x}\int_0^\infty dt\, t^{-2i\eta}\frac{J_1(t)}{t^2+x^2} + \left(\frac{1}{x^3}\frac{\partial}{\partial x} - \frac{1}{x^2}\frac{\partial^2}{\partial x^2}\right)\int_0^\infty dt\, t^{-2i\eta+1}\frac{J_0(t)}{t^2+x^2}\right] . \tag{A16}$$

By using Eqs. (A7), (A8), and (A10)-(A12) in Ref. 17, we find

$$M_2(x) = 2x^{-4}[1-(2i\eta)^2((1+i\eta)(ix)^{-2i\eta}\mathfrak{L}_{2i\eta-1,0}(ix) + (1-i\eta)(ix)^{-2i\eta+1}\mathfrak{L}_{2i\eta-2,1}(ix))] . \tag{A17}$$

Substituting Eqs. (A11) and (A17) into Eq. (A8), one gets

$$\begin{aligned} I_z(\lambda_1,\lambda_2;q) = 2^6(2i\eta)^2\lambda_1^{-4}\lambda_2^{-2}\Big[&(1+i\eta)\int_0^\infty b\,db J_0(qb)(i\lambda_1 b)^{-2i\eta}\mathfrak{L}_{2i\eta-1,0}(i\lambda_1 b) \\ &+(1-i\eta)\int_0^\infty b\,db\,J_0(qb)(i\lambda_1 b)^{-2i\eta+1}\mathfrak{L}_{2i\eta-2,1}(i\lambda_1 b) \\ &+\int_0^\infty b\,db\,J_0(qb)(i\lambda_2 b)^{-2i\eta}\mathfrak{L}_{2i\eta-1,0}(i\lambda_2 b)\Big] \\ -2^6(2i\eta)^4\lambda_1^{-4}\lambda_2^{-2}&\int_0^\infty b\,db\,J_0(qb)\,[(1+i\eta)(i\lambda_1 b)^{-2i\eta}\mathfrak{L}_{2i\eta-1,0}(i\lambda_1 b) \\ +(1-i\eta)(i\lambda_1 b)^{-2i\eta+1}&\mathfrak{L}_{2i\eta-2,1}(i\lambda_1 b)]\cdot(i\lambda_2 b)^{-2i\eta}\mathfrak{L}_{2i\eta-1,0}(i\lambda_2 b) . \end{aligned} \tag{A18}$$

With help from Eq. (A7) of Ref. 13 and Eqs. (B3) and (B5) of Ref. 17, the integrals in Eq. (A18) which involve only one modified Lommel function $\mathfrak{L}_{\mu,\nu}$ may be evaluated in close form. We therefore obtain

$$\begin{aligned} I_z(\lambda_1,\lambda_2;q) = -2^5\lambda_1^{-4}\lambda_2^{-2}(2i\eta)^2\Gamma(i\eta)\Gamma(1-i\eta)[&(1+i\eta)q^{2i\eta-2}\lambda_1^{-2i\eta}\,{}_2F_1(1-i\eta,1-i\eta;1;-\lambda_1^2/q^2) \\ &+(1-i\eta)^2q^{2i\eta-4}\lambda_1^{-2i\eta+2}\,{}_2F_1(2-i\eta,2-i\eta;2;-\lambda_1^2/q^2) \\ &+q^{2i\eta-2}\lambda_2^{-2i\eta}\,{}_2F_1(1-i\eta,1-i\eta;1;-\lambda_2^2/q^2)] \\ -2^6\lambda_1^{-4}\lambda_2^{-2}(2i\eta)^4\int_0^\infty b\,db\,J_0(qb)[&(1+i\eta)(i\lambda_1 b)^{-2i\eta}\mathfrak{L}_{2i\eta-1,0}(i\lambda_1 b)+(1-i\eta)(i\lambda_1 b)^{-2i\eta+1}\mathfrak{L}_{2i\eta-2,1}(i\lambda_1 b)] \\ &\times(i\lambda_2 b)^{-2i\eta}\mathfrak{L}_{2i\eta-1,0}(i\lambda_2 b) . \end{aligned} \tag{A19}$$

