Classical-quantum correspondence for multilevel systems

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The properties of r-mode harmonic-oscillator coherent states are reviewed. In particular, the $\mathfrak{D}$-algebra differential-operator realization of the creation and annihilation operators on the coherent states and their diagonal projectors is constructed. A homomorphism between the algebra describing r-field modes and the algebra describing r-level systems is exhibited explicitly. This homomorphism allows the projection of the multimode calculus onto the multilevel calculus. In particular, multimode coherent states and projectors can be used as generating functions for multilevel coherent states and projectors. In addition, the multilevel $\mathfrak{D}$ algebra is constructed directly from the multimode $\mathfrak{D}$ algebra under this homomorphism. For illustrative purposes, the $\mathfrak{D}$ algebra for the diagonal coherent-state projectors for two-level atomic systems is presented explicitly in terms of a parametrization in the Bloch angles θ and φ. Two classes of applications are treated: (a) the mapping of atomic-density-operator equations of motion into phase-space equations of motion for the quasiprobability weighting function P; (b) the construction of equations of motion for the diagonal elements Q of the density operator in the coherent states. It is shown that the solution to either equation with the appropriate initial condition gives complete statistical information for the atomic system. It is shown explicitly that the functions P and Q are related by a convolution integral.

I. INTRODUCTION

The mapping of quantum observables into c-number functions is an old problem in quantum mechanics[1] that has recently found a wide range of applications in quantum optics. It is well known, for example, that, in terms of the Glauber[2] coherent-state representation, virtually every density operator $W(a, a^\dagger, t)$ can be mapped into a c-number function $P(\alpha, t)$, and that precise rules of correspondence have been established to construct the c-number equations of motion for $P(\alpha, t)$.[3] This classical-quantum correspondence for Bose-Einstein observables has stimulated a number of very beautiful analyses that have led to the reformulation of the entire laser theory in terms of c-number differential equations.[4]

This classical-quantum correspondence is effected by introducing coherent states

$$|\alpha\rangle = \sum_{m=0}^{\infty} f_m(\alpha)|m\rangle, \qquad f_m(\alpha) = e^{-\alpha^*\alpha/2}\alpha^m/\sqrt{m!} \tag{1.1}$$

as an over-complete basis, and then replacing the action of creation and annihilation operators $(a^\dagger, a, [a, a^\dagger] = I)$ on these states by first-order linear differential operators acting on the function $f_m(\alpha)$:

$$\langle\alpha|\hat{A} = \mathfrak{D}^B(\hat{A})\langle\alpha|, \qquad \mathfrak{D}^B(a) = \frac{\partial}{\partial\alpha^*} + \tfrac{1}{2}\alpha, \quad \mathfrak{D}^B(a^\dagger) = \alpha^*. \tag{1.2}$$

A similar set of operator realizations may be obtained for the action on ket coherent states using the adjoint relation

$$\mathfrak{D}^K(\hat{A}^\dagger) = [\mathfrak{D}^B(\hat{A})]^*. \tag{1.3}$$

Coherent-state projectors $|\alpha\rangle\langle\alpha|$ provide a basis in terms of which most physically reasonable operators may be expanded. $\mathfrak{D}$-operator algebras also exist for projectors:

$$|\alpha\rangle\langle\alpha|\hat{A} = \mathfrak{D}^R(\hat{A})|\alpha\rangle\langle\alpha|, \qquad \mathfrak{D}^R(a) = \frac{\partial}{\partial\alpha^*} + \alpha, \quad \mathfrak{D}^R(a^\dagger) = \alpha^*. \tag{1.4}$$

A similar operator realization is obtained for the left action of the operators $\hat{A}$. The left- and right-operator realizations onto projectors are related by

$$\mathfrak{D}^L(\hat{A}^\dagger) = [\mathfrak{D}^R(\hat{A})]^*. \tag{1.5}$$

The discovery of the coherent atomic states to describe collections of two- and multi-level atomic systems[5] has provided the motivation for more recent work on the classical-quantum correspondence for angular momentum operators.[6] In particular, it has been shown that the quasi-probability-density $P(\Omega, t)$, associated with an arbitrary density operator $W(J^\pm, J_z, t)$, can be used to describe the time evolution of collections of two-level systems.[7] Furthermore, moments of the collective atomic operators, as well as multi-time correlation functions, can be represented in terms of phase-space integrals possessing a close formal similarity to classical averages and correlation functions for stochastic processes, respectively.

The main problems that we wish to pose in this paper are (i) to map multimode Bose operators and multilevel atomic-shift operators into linear differential forms; and (ii) to identify the rule of correspondence that maps the differential operators of a multimode harmonic oscillator algebra into those of the multilevel Lie algebra.

The main results of our analysis are that (a) the multimode coherent states provide a generating function for multilevel coherent atomic states, and multimode projectors provide generating functions for multilevel projectors; and (b) the multimode differential operators map homomorphically onto the multilevel differential operators for states and for projectors.

As a consequence of these results, we provide explicit expressions for the linear differential operators corresponding to the quantum observables of the multilevel algebra and prove that these differential operators also form an algebra ($\mathfrak{D}$ algebra).

This formalism finds a natural application in connection with the dynamical evolution of c-number density functions. Here we summarize the known results that have been derived in Refs. 6 and 7 for the P function, and introduce an alternative c-number description based on the diagonal elements of the density operator in the coherent atomic-state representations. The $\mathfrak{D}$-algebra formalism can also be applied to equilibrium statistical problems. This is discussed in connection with the so-called Bloch equation for the canonical density operator.

II. THE r-DIMENSIONAL HARMONIC OSCILLATOR

A. Multimode Lie algebra

We consider a system with r independent oscillator modes, each described by its creation and annihilation operators $a_j^\dagger, a_j$ $(j=1,2,\ldots,r)$. The $2r+1$ operators $a_j^\dagger, a_j, I$ span a Lie algebra with commutation relations

$$\begin{aligned}&[a_j, a_k^\dagger] = I\delta_{jk},\\ &[a_j, a_k] = 0 = [a_k^\dagger, a_j^\dagger],\\ &[a_j, I] = 0 = [I, a_j^\dagger].\end{aligned} \tag{2.1}$$

B. Multimode coherent states

Since the multimode Lie algebra is the direct sum of single-mode Lie algebras, coherent states for the multimode system are direct products of single-mode coherent states (1.1):

$$\begin{aligned}|\underline{\alpha}\rangle &= \prod_{j=1}^{r} |\alpha_j\rangle\\ &= \prod_{j=1}^{r} \exp(\alpha_j a_j^\dagger - \alpha_j^* a_j)|0\rangle\\ &= \prod \exp(-\alpha_j^*\alpha_j/2)\exp(\alpha_j a_j^\dagger)|0\rangle \qquad (2.2)\\ &= \exp(-\underline{\alpha}^\dagger\underline{\alpha}/2)\sum_{m_1=0}^{\infty}\cdots\sum_{m_r=0}^{\infty}\frac{(\alpha_1)^{m_1}\cdots(\alpha_r)^{m_r}}{(m_1!\cdots m_r!)^{1/2}}\\ &\quad\times|m_1,\ldots,m_r\rangle.\end{aligned} \tag{2.3}$$

Here $\underline{\alpha}$ is a complex r-dimensional vector, and $\underline{\alpha}^\dagger\underline{\alpha} = \sum_{j=1}^{r}\alpha_j^*\alpha_j$.

C. Multimode projectors

The diagonal "projectors" for the multimode system are simply

$$\begin{aligned}|\underline{\alpha}\rangle\langle\underline{\alpha}| &= e^{-\underline{\alpha}^\dagger\underline{\alpha}}\sum_{\underline{m}=0}^{\infty}\cdots\sum_{\underline{n}=0}^{\infty}|\underline{m}\rangle\\ &\quad\times\frac{(\alpha_1)^{m_1}\cdots(\alpha_r)^{m_r}\cdots(\alpha_1^*)^{n_1}\cdots(\alpha_r^*)^{n_r}}{(m_1!\cdots m_r!\,n_1!\cdots n_r!)^{1/2}}\langle\underline{n}|\\ &= \sum_{m_j=0}^{\infty}\cdots\sum_{n_j=0}^{\infty}\prod_{j=1}^{r}e^{-|\alpha_j|^2}|m_j\rangle\frac{(\alpha_j)^{m_j}(\alpha_j^*)^{n_j}}{(m_j!\,n_j!)^{1/2}}\langle n_j|.\end{aligned} \tag{2.4}$$

D. Multimode $\mathfrak{D}$ algebras

Since the multimode coherent states (2.2) are direct products of single-mode coherent states, the multimode $\mathfrak{D}$ algebras are obtained directly from the single-mode $\mathfrak{D}$ algebras:

$$\begin{aligned}&\mathfrak{D}^K(a_j) = \alpha_j = \mathfrak{D}^{B*}(a_j^\dagger),\\ &\mathfrak{D}^K(a_j^\dagger) = \frac{\partial}{\partial\alpha_j} + \tfrac{1}{2}\alpha_j^* = \mathfrak{D}^{B*}(a_j);\end{aligned} \tag{2.5}$$

$$\begin{aligned}&\mathfrak{D}^L(a_j) = \alpha_j = \mathfrak{D}^{R*}(a_j^\dagger),\\ &\mathfrak{D}^L(a_j^\dagger) = \frac{\partial}{\partial\alpha_j} + \alpha_j^* = \mathfrak{D}^{R*}(a_j).\end{aligned} \tag{2.6}$$

In fact, all properties of the single-mode $\mathfrak{D}$ algebras can be transferred to corresponding properties of the multimode $\mathfrak{D}$ algebras. If $\hat{A}, \hat{B}$ are elements in the multimode Lie algebra (2.1), and r, s are arbitrary complex numbers, then

$$\begin{aligned}&\mathfrak{D}^L(r\hat{A} + s\hat{B}) = r\mathfrak{D}^L(\hat{A}) + s\mathfrak{D}^L(\hat{B}), \quad \text{linearity}\\ &\mathfrak{D}^L(\hat{A}\hat{B}) = \mathfrak{D}^L(\hat{B})\mathfrak{D}^L(\hat{A}), \quad \text{antihomomorphism}\\ &\mathfrak{D}^L([\hat{A},\hat{B}]) = [\mathfrak{D}^L(\hat{B}), \mathfrak{D}^L(\hat{A})], \quad \text{antihomomorphism.}\end{aligned} \tag{2.7}$$

An identical set of properties holds for the algebra $\mathfrak{D}^K$. These properties hold even if the operators $\hat{A}, \hat{B}$ are formal polynomial products of linear elements in the multimode Lie algebra.

