

Detecting quantum resources in a semi-device-independent framework

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We investigate whether one can detect the presence of a quantum resource in some operational task or equivalently whether every quantum resource provides an advantage over its free counterpart in some black box scenarios where one does not have much information about the devices. For any dimension d , we find that for any resource theory with less than d^2 number of linearly independent free states or free operations, there exist correlations that can detect the presence of a quantum resource. For this purpose, we introduce the framework for detecting quantum resources semidevice independently by considering the prepare-and-measure scenario with the restriction on the dimension of the quantum channel connecting the preparation box with the measurement box. We then explicitly construct witnesses to observe the presence of various quantum resources. We expect these results will open avenues for detecting and finding uses of quantum resources in general operational tasks.

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Introduction. With the emergence of quantum technologies in the last few years, it has become clear that entanglement is an indispensable resource for a range of intriguing technological applications that cannot be matched by classical means [1]. The applications of entanglement are practically endless [1], spanning from quantum teleportation [2], to the secure distribution of cryptographic keys among several parties [3]. However, it has become clear in recent years that entanglement is not the only resource for quantum technological applications, and there are other applications that rely on other features of quantum systems, such as coherence [4,5], imaginarity [6–9], asymmetry [10], contextuality [11–13], purity [14–16], and so on.

To investigate the roles of these resources in a wide range of information-processing tasks, we must first understand their characteristics. Quantum resource theories [17,18] proposed in quantum information science provide a unified approach for studying all such resources, as well as their roles and limitations in various quantum technological applications. While the exploration of entanglement characterization began in the early 1990s [1,2,19–21], it was not until 2008 when Gour and Spekkens introduced the concept of resource theory and established a comprehensive framework in their investigation of the resource theory of asymmetry [10]. Soon after, a number of additional resource theories—including coherence [4,5], imaginarity [6–9], purity [14–16], stabiliser quantum

computation [22], quantum measurements [23], and numerous others—were introduced in the literature [18]. As a consequence, a generic framework for quantum resource theories has been established [18,24,25].

After identifying all of these resources, it is critical to observe their actual presence in practical scenarios. One of the most common methods for observing this is state tomography. However, in general, this method is not economical as it involves reconstructing the complete information about the given quantum state. Moreover, one also has to trust their measurement device. Similarly in the subchannel discrimination task, it was shown in Ref. [26] that any resource state belonging to a convex resource theory provides an advantage over the free states. This result was then generalized to dynamical resource theories [27], generalized probabilistic theories [28], and nonconvex resource theories [29]. However, in all these works, one must trust the subchannels along with the quantum state.

Although the above works show that quantum resources show an advantage over their free counterpart, the element of trust in these scenarios makes the problem of detecting resources not applicable in their formalism. For instance, if one already knows the quantum state then there is no point in detecting the resource inside the device. In this work, we pose a stronger question. Can quantum states or operations be detected to be resourceful without trusting them or any other operation involved in the test? To put it simply, we enquire whether one can operationally detect the resource of unknown quantum states and operations. For instance, violation of Bell inequalities serves as a witness for entanglement detection [30] where one does not trust any state or measurements. Similarly, assuming multiple independent sources, it was shown in Refs. [31,32] that one can detect whether any of the quantum states or measurements need to be imaginary, thus providing a way to detect imaginarity without trusting any state or measurements.

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Here, we focus on one of the well-known semi-device independent scenarios known as the prepare-and-measure (PM), where one assumes an upper bound on the dimension of the quantum channel connecting the preparation and measurement [33]. In a practical scenario, it can be imagined as a communication task involving two separate boxes, a preparation box responsible for creating quantum states, which are then transmitted to a measurement box. A fascinating fact of this scenario is that we can perceive those boxes as black boxes, so removing the need for trust in them. The prepare-and-measure scenario has been employed in the literature for a wide range of applications, including random access codes [34,35], informational principles [36,37], key distribution [38,39], dimension witness [33,40], self-testing [41], and more.