APPENDIX B: REDUCTION OF THE GENERATING FUNCTION $I_s(\lambda_1,\lambda_2;q)$

In cylindrical coordinates, the generating function $I_s(\lambda_1,\lambda_2;q)$ [see Eq. (11)] can be written as

$$\begin{aligned} I_s(\lambda_1,\lambda_2;q) \equiv \frac{1}{\pi^3}\int_0^\infty b\,db\int_0^{2\pi}d\varphi_b\,e^{\mp i2(\varphi_b-\varphi_q)+i\vec{q}\cdot\vec{b}}\int_0^{2\pi}d\varphi_1\,e^{\mp i2(\varphi_1-\varphi_b)}\int_0^{2\pi}d\varphi_2\int_0^\infty s_1^3\,ds_1\int_0^\infty s_2\,ds_2 \\ \times\int_{-\infty}^{+\infty}dz_1\frac{e^{-\lambda_1(s_1^2+z_1^2)^{1/2}}}{(s_1^2+z_1^2)^{1/2}}\int_{-\infty}^{+\infty}dz_2\frac{e^{-\lambda_2(s_2^2+z_2^2)^{1/2}}}{(s_2^2+z_2^2)^{1/2}} \\ \times\left[1-\left(\frac{b^2+s_1^2-2bs_1\cos(\varphi_1-\varphi_b)}{b^2}\right)^{i\eta}\left(\frac{b^2+s_2^2-2bs_2\cos(\varphi_2-\varphi_b)}{b^2}\right)^{i\eta}\right] . \end{aligned} \tag{B1}$$

We note that the first term (independent of η) gives zero contribution. Following the procedure in Appendix A and employing[32]

$$\int_0^{2\pi}d\varphi\,e^{\mp i2\varphi+iqb\cos\varphi} = -2\pi J_2(qb) , \tag{B2}$$

we find that $I_s(\lambda_1,\lambda_2;q)$ can be written as

$$I_s(\lambda_1,\lambda_2;q) = 2^5\int_0^\infty b^7\,db\,J_2(qb)M_1(\lambda_2 b)M_3(\lambda_1 b) , \tag{B3}$$

where

$$M_3(x)=\int_0^\infty s^3\,ds\,K_0(xs)\frac{1}{2\pi}\int_0^{2\pi}d\varphi\,e^{\mp i2\varphi}(1+s^2-2s\cos\varphi)^{i\eta}\,. \tag{B4}$$

Again, by introducing the integral representation of Thomas and Gerjuoy, we have

$$M_3(x)=-2^{2i\eta}\frac{\Gamma(1+i\eta)}{\Gamma(1-i\eta)}\int_0^\infty s^3\,ds\,K_0(xs)\int_0^\infty dt\,t^{-2i\eta}\frac{\partial}{\partial t}[J_2(t)J_2(st)]\,. \tag{B5}$$

The integral over s may be done immediately via[34]

$$\int_0^\infty s^3 J_2(ts)K_0(xs)\,ds=8t^2(t^2+x^2)^{-3}\,. \tag{B6}$$

Since

$$\frac{\partial}{\partial t}\left(\frac{8t^2J_2(t)}{(t^2+x^2)^3}\right)=-\left(\frac{3}{x}\frac{\partial}{\partial x}+\frac{\partial^2}{\partial x^2}\right)\frac{J_1(t)}{t^2+x^2}+\left(\frac{3}{x^5}\frac{\partial}{\partial x}-\frac{3}{x^4}\frac{\partial^2}{\partial x^2}+\frac{1}{x^3}\frac{\partial^3}{\partial x^3}\right)\frac{t^3J_2(t)}{t^2+x^2}\,, \tag{B7}$$

one has

$$M_3(x)=-2^{2i\eta}\frac{\Gamma(1+i\eta)}{\Gamma(1-i\eta)}\left[-\left(\frac{3}{x}\frac{\partial}{\partial x}+\frac{\partial^2}{\partial x^2}\right)\int_0^\infty dt\,t^{-2i\eta}\frac{J_1(t)}{t^2+x^2}+\left(\frac{3}{x^5}\frac{\partial}{\partial x}-\frac{3}{x^4}\frac{\partial^2}{\partial x^2}+\frac{1}{x^3}\frac{\partial^3}{\partial x^3}\right)\int_0^\infty dt\,t^{-(2i\eta-3)}\frac{J_2(t)}{t^2+x^2}\right]. \tag{B8}$$