Furthermore, the relations between the single-mode $\mathfrak{D}$ algebras can be transferred to the multimode system:

$$\mathfrak{D}^{L*}(\hat{A}) = \mathfrak{D}^R(\hat{A}^\dagger), \quad \mathfrak{D}^{K*}(\hat{A}) = \mathfrak{D}^B(\hat{A}^\dagger). \tag{2.8}$$

Relations (2.7) and (2.8) immediately lead to the properties

$$\begin{aligned} &\mathfrak{D}^R(r\hat{A} + s\hat{B}) = r\mathfrak{D}^R(\hat{A}) + s\mathfrak{D}^R(\hat{B}), \quad \text{linearity} \\ &\mathfrak{D}^R(\hat{A}\hat{B}) = \mathfrak{D}^R(\hat{A})\mathfrak{D}^R(\hat{B}), \quad \text{homomorphism} \\ &\mathfrak{D}^R([\hat{A}, \hat{B}]) = [\mathfrak{D}^R(\hat{A}), \mathfrak{D}^R(\hat{B})], \quad \text{homomorphism.} \end{aligned} \tag{2.9}$$

Once again, these properties are valid for the algebra $\mathfrak{D}^B$, and the operators $\hat{A}, \hat{B}$ may be formal polynomial products of the elements in the Lie algebra (2.1).

Finally, from (2.5) and (2.6),

$$[\mathfrak{D}^K(\hat{A}), \mathfrak{D}^B(\hat{B})] = 0, \quad [\mathfrak{D}^L(\hat{A}), \mathfrak{D}^R(\hat{B})] = 0 \tag{2.10}$$

for arbitrary $\hat{A}, \hat{B}$.

III. THE r-LEVEL ATOMIC SYSTEM

A. Multilevel Lie algebra

We now consider a single system with r internal degrees of freedom $|j\rangle$, $(j = 1, 2, \ldots, r)$. The operators E_{kj} effecting transitions from state $|j\rangle$ to state $|k\rangle$ obey the commutation relations

$$[E_{jk}, E_{mn}] = E_{jn}\delta_{km} - E_{mk}\delta_{nj}. \tag{3.1}$$

These are the commutation relations for the Lie algebra $u(r)$ of the group $U(r)$. As a result, the multilevel Lie algebra is isomorphic to $u(r)$.

It is very useful to observe that the Lie algebra $u(r)$ can be realized in terms of bilinear combinations of boson creation and annihilation operators, using the identification

$$E_{jk} = a_j^\dagger a_k. \tag{3.2}$$

As a consequence, we may expect a rather deep and beautiful connection between multimode properties and multilevel properties.

B. Multilevel coherent states

In general, the ensemble of coherent states for a system is obtained by applying the ensemble of allowed unitary transformations to the ground state of the system.[5,8] For a single r-level atom with ground state $|1\rangle = \mathrm{col}(0, \ldots, 0, 1)$, the most general allowed unitary transformation is an $r \times r$ unitary matrix. The action of an arbitrary $r \times r$ unitary transformation on the ground state can always be written

$$\exp\left(\sum_{u=2}^{r} (y_{u1}E_{u1} + y_{1u}E_{1u})\right) \times \exp\left(\sum_{u=2}^{r}\sum_{v=2}^{r} y_{uv}E_{uv} + y_{11}E_{11}\right)|1\rangle. \tag{3.3}$$

Since $E_{jk} = E_{kj}^\dagger$, the y_{jk} must obey the relation $y_{jk}^* = -y_{kj}$ in order for the argument of the exponential to be anti-Hermitian.[9]

The exponentials in (3.3) may be evaluated,[5] resulting in the expression

$$\left[\begin{array}{c|c} [I_{r-1} - \underline{x}\,\underline{x}^\dagger]^{1/2} & \underline{x} \\ \hline -\underline{x}^\dagger & x_1 \end{array}\right] \left[\begin{array}{c|c} U(r-1) & 0 \\ \hline 0 & e^{i\phi} \end{array}\right] \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} x_r \\ \vdots \\ x_2 \\ x_1 \end{bmatrix} e^{i\phi}. \tag{3.4}$$

In this expression $\underline{x}$ is the $(r-1)\times 1$ matrix $\mathrm{col}(x_r, \ldots, x_2)$, and the parameters x_u and $y_u \equiv y_{u1}$ are related by

$$\begin{aligned} &x_u = [y_u \sin(\underline{y}^\dagger \underline{y})^{1/2}]/(\underline{y}^\dagger\underline{y})^{1/2}, \\ &x_1 = (1 - \underline{x}^\dagger \underline{x})^{1/2}, \end{aligned} \tag{3.5}$$

where $\underline{y}^\dagger\underline{y} = \sum y_u^* y_u$. The phase factor $e^{i\phi}$ has been explicitly removed from the column vector on the right. We define the vector[5] $\mathrm{col}(x_r, \ldots, x_2, x_1)$ to be the coherent state for a single r-level system obtained by applying the unitary transformation (3.3) to the ground state of the system.

It is clear that r-level coherent atomic states exist in one-to-one correspondence with the space of coset representatives $U(r)/U(r-1)\otimes U(1) = SU(r)/U(r-1)$. This space may be identified with the $2(r-1)$-dimensional sphere $S^{2(r-1)}$:

$$x_1^2 + \sum_{u=2}^{r} x_u^* x_u = 1. \tag{3.6}$$

A third parametrization for coherent states that is sometimes useful is given by

$$\tau_u = x_u/x_1. \tag{3.7}$$

Each of the three coherent-state parametrizations described above involves $r-1$ complex parameters or $2(r-1)$ independent (real) parameters: x_u, x_u^*; y_u, y_u^*; τ_u, τ_u^*. The coordinatization involving the x_u will be called the group parametrization, since the x_u are matrix elements in a unitary group matrix. The complex numbers y_u will be called algebraic coordinates, since they parametrize an element in the Lie algebra $u(r)$. The parameters τ_u will be called projective coordinates, since they are obtained by stereographic projection from the sphere.[10]

The three parameter spaces are bounded as follows: $\underline{x}^\dagger \underline{x} \leq 1$, $\underline{y}^\dagger \underline{y} \leq (\pi/2)^2$, $\underline{\tau}^\dagger \underline{\tau} < \infty$. These three parametrizations are generalizations of (3.13), (3.7), and (3.11) of ACGT,[5] respectively. The relationship between these parametrizations is shown in Table I.

The coherent states for an ensemble of N identical r-level atoms, evolving simultaneously from the ground state under identical unitary transformations, are given by

$$\left|\begin{matrix} N \\ \underline{x} \end{matrix}\right\rangle = \prod_{\mu=1}^{N} \otimes \{U_\mu(\underline{x}) |1\rangle_\mu\} = \sum_{\underline{m}=N} |\underline{m}\rangle \Gamma^N_{\underline{m},g}(\underline{x}). \quad (3.8)$$

In this expression, the tensor product $\otimes$ is taken over the N identical r-level atoms, $\Gamma^N(\underline{x})$ is the fully symmetrized Nth-order tensor representation of $U(r)$ of dimensionality $(N+r-1)!/N!(r-1)!$, and the states $|\underline{m}\rangle$ are the basis vectors for this representation, defined by

$$\begin{aligned} |\underline{m}\rangle &= |m_1, m_2, \ldots, m_r\rangle \\ &= \sum_{\text{all } P} \otimes \frac{|1\rangle^{m_1} |2\rangle^{m_2} \cdots |r\rangle^{m_r}}{(m_1! m_2! \cdots m_r!)^{1/2}} . \end{aligned} \quad (3.9)$$

The sum extends over all $N!$ permutations of the $m_1 + m_2 + \cdots + m_r = N$ single-particle basis vectors involved in the tensor product. In particular, the N-particle ground state is $|N, 0, 0, \ldots, 0\rangle$.

The only important matrix elements in the symmetric representation $\Gamma^N[U(r)]$ are those belonging to the column acting on the ground state of the total system. These are homogeneous monomials.[5] The coherent states for N identical r-level particles in the group parametrization are then

$$\left|\begin{matrix} N \\ \underline{x} \end{matrix}\right\rangle = \sum_{m_1 + \cdots + m_r = N} |m_1, \ldots, m_r\rangle \times C(m_1, \ldots, m_r)(x_1)^{m_1} \cdots (x_r)^{m_r}, \quad (3.10)$$

$$C(m_1, \ldots, m_r) = \left(\frac{(m_1 + \cdots + m_r)!}{m_1! \cdots m_r!}\right)^{1/2}.$$

These coherent states can be expressed in terms of the algebraic and projective parametrizations using the correspondence given in Table I.

C. Multilevel projectors

The diagonal projectors for the multilevel system are simply

$$\left|\begin{matrix} N \\ \underline{x} \end{matrix}\right\rangle \left\langle\begin{matrix} N \\ \underline{x} \end{matrix}\right| = \sum_{\underline{m}=N} \sum_{\underline{n}=N} |\underline{m}\rangle C(\underline{m}) \times (x_1)^{m_1} \cdots (x_r)^{m_r} (x_1^*)^{n_1} \cdots (x_r^*)^{n_r} \times C(\underline{n}) \langle \underline{n}| . \quad (3.11)$$

Here the summation $\sum_{\underline{m}=N}$ indicates a sum over all values of all m_j subject to the constraint $m_1 + \cdots + m_r = N$.

D. Multilevel $\mathfrak{D}$ algebras

The $\mathfrak{D}$ algebras for atomic coherent states can be constructed by the same techniques as the $\mathfrak{D}$ algebras for the single-mode oscillator coherent states. That is, we construct first-order linear differential operators $\mathfrak{D}^{\mathfrak{X}}(\hat{A})$ ($\mathfrak{X} = B, K, L, R$) having the same effect on the states or the projectors as the operator $\hat{A}$ in $u(r)$. In this case, we must construct differential operator realizations for the shift operators E_{jk} which span the Lie algebra $u(r)$. We shall illustrate this construction explicitly for $\mathfrak{D}^L(E_{jk})$.

The effect of the shift operator E_{jk} on the basis vector $|m_1, \ldots, m_r\rangle$ can be determined using the boson operator realization (3.2) on the symmetrized states (3.9):

TABLE I. Relationships between the group, algebraic, and projective multilevel atomic coherent-state parametrizations.

Group	Algebraic	Projective	Bound
x_u	$y_u \dfrac{\sin(\underline{y}^\dagger \underline{y})^{1/2}}{(\underline{y}^\dagger \underline{y})^{1/2}}$	$\dfrac{\tau_u}{(1+\underline{\tau}^\dagger \underline{\tau})^{1/2}}$	$\underline{x}^\dagger \underline{x} \leq 1$
$x_u \dfrac{in^{-1}(\underline{x}^\dagger \underline{x})^{1/2}}{(\underline{x}^\dagger \underline{x})^{1/2}}$	y_u	$\tau_u \dfrac{\tan^{-1}(\underline{\tau}^\dagger \underline{\tau})^{1/2}}{(\underline{\tau}^\dagger \underline{\tau})^{1/2}}$	$\underline{y}^\dagger \underline{y} \leq (\frac{1}{2}\pi)^2$
$\dfrac{x_u}{(1-\underline{x}^\dagger \underline{x})^{1/2}}$	$y_u \dfrac{\tan(\underline{y}^\dagger \underline{y})^{1/2}}{(\underline{y}^\dagger \underline{y})^{1/2}}$	τ_u	$\underline{\tau}^\dagger \underline{\tau} < \infty$

$$E_{jk}|m_1,\ldots,m_j,\ldots,m_k,\ldots,m_r\rangle = (m_j+1)^{1/2}m_k^{1/2} \times|m_1,\ldots,m_j+1,\ldots,m_k-1,\ldots,m_r\rangle. \tag{3.12}$$

As a result, the effect of E_{jk} on the coherent-state projector

$$\left|\begin{matrix}N\\ \underline{x}\end{matrix}\right\rangle\left\langle\begin{matrix}N\\ \underline{x}\end{matrix}\right| = \sum_{\underline{m}}\sum_{\underline{n}}|\underline{m}\rangle M_{\underline{m},\underline{n}}(\underline{x})\langle\underline{n}|$$

of (3.11) is

$$E_{jk}\sum_{\underline{m}}\sum_{m}|\underline{m}\rangle M_{\underline{m},\underline{n}}(\underline{x})\langle\underline{n}| = \sum_{\underline{m}}\sum_{\underline{n}}|\underline{m}\rangle\frac{m_j x_k}{x_j}M_{\underline{m},\underline{n}}(\underline{x})\langle\underline{n}|. \tag{3.13}$$

The differential operator producing the same effect on the functions $M_{\underline{m},\underline{n}}(\underline{x})$ can be determined to be

$$\mathfrak{D}^L(E_{jk}) = x_k\frac{\partial}{\partial x_j} + x_k x_j^*[N - \tfrac{1}{2}(\underline{x}\cdot\underline{\nabla} + \underline{x}^*\cdot\underline{\nabla}^*)]. \tag{3.14}$$

In this expression, we have defined

$$\underline{x}\cdot\underline{\nabla} + \underline{x}^*\cdot\underline{\nabla}^* = \sum_{u=2}^{r}\left(x_u\frac{\partial}{\partial x_u} + x_u^*\frac{\partial}{\partial x_u^*}\right). \tag{3.15}$$

In addition, since $x_1 = (1-\underline{x}^*\cdot\underline{x})^{1/2}$ is not an independent variable, $\partial/\partial x_1$ is not defined, and $\mathfrak{D}^L(E_{1k})$ must be discussed separately. One finds that $\mathfrak{D}^L(E_{1k})$ is given also by (3.14), provided one defines the symbol $\partial/\partial x_1$ to be

$$\frac{\partial}{\partial x_1} \equiv \frac{-1}{2x_1}(\underline{x}\cdot\underline{\nabla} - \underline{x}^*\cdot\underline{\nabla}^*) \equiv -\frac{\partial}{\partial x_1^*}. \tag{3.16a}$$

The realization $\mathfrak{D}^K(E_{jk})$ is obtained from $\mathfrak{D}^L(E_{jk})$ by the substitution $N\to\frac{1}{2}N$ and the definition

$$\frac{\partial}{\partial x_1} \equiv \tfrac{1}{2}x_1[N - (\underline{x}\cdot\underline{\nabla} + \underline{x}^*\cdot\underline{\nabla}^*)]. \tag{3.16b}$$

This definition is not unique in this case; however, it leads to the simplest algebraic form of the $\mathfrak{D}$ operators. In addition,

$$\mathfrak{D}^{B*}(\hat{A}) = \mathfrak{D}^K(\hat{A}^\dagger), \quad \mathfrak{D}^{R*}(\hat{A}) = \mathfrak{D}^L(\hat{A}^\dagger). \tag{3.17}$$

The construction of the multilevel $\mathfrak{D}$ algebras by these considerations is anything but straightforward. Furthermore, the direct verification of properties (2.7)–(2.10) for these $\mathfrak{D}$ algebras is an extremely involved procedure.

For the reasons cited, it is useful to observe that the multimode calculus can be used as a generating function for the multilevel calculus. In the next section we will establish a homomorphic mapping of the multimode calculus (Lie algebra, coherent states, diagonal "projectors," $\mathfrak{D}$ algebras) onto the multilevel calculus. As a consequence of the results of the following section, all the results of this section will be seen to follow as trivial consequences of the corresponding results of Sec. II.

IV. MULTIMODE TO MULTILEVEL MAPPING

A. Lie algebra homomorphism

The multimode Lie algebra described by (2.1) is actually only the spectrum-generating part of the full multimode Lie algebra. The full Lie algebra contains homogeneous polynomial products of the boson creation and annihilation operators of order 0, 1, and 2. In addition to the identity I and the operators $a_j^\dagger, a_j$ which add or remove one photon from mode j, there are r^2 second-order operators of the form $a_j^\dagger a_k$, which transfer a photon from mode k to mode j, $\frac{1}{2}r(r+1)$ operators of the form $a_j^\dagger a_k^\dagger$ $(j\le k)$, which add one photon each to modes j and k or two photons to mode $j=k$, and their adjoints $a_k a_j$, which remove pairs of photons. The set of operators I, $a_j^\dagger$, a_j, $a_j^\dagger a_k$, $a_j^\dagger a_k^\dagger$, $a_k a_j$ closes under commutation and therefore spans a Lie algebra.

This Lie algebra has a variety of subalgebras of interest. One such subalgebra is spanned by the r^2 photon number-preserving operators $a_j^\dagger a_k$. This subalgebra is isomorphic with $u(r)$. The homomorphism h of the full multimode Lie algebra onto the number-preserving subalgebra, given by

$$\begin{aligned} &h(a_j^\dagger a_k) = a_j^\dagger a_k, \\ &h(a_j^\dagger a_k^\dagger) = h(a_k a_j) = 0, \\ &h(a_j^\dagger) = h(a_j) = h(I) = 0, \end{aligned} \tag{4.1}$$

provides also a homomorphism of the full multimode Lie algebra onto the multilevel Lie algebra given by $a_j^\dagger a_k \simeq E_{jk}$ (cf. 3.2).

Under the restriction (i.e., homomorphism) of the full multimode Lie algebra to the number-preserving subalgebra, we should expect the irreducible representations of the full algebra, with corresponding basis vectors $|m_1,\ldots,m_r\rangle$, to decompose into a direct sum of irreducible representations of the $u(r)$ subalgebra, characterized by the constraints $m_1+\cdots+m_r=N$, $N=0,1,2,\ldots$. We should therefore also expect that the multimode coherent states decompose into a direct sum of multilevel coherent states; that the multimode projectors provide a generating function for the multilevel projectors; and that the multimode $\mathfrak{D}$

algebras map homomorphically onto the multilevel $\mathfrak{D}$ algebras.

B. Coherent state homomorphism

The relationship between the oscillator coherent states and the atomic coherent states is made manifest by the change of variables

$$\alpha_j = x_j \alpha e^{i\phi}, \quad \alpha = (\underline{\alpha}^\dagger \underline{\alpha})^{1/2}, \quad e^{i\phi} = \alpha_1/|\alpha_1|. \tag{4.2}$$

The effect of this transformation is to express the r independent complex parameters α_j $(j=1,2,\ldots,r)$ describing multimode coherent states in terms of the $r-1$ independent complex parameters x_u $(u=2,3,\ldots,r)$ describing multilevel coherent states. Two additional real parameters are required to make the transformation one to one. These are α, the modulus of $\underline{\alpha}$, which provides the spherical condition (3.6) on the variables x_j, and the phase φ which is explicitly extracted from α_1 to make x_1 real and thus completely dependent: $x_1 = (1 - \underline{x}^* \cdot \underline{x})^{1/2}$ [cf. (3.4) and (3.5)].