By using Eqs. (A7), (A8), (A10)–(A12) in Ref. 17, we obtain

$$\begin{aligned}M_3(x)=&-2(2i\eta)^2(ix)^{-2i\eta-2}\mathcal{L}_{2i\eta-1,0}(ix)+4i\eta(1+i\eta)(ix)^{-2i\eta-3}\mathcal{L}_{2i\eta,1}(ix)-4i\eta(1-i\eta)(ix)^{-2i\eta-1}\mathcal{L}_{2i\eta-2,1}(ix)\\&+8i\eta(1-i\eta)(2-i\eta)[24i\eta(1-i\eta)(2-i\eta)(ix)^{-2i\eta-2}\mathcal{L}_{2i\eta-5,0}(ix)+24i\eta(1-i\eta)(1+i\eta)(ix)^{-2i\eta-3}\mathcal{L}_{2i\eta-4,1}(ix)\\&+8(1-i\eta)(2-i\eta)(3-i\eta)(ix)^{-2i\eta-1}\mathcal{L}_{2i\eta-6,1}(ix)\\&+8i\eta(1+i\eta)(2+i\eta)(ix)^{-2i\eta-4}\mathcal{L}_{2i\eta-3,2}(ix)]\,.\end{aligned} \tag{B9}$$

By applying the recurrence relation for the Lommel function $\mathcal{L}_{\mu,\nu}$ [i.e., Eq. (A11) in Ref. 17], Eq. (B9) can be further simplified,

$$M_3(x)=(2i\eta)^2(ix)^{-2i\eta-2}\mathcal{L}_{2i\eta-1,0}(ix)-8i\eta(1+i\eta)(ix)^{-2i\eta-3}\mathcal{L}_{2i\eta,1}(ix)+4i\eta(2+i\eta)(ix)^{-2i\eta-4}\mathcal{L}_{2i\eta+1,2}(ix)\,. \tag{B10}$$

Substituting Eqs. (A11) and (B10) into Eq. (B3) and carrying out the integrals involving only one modified Lommel function $\mathcal{L}_{\mu,\nu}$ we finally obtain

$$\begin{aligned}I_s(\lambda_1,\lambda_2;q)=&\,2^5(\lambda_1\lambda_2)^{-2}(2i\eta)^2[\Gamma(i\eta)\Gamma(1-i\eta)](1-i\eta)(2-i\eta)q^{2i\eta-4}\lambda_1^{-2i\eta}\\&\times[2\,{}_2F_1(3-i\eta,1-i\eta;1;-\lambda_1^2q^{-2})-4(1+i\eta)\,{}_2F_1(3-i\eta,1-i\eta;2;-\lambda_1^2q^{-2})\\&+(1+i\eta)(2+i\eta)\,{}_2F_1(3-i\eta,1-i\eta;3;-\lambda_1^2q^{-2})]\\&+2^5(\lambda_1\lambda_2)^{-2}(2i\eta)^3\int_0^\infty b^3\,db\,J_2(qb)[2i\eta(i\lambda_1b)^{-2i\eta}\mathcal{L}_{2i\eta-1,0}(i\lambda_1b)-4(1+i\eta)(i\lambda_1b)^{-2i\eta-1}\mathcal{L}_{2i\eta,1}(i\lambda_1b)\\&+2(2+i\eta)(i\lambda_1b)^{-2i\eta-2}\mathcal{L}_{2i\eta+1,2}(i\lambda_1b)](i\lambda_2b)^{-2i\eta}\mathcal{L}_{2i\eta-1,0}(i\lambda_2b)\,.\end{aligned} \tag{B11}$$

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