This change of variables relates the multimode (2.3) and multilevel (3.10) coherent states:

$$\begin{aligned}|\underline{\alpha}\rangle &= \sum_{m_1=0}^{\infty} \cdots \sum_{m_r=0}^{\infty} e^{-\underline{\alpha}^\dagger \underline{\alpha}/2} \\ &\qquad \times \frac{(\alpha_1)^{m_1} \cdots (\alpha_r)^{m_r}}{(m_1! \cdots m_r!)^{1/2}} |m_1, \ldots, m_r\rangle \\ &= \sum_{N=0}^{\infty} \sum_{\underline{m}=N} e^{-\alpha^2/2} \frac{(\alpha e^{i\phi})^N}{\sqrt{N!}} \left(\frac{(m_1 + \cdots + m_r)!}{m_1! \cdots m_r!}\right)^{1/2} \\ &\qquad \times (x_1)^{m_1} \cdots (x_r)^{m_r} |m_1, \ldots, m_r\rangle \\ &= \sum_{N=0}^{\infty} e^{-\alpha^2/2} \frac{(\alpha e^{i\phi})^N}{\sqrt{N!}} \left| \begin{matrix} N \\ \underline{x} \end{matrix} \right\rangle . \end{aligned} \tag{4.3}$$

As a consequence, the multimode coherent states can be used as generating functions for the multilevel coherent states. The explicit homomorphism from the multimode coherent state $|\underline{\alpha}\rangle$ onto the multilevel coherent state $|{N \atop \underline{x}}\rangle$ is given by

$$\lim_{\alpha \to 0} e^{-iN\varphi} \left(\frac{\partial}{\partial \alpha}\right)^N \frac{e^{\alpha^2/2}}{\sqrt{N!}} |\underline{\alpha}\rangle = \left| \begin{matrix} N \\ \underline{x} \end{matrix} \right\rangle . \tag{4.4}$$

C. Projector homomorphism

The coherent-state homomorphism provides a projector homomorphism as follows:

$$|\underline{\alpha}\rangle\langle\underline{\alpha}| = \sum_{M=0}^{\infty} \sum_{N=0}^{\infty} e^{-\alpha^2} \frac{\alpha^{M+N} (e^{i\phi})^{M-N}}{(M!N!)^{1/2}} \left| \begin{matrix} M \\ \underline{x} \end{matrix} \right\rangle \left\langle \begin{matrix} N \\ \underline{x} \end{matrix} \right| , \tag{4.5}$$

$$\oint |\underline{\alpha}\rangle\langle\underline{\alpha}| \frac{d\phi}{2\pi} = \sum_{N=0}^{\infty} e^{-\alpha^2} \frac{(\alpha^2)^N}{N!} \left| \begin{matrix} N \\ \underline{x} \end{matrix} \right\rangle \left\langle \begin{matrix} N \\ \underline{x} \end{matrix} \right| . \tag{4.6}$$

The multilevel coherent states are Poisson distributed in the ϕ average. The homomorphism onto a specific multilevel projector is provided by the limit

$$\lim_{\alpha^2 \to 0} \left(\frac{\partial}{\partial \alpha^2}\right)^M e^{\alpha^2} \oint |\underline{\alpha}\rangle\langle\underline{\alpha}| \frac{d\phi}{2\pi} = \left| \begin{matrix} M \\ \underline{x} \end{matrix} \right\rangle \left\langle \begin{matrix} M \\ \underline{x} \end{matrix} \right| . \tag{4.7}$$

This homomorphism can also be constructed from (4.4) and its adjoint.

D. $\mathfrak{D}$ algebra homomorphisms

The homomorphisms (4.4) and (4.7) also provide homomorphisms from the multimode $\mathfrak{D}$ algebras onto the multilevel $\mathfrak{D}$ algebras. This connection is made very simply; as an example, we construct the homomorphism explicitly for the $\mathfrak{D}^L$ algebra:

$$\begin{aligned}\mathfrak{D}^L(E_{jk}) &= \mathfrak{D}^L(a_j^\dagger a_k) && \text{by (3.2)} \\ &= \mathfrak{D}^L(a_k)\mathfrak{D}^L(a_j^\dagger) && \text{by (2.7)} \\ &= \alpha_k\left(\frac{\partial}{\partial \alpha_j} + \alpha_j^*\right) && \text{by (2.6).}\end{aligned} \tag{4.8}$$

The $\partial/\partial\alpha_k$ are expressed in terms of $\partial/\partial\alpha$, $\partial/\partial\phi$, $\partial/\partial x_k$, and $\partial/\partial x_k^*$ using (4.2) and the chain rule (cf. Appendix A):

$$\begin{aligned}\frac{\partial}{\partial \alpha_1} &= \tfrac{1}{2} x_1^* e^{-i\phi} \frac{\partial}{\partial \alpha} \\ &\quad + \frac{1}{\alpha e^{i\phi}} \frac{1}{2x_1} \left(\frac{\partial}{\partial i\phi} - \underline{x} \cdot \underline{\nabla} + \underline{x}^* \cdot \underline{\nabla}^*\right) \\ &\quad - \frac{x_1 e^{-i\phi}}{2} (\underline{x} \cdot \underline{\nabla} + \underline{x}^* \cdot \underline{\nabla}^*), \\ \frac{\partial}{\partial \alpha_u} &= \tfrac{1}{2} x_u^* e^{-i\phi} \frac{\partial}{\partial \alpha} + \frac{1}{\alpha e^{i\phi}} \frac{\partial}{\partial x_u} \\ &\quad - \frac{x_u^* e^{-i\phi}}{2\alpha} (\underline{x} \cdot \underline{\nabla} + \underline{x}^* \cdot \underline{\nabla}^*). \end{aligned} \tag{4.9}$$

As a result, we find for all values of j, k:

$$\begin{aligned}\mathfrak{D}^L(E_{jk}) &= x_k \frac{\partial}{\partial x_j} + x_k x_j^* \left(\frac{\alpha}{2} \frac{\partial}{\partial \alpha} + \alpha^2\right) \\ &\quad - \tfrac{1}{2} x_k x_j^* (\underline{x} \cdot \underline{\nabla} + \underline{x}^* \cdot \underline{\nabla}^*), \end{aligned} \tag{4.10}$$

where we have defined

$$\begin{aligned}\frac{\partial}{\partial x_1} &\equiv \frac{1}{2x_1} \left(\frac{\partial}{\partial (i\phi)} - \underline{x} \cdot \underline{\nabla} + \underline{x}^* \cdot \underline{\nabla}^*\right) \\ &\equiv -\frac{\partial}{\partial x_1^*} . \end{aligned} \tag{4.11}$$

We now apply the operator $\mathfrak{D}^L(E_{jk}) = \mathfrak{D}^L(a_j^\dagger a_k)$ to the diagonal projectors (2.4), and follow by projecting the result onto the diagonal projectors (3.11) using the homomorphism (4.7). Under the φ average, the derivative $\partial/\partial(i\varphi)$ vanishes. In addition, we have

$$\left(\alpha^2\frac{\partial}{\partial\alpha^2}+\alpha^2\right)e^{-\alpha^2}\frac{(\alpha^2)^N}{N!}\to Ne^{-\alpha^2}\frac{(\alpha^2)^N}{N!}\,. \quad (4.12)$$

As a result, the $\mathfrak{D}^L$ algebra homomorphism is given explicitly by

$$\lim_{\alpha^2\to 0}\left(\frac{\partial}{\partial\alpha^2}\right)^N e^{\alpha^2}\oint \mathfrak{D}^L(a_j^\dagger a_k)|\underline{\alpha}\rangle\langle\underline{\alpha}|\,\frac{d\phi}{2\pi}$$
$$=\left(x_k\frac{\partial}{\partial x_j}+x_k x_j^*[N-\tfrac{1}{2}(\underline{x}\cdot\underline{\nabla}+\underline{x}^*\cdot\underline{\nabla}^*)]\right)\left|\begin{matrix}N\\ \underline{x}\end{matrix}\right\rangle\left\langle\begin{matrix}N\\ \underline{x}\end{matrix}\right|, \quad (4.13)$$

where we have defined

$$\frac{\partial}{\partial x_1}\equiv-\frac{1}{2x_1}(\underline{x}\cdot\underline{\nabla}-\underline{x}^*\cdot\underline{\nabla}^*)\equiv-\frac{\partial}{\partial x_1^*}\,. \quad (4.14)$$

The result (4.13) is obtained simply but indirectly; it is identical to the result (3.14) which is obtained more directly but less simply.

Analogous calculations can be carried out for the algebras $\mathfrak{D}^K$, $\mathfrak{D}^R$, $\mathfrak{D}^B$. The realization $\mathfrak{D}^K(E_{jk})$ is obtained from $\mathfrak{D}^L(E_{jk})$ by the substitution $N\to\frac{1}{2}N$ and the definition (3.16b) for $\partial/\partial x_1$. The algebras $\mathfrak{D}^R$, $\mathfrak{D}^B$ are constructed from $\mathfrak{D}^L$, $\mathfrak{D}^K$ using (3.17).

The homomorphism (4.4) and (4.7) preserve the properties (2.7)–(2.10) of the multimode $\mathfrak{D}$ algebras. As a trivial consequence, the multilevel $\mathfrak{D}$ algebras possess these properties as well.

V. $\mathfrak{D}$ ALGEBRA FOR TWO-LEVEL ATOMIC SYSTEMS

As an illustration, we specialize some of the general considerations developed in Secs. III and IV to the case of two-level systems. First, we observe that the coherent states (3.10) for N identical two-level systems can be expressed in the familiar form

$$|\Omega\rangle=\sum_{m=-J}^{J}\binom{2J}{m+J}^{1/2}(\sin\tfrac{1}{2}\theta)^{m+J}(\cos\tfrac{1}{2}\theta)^{J-m}e^{-i(J+m)\varphi}\times|Jm\rangle \quad (5.1)$$

using the algebraic parametrization

$$x_1=\cos\tfrac{1}{2}\theta,\quad x_2=e^{-i\varphi}\sin\tfrac{1}{2}\theta. \quad (5.2)$$

The states $|\Omega\rangle$ in Eq. (5.1) are the coherent atomic states extensively discussed in Ref. 5 and parametrized on the surface of the Bloch sphere. The diagonal projector $\Lambda(\Omega)=|\Omega\rangle\langle\Omega|$ takes the form

$$|\Omega\rangle\langle\Omega|=\sum_{m=-J}^{J}\sum_{m'=-J}^{J}|J,m\rangle\langle J,m'|f_m(\Omega)f_{m'}^*(\Omega), \quad (5.3)$$

where the c-number functions $f_m(\Omega)$ are defined by

$$f_m(\Omega)=\binom{2J}{m+J}^{1/2}(\sin\tfrac{1}{2}\theta)^{J+m}(\cos\tfrac{1}{2}\theta)^{J-m}e^{-i(J+m)\varphi}. \quad (5.4)$$

As indicated in Sec. III, we are interested in mapping transformations of the form

$$\hat{X}\Lambda(\Omega)\equiv\sum_{m,m'}f_m(\Omega)f_{m'}^*(\Omega)\hat{X}|J,m\rangle\langle J,m'|=\mathfrak{D}^L(\hat{X})\Lambda(\Omega) \quad (5.5)$$

and

$$\Lambda(\Omega)\hat{X}\equiv\sum_{m,m'}f_m(\Omega)f_{m'}^*(\Omega)|J,m\rangle\langle J,m'|\hat{X}=\mathfrak{D}^R(\hat{X})\Lambda(\Omega), \quad (5.6)$$

where the superscripts L and R refer to the position of the operator $\hat{X}$ with respect to the projector $\Lambda(\Omega)$. The differential operators $\mathfrak{D}^L(\hat{X})$ and $\mathfrak{D}^R(\hat{X})$ are first-order differential operators acting on the c-number functions $f_m(\Omega)$ and $f_{m'}^*(\Omega)$. As shown in Sec. III [Eq. (3.14)], the left differential operator corresponding to E_{jk} ($j,k=1,2$) is given by

$$\mathfrak{D}^L(E_{jk})=x_k\frac{\partial}{\partial x_j}+x_k x_j^*[N-\tfrac{1}{2}(\underline{x}\cdot\underline{\nabla}+\underline{x}^*\cdot\underline{\nabla})], \quad (5.7)$$

where $\partial/\partial x_1$ is defined by

$$\frac{\partial}{\partial x_1}=-\frac{1}{2x_1}(\underline{x}\cdot\underline{\nabla}-\underline{x}^*\cdot\underline{\nabla}^*)\equiv-\frac{\partial}{\partial x_1^*}\,. \quad (5.8)$$

In the present case, a familiar realization of the operators E_{jk} is provided by the set of angular momentum operators $\hat{J}^\pm$, $\hat{J}_z$

$$E_{21}=\hat{J}^+,\quad E_{12}=\hat{J}^-,\quad E_{22}-E_{11}=2\hat{J}_z\,. \quad (5.9)$$

Keeping in mind the parametrization (5.2) and the explicit form of the partial derivative $\partial/\partial x_2$

$$\frac{\partial}{\partial x_2}=e^{i\varphi}\left(\frac{1}{\cos(\theta/2)}\frac{\partial}{\partial\theta}+\frac{1}{2}\frac{i}{\sin(\theta/2)}\frac{\partial}{\partial\varphi}\right), \quad (5.10)$$

Eq. (5.7), for $j=2$ and $k=1$, reduces to

$$\mathfrak{D}^L(E_{21})\equiv\mathfrak{D}^L(\hat{J}^+)=e^{i\varphi}\left[J\sin\theta+\cos^2\left(\frac{\theta}{2}\right)\frac{\partial}{\partial\theta}+\frac{i}{2}\cot\left(\frac{\theta}{2}\right)\frac{\partial}{\partial\varphi}\right]. \quad (5.11)$$

We now consider the indices $j=1$ and $k=2$. In this case, Eq. (5.7) must be supplemented by the definition of $\partial/\partial x_1$ given by Eq. (5.8). The result is

$$\mathfrak{D}^L(E_{12}) = \mathfrak{D}^L(\hat{J}^-) = e^{-i\varphi}\left[J\sin\theta - \sin^2\left(\frac{\theta}{2}\right)\frac{\partial}{\partial\theta} - \frac{i}{2}\tan\left(\frac{\theta}{2}\right)\frac{\partial}{\partial\varphi}\right]. \quad (5.12)$$

In a similar way we can construct $\mathfrak{D}^L(E_{22})$ and $\mathfrak{D}^L(E_{11})$:

$$\mathfrak{D}^L(E_{22}) = 2J\sin^2\left(\frac{\theta}{2}\right) + \tfrac{1}{2}\sin\theta\frac{\partial}{\partial\theta} + \frac{i}{2}\frac{\partial}{\partial\varphi}, \quad (5.13)$$

$$\mathfrak{D}^L(E_{11}) = 2J\cos^2\left(\frac{\theta}{2}\right) - \tfrac{1}{2}\sin\theta\frac{\partial}{\partial\theta} - \frac{i}{2}\frac{\partial}{\partial\varphi}. \quad (5.14)$$

Thus it follows that

$$\mathfrak{D}^L(\hat{J}_3) = \tfrac{1}{2}[\mathfrak{D}^L(E_{22}) - \mathfrak{D}^L(E_{11})] = -J\cos\theta + \tfrac{1}{2}\sin\theta\frac{\partial}{\partial\theta} + \frac{i}{2}\frac{\partial}{\partial\varphi}. \quad (5.15)$$

The explicit form of the right operators $\mathfrak{D}^R(\hat{X})$ follows from the symmetry relation

$$\mathfrak{D}^R(\hat{X}) = [\mathfrak{D}^L(\hat{X}^\dagger)]^*. \quad (5.16)$$

VI. THE QUASI-PROBABILITY FUNCTIONS $P(\Omega,t)$ AND $Q(\Omega,t)$

A useful application of the $\mathfrak{D}$ algebra is the mapping of operator equations into c-number differential equations. Here we consider two types of quasi-probability functions associated with the density operator W in the Hilbert space of angular momentum: the P function, defined by the integral representation

$$W(t) = \int d\Omega\, P(\Omega, t)\Lambda(\Omega), \quad \Lambda(\Omega) \equiv |\Omega\rangle\langle\Omega|, \quad (6.1)$$

and the Q function defined by the diagonal elements of the density operator in the atomic coherent-state representation

$$Q(\Omega, t) = \langle\Omega|W(t)|\Omega\rangle = \int d\Omega' P(\Omega', t)|\langle\Omega|\Omega'\rangle|^2. \quad (6.2)$$

The P function has been used to describe the evolution of a single-mode superradiant system,[7] as well as the approach to thermal equilibrium of a $(2J+1)$-level atom.[6] The Q function is analogous to the function $Q(\alpha, t) = \langle\alpha|W|\alpha\rangle$ discussed in the context of the Glauber coherent-state representation.[11]

Here we focus on the formal derivation of the equations of motion for both $P(\Omega, t)$ and $Q(\Omega, t)$, starting from a fairly general class of master equations, and then relate the physically relevant expectation values of the dynamical variables to these density functions. In Sec. VII we provide an inversion formula for Eq. (6.2). For future use, consider the following identity

$$\int d\Omega\, P(\Omega, t)\mathfrak{D}^L(\hat{J}^\alpha)[\Lambda(\Omega)] = \int d\Omega\, \Lambda(\Omega)\tilde{\mathfrak{D}}^L(\hat{J}^\alpha)[P(\Omega)], \quad (6.3)$$

where α stands for $\pm$, or z and where

$$\tilde{\mathfrak{D}}^L(\hat{J}^+) = e^{i\varphi}\left[(J+1)\sin\theta - \cos^2\left(\frac{\theta}{2}\right)\frac{\partial}{\partial\theta} - \frac{i}{2}\cot\left(\frac{\theta}{2}\right)\frac{\partial}{\partial\varphi}\right], \quad (6.4)$$

$$\tilde{\mathfrak{D}}^L(\hat{J}^-) = e^{-i\varphi}\left[(J+1)\sin\theta + \sin^2\left(\frac{\theta}{2}\right)\frac{\partial}{\partial\theta} + \frac{i}{2}\tan\left(\frac{\theta}{2}\right)\frac{\partial}{\partial\varphi}\right], \quad (6.5)$$

$$\tilde{\mathfrak{D}}^L(\hat{J}_z) = -(J+1)\cos\theta - \tfrac{1}{2}\sin\theta\frac{\partial}{\partial\theta} - \frac{i}{2}\frac{\partial}{\partial\varphi}. \quad (6.6)$$

(The general properties of the $\tilde{\mathfrak{D}}$ operators for multilevel atomic systems are discussed in Appendix B.) We consider a class of master equations of the form

$$\dot{\hat{W}} = \sum_{n,m} C_{n,m}\hat{A}_n\hat{W}\hat{B}_m, \quad (6.7)$$

where $C_{n,m}$ are c-number coefficients and $\hat{A}_n$ and $\hat{B}_m$ are arbitrary products of angular momentum operators. In terms of the P representation, Eq. (6.7) becomes

$$\int d\Omega\,\Lambda(\Omega)\frac{\partial P}{\partial t} = \sum_{n,m} C_{n,m}\int d\Omega\, P(\Omega)\hat{A}_n\Lambda(\Omega)\hat{B}_m = \sum_{n,m} C_{n,m}\int d\Omega\, P(\Omega)\mathfrak{D}^L(\hat{A}_n)\mathfrak{D}^R(\hat{B}_m)\Lambda(\Omega) = \int d\Omega\,\Lambda(\Omega)\sum_{n,m} C_{n,m}\tilde{\mathfrak{D}}^L(\hat{A}_n)\tilde{\mathfrak{D}}^R(\hat{B}_m)P(\Omega, t), \quad (6.8)$$

where we have used the identity (6.3). The equation of motion for the function $P(\Omega, t)$ follows at once from Eq. (6.8):

$$\frac{\partial P}{\partial t} = \sum_{n,m} C_{n,m}\tilde{\mathfrak{D}}^L(\hat{A}_n)\tilde{\mathfrak{D}}^R(\hat{B}_m)P(\Omega). \quad (6.9)$$

Notice that the order of the left and right operators in Eq. (6.9) is immaterial, since the left and right operators $\tilde{\mathfrak{D}}$ commute with one another. In general, we may expect the operators $\hat{A}_n$ and $\hat{B}_m$ to be products of elementary angular momentum operators $\hat{J}^\alpha$. In this case, it is easy to verify that the differential operator $\tilde{\mathfrak{D}}^L$ corresponding to

$$\hat{A}_n = \hat{J}^{\alpha_1}\hat{J}^{\alpha_2}\cdots\hat{J}^{\alpha_n}$$

is given by

$$\tilde{\mathfrak{D}}^L(\hat{A}_n)=\tilde{\mathfrak{D}}^L(\hat{J}^{\alpha_1})\cdots\tilde{\mathfrak{D}}^L(\hat{J}^{\alpha_n}). \tag{6.10}$$

Similarly, if $\hat{B}_m=\hat{J}^{\alpha_1}\cdots\hat{J}^{\alpha_m}$, we have

$$\tilde{\mathfrak{D}}^R(\hat{B}_m)=\tilde{\mathfrak{D}}^R(\hat{J}^{\alpha_m})\cdots\tilde{\mathfrak{D}}^R(\hat{J}^{\alpha_1}). \tag{6.11}$$

In conclusion, the mapping of a master equation of the form given by Eq. (6.7) follows from the formal replacements

$$\begin{aligned}&\hat{W}\to P(\Omega,t),\\ &\hat{A}_n\equiv\hat{J}^{\alpha_1}\cdots\hat{J}^{\alpha_n}\to\tilde{\mathfrak{D}}^L(\hat{A}_n)=\tilde{\mathfrak{D}}^L(\hat{J}^{\alpha_1})\cdots\tilde{\mathfrak{D}}^L(\hat{J}^{\alpha_n}),\\ &\hat{B}_m\equiv\hat{J}^{\alpha_1}\cdots\hat{J}^{\alpha_m}\to\tilde{\mathfrak{D}}^R(\hat{B}_m)=\tilde{\mathfrak{D}}^R(\hat{J}^{\alpha_m})\cdots\tilde{\mathfrak{D}}^R(\hat{J}^{\alpha_1}),\\ &\hat{A}_n\hat{W}\hat{B}_m\to\tilde{\mathfrak{D}}^L(\hat{A}_n)\tilde{\mathfrak{D}}^R(\hat{B}_m)P(\Omega,t).\end{aligned} \tag{6.12}$$

It follows by inspection that the order of the partial differential equation for $P(\Omega,t)$ is the same as the largest number of elementary angular momentum operators $\hat{J}^{\alpha_i}$ contained in the term $\hat{A}_n\hat{W}\hat{B}_m$ of the master equation.

In a similar manner we can construct the differential equation of motion for $Q(\Omega,t)$. To this purpose we multiply Eq. (6.7) by the diagonal projector $\Lambda(\Omega)$ and take the trace over the angular momentum degrees of freedom.

It follows that

$$\begin{aligned}\mathrm{tr}[\dot{\hat{W}}\Lambda(\Omega)]&\equiv\frac{\partial Q(\Omega,t)}{\partial t}\\ &=\sum_{n,m}C_{n,m}\mathrm{tr}[\hat{W}\mathfrak{D}^L(\hat{B}_m)\mathfrak{D}^R(\hat{A}_n)\Lambda(\Omega)].\end{aligned} \tag{6.13}$$

The equation of motion for the Q function is

$$\frac{\partial Q(\Omega,t)}{\partial t}=\sum_{n,m}C_{n,m}\mathfrak{D}^L(\hat{B}_m)\mathfrak{D}^R(\hat{A}_n)Q(\Omega,t). \tag{6.14}$$

Thus, the mapping procedure for constructing the equation of motion for $Q(\Omega,t)$ from the master equation (6.7) can be summarized by the following formal replacements:

$$\begin{aligned}&\hat{W}(t)\to Q(\Omega,t),\\ &\hat{A}_n=\hat{J}^{\alpha_1}\cdots\hat{J}^{\alpha_n}\to\mathfrak{D}^R(\hat{A}_n)=\mathfrak{D}^R(\hat{J}^{\alpha_1})\cdots\mathfrak{D}^R(\hat{J}^{\alpha_n}),\\ &\hat{B}_m=\hat{J}^{\alpha_1}\cdots\hat{J}^{\alpha_m}\to\mathfrak{D}^L(\hat{B}_m)=\mathfrak{D}^L(\hat{J}^{\alpha_m})\cdots\mathfrak{D}^L(\hat{J}^{\alpha_1}),\\ &\hat{A}_n\hat{W}\hat{B}_m\to\mathfrak{D}^L(\hat{B}_m)\mathfrak{D}^R(\hat{A}_n)Q(\Omega,t).\end{aligned} \tag{6.15}$$

Once again, the order of the right and left operators is immaterial, because $\mathfrak{D}^L$ and $\mathfrak{D}^R$ commute with one another.

The expectation values of arbitrary functions of angular momentum operators can be calculated in terms of the P function from the integral form

$$\langle\hat{X}(t)\rangle=\int d\Omega\,P(\Omega,t)\langle\Omega|\hat{X}(t)|\Omega\rangle. \tag{6.16}$$

The diagonal matrix element of $\hat{X}$ has been evaluated in Ref. 5 and is usually no more involved than trigonometric polynomials for physically relevant operators.

In terms of the Q representation we have instead

$$\begin{aligned}\langle\hat{X}\rangle&=\mathrm{tr}\left(\frac{2J+1}{4\pi}\int d\Omega\,\hat{W}(t)\hat{X}\Lambda(\Omega)\right)\\ &=\frac{2J+1}{4\pi}\int d\Omega\,\mathfrak{D}^L(\hat{X})Q(\Omega,t).\end{aligned} \tag{6.17}$$

Thus, in the Q representation the trigonometric function $\langle\Omega|\hat{X}|\Omega\rangle$ is replaced by the differential operator $\mathfrak{D}^L(\hat{X})$.

As a final application of the techniques developed in this paper, we consider the mapping of the Bloch equation of equilibrium statistical mechanics. Let $\hat{W}=\exp(-\beta\hat{H})$ be the (un-normalized) canonical density operator corresponding to the formal solution of the Bloch equation

$$\frac{\partial\hat{W}}{\partial\beta}=-\tfrac{1}{2}(\hat{H}\hat{W}+\hat{W}\hat{H}). \tag{6.18}$$

According to the mapping rules summarized in Eqs. (6.12) and (6.15), the equilibrium c-number functions $P(\Omega,\beta)$ and $Q(\Omega,\beta)$ satisfy the following differential equations:

$$\frac{\partial P(\Omega,\beta)}{\partial\beta}=-\mathrm{Re}[\tilde{\mathfrak{D}}^L(\hat{H})P(\Omega,\beta)] \tag{6.19}$$

and

$$\frac{\partial Q(\Omega,\beta)}{\partial\beta}=-\mathrm{Re}[\mathfrak{D}^L(\hat{H})Q(\Omega,\beta)], \tag{6.20}$$

where $\tilde{\mathfrak{D}}^L$ and $\mathfrak{D}^L$ are constructed as indicated in Eqs. (6.12) and (6.15), respectively, and where the "initial conditions" for $\beta=0$ are given by

$$P(\Omega,0)=1,\quad Q(\Omega,0)=1. \tag{6.21}$$

As an elementary illustration, consider the equilibrium density operator for a $(2J+1)$-level system described by the Zeeman Hamiltonian

$$\hat{H}=-g\mu_B\mathcal{H}_0\hat{J}_z. \tag{6.22}$$

The Q function satisfies the differential equation

$$\begin{aligned}&\frac{\partial Q}{\partial\beta'}=-\tfrac{1}{2}\sin\theta\frac{\partial Q}{\partial\theta}=-J\cos\theta\,Q\\ &\beta'=\beta g\mu_B\mathcal{H}_0\end{aligned} \tag{6.23}$$

whose solution takes the form

$$\begin{aligned}Q(\theta,\beta)&=e^{-\beta' J}\left(\frac{(1+e^{\beta'})+(1-e^{\beta'})\cos\theta}{2}\right)^{2J}\\ &=\left[\cosh\left(\frac{\beta'}{2}\right)-\sinh\left(\frac{\beta'}{2}\right)\cos\theta\right]^{2J}.\end{aligned} \tag{6.24}$$

After imposing the normalization condition

$$\frac{2J+1}{4\pi}\int d\Omega\, Q(\Omega,\beta)=1, \tag{6.25}$$

the moments of interest can be calculated according to Eq. (6.17). It goes without saying that in this elementary case, the partition function of interest can be calculated much more directly. The differential equations (6.19) and (6.20), however, appear to offer some advantage over the operator equation (6.18) in more complicated situations where the partition function is not so readily accessible.

VII. TRANSFORM BETWEEN P AND Q

In this section we derive the explicit transform relations between the Q and P representations discussed in Sec. VI. The generalization to multilevel systems ($r>2$) is shown to be straightforward.

It is useful to compare the relations between the Q and P representations for field systems and for two-level atomic systems, which are given respectively by

$$Q(\alpha)=\int d^2\beta P(\beta)|\langle\beta|\alpha\rangle|^2 \tag{7.1a}$$

and

$$Q(\Omega)=\int d_\mu\Omega' P(\Omega')|\langle\Omega'|\Omega\rangle|^2. \tag{7.1b}$$

Since $|\langle\beta|\alpha\rangle|^2=\exp(-|\alpha-\beta|^2)$, (7.1a) is a convolution and may therefore be inverted by standard techniques. If $q(k)$ is the Fourier transform of $Q(\alpha)$, i.e., $F\{Q\}=q(k)$, then the convolution transformation theorem gives

$$q(k)=p(k)\exp(-|k|^2/4), \tag{7.2a}$$

where we have used $F\{e^{-|\alpha|^2}\}=e^{-|k|^2/4}$. As a result, $P(\beta)$ is given explicitly by

$$P(\beta)=F^{-1}\{q(k)\exp(|k|^2/4)\}. \tag{7.3a}$$

A similar procedure is used to invert (7.1b). Since

$$\langle\Omega'|\Omega\rangle=\sum_m[\Gamma^j_{m,-j}(\Omega')]^*\Gamma^j_{m,-j}(\Omega)=\Gamma^j_{-j,-j}(\Omega'^{-1}\Omega),$$

Eq. (7.1b) has the form of a convolution on the Bloch sphere $SU(2)/U(1)$. As a result, (7.1b) may also be regarded as a convolution on the group $SU(2)$. If $\tilde{F}$ represents the Fourier transform on $SU(2)$, then the convolution theorem assumes the form[12]

$$\tilde{F}[Q(\Omega)]=\tilde{F}[P(\Omega)]\cdot\tilde{F}(|\langle Id|\Omega\rangle|^2). \tag{7.2b}$$

As a result, $P(\Omega)$ is given formally by

$$P(\Omega)=\tilde{F}^{-1}\{\tilde{F}[Q]/\tilde{F}[|\langle Id|\Omega\rangle|^2]\}. \tag{7.3b}$$

To make (7.3b) more explicit, we resolve the functions $Q(\Omega)$, $P(\Omega)$, and $|\langle\Omega'|\Omega\rangle|^2$ in terms of a complete set of functions defined on $SU(2)$:

$$P(\Omega)=P(\Omega h)=\sum\frac{d(J)}{V(G)}P^J_{m'm''}\Gamma^J_{m'm''}(\Omega h), \tag{7.4}$$

where $d(J)=2J+1$ is the dimensionality of the unitary irreducible representation Γ^J of $SU(2)$, $V(G)$ is the volume of $SU(2)$, $h\in U(1)$, $\Omega\in SU(2)/U(1)$, and Ωh represents an arbitrary group element in $SU(2)$. The Fourier coefficients of $P(\Omega)$ are

$$P^J_{m'm''}=\int P(\Omega h)[\Gamma^J_{m'm''}(\Omega h)]^*d_\mu(\Omega h). \tag{7.5}$$

The Fourier coefficients of the weighting function $|\langle\Omega'|\Omega\rangle|^2$ are

$$R^J_{m'm''}=\int\Gamma^j_{-j,-j}(\Omega h)[\Gamma^j_{-j,-j}(\Omega h)]^*[\Gamma^J_{m'm''}(\Omega h)]^*\times\frac{d_\mu(\Omega)d_\mu(h)}{V(H)}. \tag{7.6}$$

This integral can be expressed in terms of Clebsch-Gordan coefficients[13]

$$R^J_{m'm''}=\frac{V(G)/V(H)}{(2j+1)}\left\langle\begin{matrix}j & J\\ -j & m'\end{matrix}\middle|\begin{matrix}j\\ -j\end{matrix}\right\rangle\left\langle\begin{matrix}j\\ -j\end{matrix}\middle|\begin{matrix}j & J\\ -j & m''\end{matrix}\right\rangle. \tag{7.7}$$

The only nonvanishing terms in this expression are those for which $J=L$ (integer), $0\leq L\leq 2j$, $m'=m''=0$. Then,[14] using $V(G)/V(H)=V[SU(2)/U(1)]=4\pi$,

$$R^L_{00}=\frac{4\pi}{2j+1}\left|\left\langle\begin{matrix}j\\ -j\end{matrix}\middle|\begin{matrix}j & L\\ -j & 0\end{matrix}\right\rangle\right|^2,$$

$$\left|\left\langle\begin{matrix}j\\ -j\end{matrix}\middle|\begin{matrix}j & L\\ -j & 0\end{matrix}\right\rangle\right|^2=\frac{(2j+1)!\,(2j)!}{(2j+1+L)!\,(2j-L)!}. \tag{7.8}$$

The explicit form of (7.2b) is then $Q^L_{m0}=P^L_{m0}R^L_{00}$. As a result,

$$P(\Omega h)=\sum_{L=0}^{2j}\sum_{m=-L}^{+L}\frac{d(L)}{V(G)}\frac{Q^L_{m0}}{R^L_{00}}\Gamma^L_{m0}(\Omega h). \tag{7.9}$$

Since $P(\Omega h)=P(\Omega)$ and $\Gamma^L_{m0}(\Omega h)=\Gamma^L_{m0}(\Omega)$, the argument Ωh in (7.9) can be replaced by the coset representative Ω. In addition, (7.9) can be expressed in terms of the more familiar spherical harmonics using[15]

$$Y^L_m(\Omega)=\left(\frac{d(L)}{V(G)/V(H)}\right)^{1/2}\Gamma^L_{m0}(\Omega), \tag{7.10}$$

$$Q^L_{m0} = \int Q(\Omega h)[\Gamma^L_{m0}(\Omega h)]^* d_\mu(\Omega h)$$
$$= V(H)\int Q(\Omega)[\Gamma^L_{m0}(\Omega)]^* d_\mu(\Omega)$$
$$= V(H)\left(\frac{V(G)/V(H)}{d(L)}\right)^{1/2}\int Q(\Omega)[Y^L_m(\Omega)]^* d_\mu(\Omega). \tag{7.11}$$

As a result,

$$P(\Omega) = \int k(\Omega,\Omega')Q(\Omega')\, d_\mu(\Omega'), \tag{7.12}$$

where

$$k(\Omega,\Omega') = \frac{(2j+1)}{4\pi}\sum_{L=0}^{2j}\left|\left\langle \begin{matrix} j \\ -j \end{matrix}\middle|\begin{matrix} j & L \\ -j & 0\end{matrix}\right\rangle\right|^{-2} \times \sum_{m=-L}^{+L} Y^L_m(\Omega)[Y^L_m(\Omega')]^*.$$

Since the kernel $k(\Omega,\Omega')$ is a function of the group operation $\Omega'^{-1}\Omega$ only (by the addition theorem for spherical harmonics), the inverting relation (7.12) has the form of a convolution. In fact, in the field case, (7.3a) can also be put into the form of a convolution.

Finally, we exhibit explicitly the symmetry between the convolutions in (7.1b) and (7.12) as follows:

$$Q(\Omega) = \int k_+(\Omega,\Omega')P(\Omega')\, d_\mu(\Omega'),$$
$$P(\Omega) = \int k_-(\Omega,\Omega')Q(\Omega')\, d_\mu(\Omega'),$$
$$k_\pm(\Omega,\Omega') = \sum_{L=0}^{2j}(R^L_{00})^{\pm1}\sum_{m=-L}^{+L}Y^L_m(\Omega')[Y^L_m(\Omega')]^*. \tag{7.13}$$

This result generalizes immediately to the case of the multilevel coherent states for $r > 2$.

VIII. CONCLUSION

Coherent states for an ensemble of N identical r-level atoms[5] are defined in terms of the coherent state for a single r-level atom generated by a unitary transformation applied to its ground state [Eq. (3.4)]. The coherent states for an ensemble of identical r-level atoms are given as the tensor product of the individual atomic coherent states [Eq. (3.8)]. A linear combination of these r-level coherent states was shown to represent the r-mode harmonic oscillator coherent state [Eq. (4.3)], and in particular, the diagonal projectors for the r-level coherent states were shown to be Poisson-distributed in a phase average of the diagonal r-mode coherent-state projectors [Eq. (4.6)]. We have exploited the multimode-to-multilevel homomorphism (4.1) and used the multimode coherent states, the projectors, and the $\mathfrak{D}$ algebras as generating functions for multilevel coherent states, projectors, and $\mathfrak{D}$ algebras [(4.3), (4.5), and (4.13), respectively].

Diagonal projectors appear frequently in quantum statistical calculations, where one represents operators, and particularly density operators, using the coherent states as a basis.[6,7] The $\mathfrak{D}$ algebra for r-level systems allows the replacement of any operator by an equivalent c-number differential operator. The $\mathfrak{D}$ algebra has already proven extremely useful in mapping reduced atomic density operator equations of motion into differential equations for the quasi-probability distribution function $P(\Omega)$.[6,7]

For the purpose of establishing the $\mathfrak{D}$ algebra in the most familiar terms, we have specialized in Sec. V to operators and coherent states in $SU(2)$, with the natural parametrization[5] of the coherent states in the Bloch angles θ and φ. The $\mathfrak{D}$ operators corresponding to the angular momentum operators $\hat{J}^+$, $\hat{J}^-$, and $\hat{J}_z$ are given explicitly in these variables by Eqs. (5.11), (5.12), (5.15), and (5.16).

These results were applied in Sec. VI to the mapping of a rather large class of density operator equations of motion [cf. (6.7)] into c-number partial differential equations [cf. (6.1) and (6.9)]. This we call the P representation. In a similar way, a mapping of the equations of motion for the diagonal elements of the density operator in the coherent states, called the Q representation, was derived [cf. (6.2) and (6.14)]. It was shown explicitly that the P and Q representations are equivalent. A simple illustration was given in terms of the mapping of the Bloch equation of equilibrium statistical mechanics.

Finally, we have shown in Sec. VII the explicit connection between the quasi-probability distribution functions P and Q. The two functions were shown to be related to one another by a convolution integral.

APPENDIX A: CHANGE OF VARIABLE

It is necessary to express the r complex variables α_u and the derivatives $\partial/\partial\alpha_u$ $(u = 1,2,\ldots,r)$ in terms of the $r-1$ complex variables x_j, the derivatives $\partial/\partial x_j$ $(j = 2,3,\ldots,r)$, and the two real variables α, φ, and $\partial/\partial\alpha$, $\partial/\partial\varphi$. The relations between the variables themselves are straightforward:

$$\alpha_j = x_j\,\alpha\, e^{i\varphi},$$
$$\sum_{u=1}^{r}\alpha_u^*\alpha_u = \alpha^2, \tag{A1}$$
$$\alpha_1/|\alpha_1| = e^{i\varphi}.$$

The relationship between the derivatives is determined using the chain rule. For example,

$$\frac{\partial}{\partial \alpha_u}=\frac{\partial \alpha}{\partial \alpha_u}\frac{\partial}{\partial \alpha}+\frac{\partial \varphi}{\partial \alpha_u}\frac{\partial}{\partial \varphi}+\sum_{j=2}^{r}\left(\frac{\partial x_j}{\partial \alpha_u}\frac{\partial}{\partial x_j}+\frac{\partial x_j^*}{\partial \alpha_u}\frac{\partial}{\partial x_j^*}\right). \tag{A2}$$

The partial derivatives in (A2) and their complex conjugates are most conveniently determined implicitly, according to the following example:

$$\alpha^2=\sum_{n=1}^{r}\alpha_u^*\alpha_u,$$
$$2\alpha\, d\alpha=\sum_{u=1}^{r}(\alpha_u\, d\alpha_u^*+\alpha_u^*\, d\alpha_u), \tag{A3}$$
$$\frac{\partial \alpha}{\partial \alpha_u}=\frac{\alpha_u^*}{2\alpha},\quad \frac{\partial \alpha}{\partial \alpha_u^*}=\frac{\alpha_u}{2\alpha}.$$

The other partial derivatives may be determined in a similar way. It is possible to establish the following identities:

$$0=\frac{\alpha_1^*}{2\alpha^2}x_j+\frac{\alpha_1^*}{2\alpha_1^*\alpha_1}x_j+\frac{\partial x_j}{\partial\alpha_1},$$
$$\frac{\delta_{jk}}{\alpha e^{i\varphi}}=\frac{\alpha_k^*}{2\alpha^2}x_j+0+\frac{\partial x_j}{\partial\alpha_k},$$
$$0=\frac{\alpha_1}{2\alpha^2}x_j-\frac{\alpha_1}{2\alpha_1^*\alpha_1}+\frac{\partial x_j}{\partial\alpha_1^*}, \tag{A4}$$
$$0=\frac{\alpha_k}{2\alpha^2}x_j+0+\frac{\partial x_j}{\partial\alpha_k^*}.$$

A similar set of equations involving x_j^* is obtained from the set (A4) by complex conjugation. As a result we obtain, after a little algebra,

$$\frac{\partial}{\partial \alpha_1}=\tfrac{1}{2}x_1 e^{-i\varphi}\frac{\partial}{\partial\alpha}+\frac{1}{2x_1e^{i\varphi}}\frac{\partial}{\partial(i\varphi)}-\frac{x_1e^{-i\varphi}}{2\alpha}(\underline{x}\cdot\underline{\nabla}+\underline{x}^*\cdot\underline{\nabla}^*)-\frac{1}{2x_1e^{i\varphi}}(x\cdot\nabla-x^*\cdot\nabla^*), \tag{A5}$$

$$\frac{\partial}{\partial \alpha_j}=\tfrac{1}{2}x_j^* e^{-i\varphi}\frac{\partial}{\partial\alpha}+\frac{1}{\alpha e^{i\varphi}}\frac{\partial}{\partial x_j}-\frac{x_je^{-i\varphi}}{2\alpha}(\underline{x}\cdot\underline{\nabla}+\underline{x}^*\cdot\underline{\nabla}^*). \tag{A6}$$

In these equations, we have defined

$$\underline{x}\cdot\underline{\nabla}\pm\underline{x}^*\cdot\underline{\nabla}^*=\sum_{j=2}^{r}\left(x_j\frac{\partial}{\partial x_j}\pm x_j^*\frac{\partial}{\partial x_j^*}\right). \tag{A7}$$

Expressions for $\partial/\partial\alpha_u^*$ are obtained from (A5) and (A6) by complex conjugation.

APPENDIX B: THE $\tilde{\mathfrak{D}}$ OPERATORS

The $\tilde{\mathfrak{D}}$ operators are defined by the identity

$$\int d^2x_2\cdots d^2x_r P(\underline{x},t)\,\mathfrak{D}^L(\hat{X})\Lambda(\underline{x})=\int d^2x_2\cdots d^2x_r\,\Lambda(\underline{x})\tilde{\mathfrak{D}}^L(\hat{X})P(\underline{x},t), \tag{B1}$$

where $\hat{X}$ is an arbitrary operator in the Hilbert space of the r-level atoms and $\Lambda(\underline{x})$ is the coherent-state projector defined by Eq. (3.11).

A special set of $\tilde{\mathfrak{D}}$ operators appropriate for two-level coherent states was introduced in Sec. VI to construct the equation of motion for the quasiprobability-function $P(\Omega,t)$. Here we discuss the $\tilde{\mathfrak{D}}$ operators for multilevel systems and summarize their properties. First we observe that if $\hat{A}$ and $\hat{B}$ are elements in the multilevel Lie algebra, and r and s are arbitrary complex numbers, the following properties hold for the left operators:

$$\tilde{\mathfrak{D}}^L(r\hat{A}+s\hat{B})=r\tilde{\mathfrak{D}}^L(\hat{A})+s\tilde{\mathfrak{D}}^L(\hat{B}),\quad \text{linearity}$$
$$\tilde{\mathfrak{D}}^L(\hat{A}\hat{B})=\tilde{\mathfrak{D}}^L(\hat{A})\tilde{\mathfrak{D}}^L(\hat{B}),\quad \text{homomorphism}$$
$$\tilde{\mathfrak{D}}^L[\hat{A},\hat{B}]=[\tilde{\mathfrak{D}}^L(\hat{A}),\tilde{\mathfrak{D}}^L(\hat{B})],\quad \text{covariant commutation relations.} \tag{B2}$$

The right operators $\tilde{\mathfrak{D}}^R$ [also defined by Eq. (B1)] satisfy the properties

$$\tilde{\mathfrak{D}}^R(r\hat{A}+s\hat{B})=r\tilde{\mathfrak{D}}^R(\hat{A})+s\tilde{\mathfrak{D}}^R(\hat{B}),\quad \text{linearity}$$
$$\tilde{\mathfrak{D}}^R(\hat{A}\hat{B})=\tilde{\mathfrak{D}}^R(\hat{B})\tilde{\mathfrak{D}}^R(\hat{A}),\quad \text{antihomomorphism}$$
$$\tilde{\mathfrak{D}}^R[\hat{A},\hat{B}]=[\tilde{\mathfrak{D}}^R(\hat{B}),\tilde{\mathfrak{D}}^R(\hat{A})],\quad \text{contravariant commutation relations.} \tag{B3}$$

The construction of the explicit form of the $\tilde{\mathfrak{D}}$ operators is a straightforward but lengthy process which is based on an integration by parts of the left-hand side of Eq. (B1). In view of the properties (B2) and (B3), we only need to restrict our attention to the operators $\mathfrak{D}(E_{jk})$ given by Eqs. (3.14) and (3.16). We notice that the element of volume contains $2(r-1)$ differential elements $d(\text{Re}x_j)$ and $d(\text{Im}x_j)$ with $j\neq 1$. Accordingly, one must treat $\mathfrak{D}(E_{1k})$ separately from $\mathfrak{D}(E_{jk})$ $(j\neq 1)$. The operators $\tilde{\mathfrak{D}}(E_{1k})$ and $\tilde{\mathfrak{D}}(E_{jk})$, however, can be expressed in the compact form

$$\tilde{\mathfrak{D}}^L(E_{jk})=-x_k\frac{\partial}{\partial x_j}+x_kx_j^*[(N+r)+\tfrac{1}{2}(\underline{x}\cdot\underline{\nabla}+\underline{x}^*\cdot\underline{\nabla}^*)]-\delta_{kj}, \tag{B4}$$
$$\tilde{\mathfrak{D}}^R(E_{jk})=[\tilde{\mathfrak{D}}^L(E_{jk}^\dagger)]^*,$$

where $\partial/\partial x_1$ is defined in the usual way, i.e.,

$$\frac{\partial}{\partial x_1}=\frac{1}{2x_1}\,(\underline{x}\cdot\underline{\nabla}-\underline{x}^*\cdot\underline{\nabla}^*). \tag{B5}$$

The algorithm connecting $\mathfrak{D}$ and $\tilde{\mathfrak{D}}$ operators can be summarized by the following elementary replacements:

$$N\to N+r,$$

$$\frac{\partial}{\partial x_j}\to-\frac{\partial}{\partial x_j}\quad (j=1\cdots r), \tag{B6}$$

and add $-\delta_{k,j}$.

Note added in proof. Many-atom systems have been previously described by Bonifacio, Kim, and Scully in terms of coherent boson states.[16] The formal connection between their representation and that of Arecchi *et al.*[5] is provided by Eqs. (4.3) and (4.7) specialized to the case $\underline{\alpha}\equiv(\alpha_1,\alpha_2)$ and $\underline{x}=(\cos\theta/2,\, e^{-i\varphi}\sin\theta/2)$. It is worthwhile to point out that, while the atomic coherent states $|\theta,\varphi\rangle_J$ correspond to a fixed specification of the total cooperation number (or angular momentum), the states $|\alpha_1,\alpha_2\rangle$ correspond to an undetermined value of the angular momentum, i.e.,

$$|\alpha_1,\alpha_2\rangle=\sum_J C_J(\alpha_1,\alpha_2)\,|\theta,\varphi\rangle_J\,.$$

The coefficients $|C_J|^2$ form a Poisson distribution with parameter $|\alpha_1|^2+|\alpha_2|^2$. The relative dispersion of the angular momentum $\Delta J/J$ for a state $|\alpha_1,\alpha_2\rangle$ is $(|\alpha_1|^2+|\alpha_2|^2)^{-1/2}$. We are grateful to Professors Bonifacio and Scully for their stimulating comments.

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