As we are concerned here with detecting resourceful states and operations, we treat operations as generalized instruments that can generate quantum or classical outputs. Restricting the quantum channel to transmit d -dimensional states, we then show that for any resource theory with strictly less than d^2 number of linearly independent free states or free operations, one can always detect whether the preparation box produces at least one resource state and the operation box performs at least one resourceful operation. Then considering particular resource theories, we construct explicit witnesses to detect their presence inside the preparation as well as the measurement box.

Quantum resource theories. Any quantum resource theory is built on two basic components: a set of free states and a set of free operations [18]. The quantum states that do not belong to the set of free states are referred to as resource states. Any resource theory must adhere to the fundamental constraint that the set of free operations must not be able to generate resource states from the set of free states. Hence, quantum resource theories allow us to study quantum information processing tasks under a restricted set of operations. As an example, in the resource theory of entanglement, separable states denote free states, local operations and classical communications (LOCC) denote free operations, and information processing tasks are studied under LOCC [1].

Let us now restrict to states and operations acting on \mathbb{C}^d for any $d \geq 2$, where d is the dimension of the Hilbert space. The free states are denoted as σ_d and the collection of these states is denoted by the set \mathcal{S}_d . Now, consider a set of linearly independent density matrices $|\Gamma_{d,i}\rangle\langle\Gamma_{d,i}| \in \mathcal{S}_d$ using which any state σ_d can be written via their linear combination. Equivalently, the elements of the set $\mathcal{S}_{\text{ext},d} = \{|\Gamma_{d,i}\rangle\langle\Gamma_{d,i}|\}_i$ forms a basis of \mathcal{S}_d . However, it is important to note here that \mathcal{S}_d might not contain all the density matrices which can be expressed as linear combinations of $|\Gamma_{d,i}\rangle\langle\Gamma_{d,i}|$. Moreover, it is also important to realize that not every linear combination of these states would result in a valid density matrix. Furthermore, the set $\mathcal{S}_{\text{ext},d}$ might not be unique. Let us now denote the cardinality of $\mathcal{S}_{\text{ext},d}$ as $n(\mathcal{S}_{\text{ext},d})$. For instance, in the resource theory of coherence [5], the set $\mathcal{S}_{\text{ext},d}$ is given by $\mathcal{S}_{\text{ext},d} = \{|0\rangle\langle 0|, |1\rangle\langle 1|, \dots, |d-1\rangle\langle d-1|\}$ and thus $n(\mathcal{S}_{\text{ext},d}) = d$. Similarly, for the resource theory of athermality or asymmetry [10,42], we have that $n(\mathcal{S}_{\text{ext},d}) \leq (d-1)^2 + 1$. Likewise, for the resource theory of magic or imaginarity in dimension two [7,22], we have that $n(\mathcal{S}_{\text{ext},2}) =$

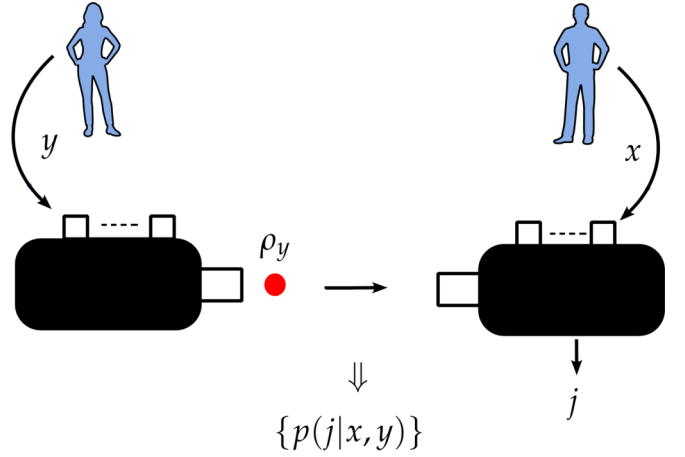


FIG. 1. We consider two parties, a producer and an observer, each situated in spatially separated labs. They can communicate solely via a quantum channel supporting a quantum system of dimension d . The producer freely selects y inputs to prepare quantum states ρ_y , which are then sent to the observer. The observer, in turn, selects x inputs to apply quantum operations on the received states, yielding outcomes j . After all the runs of the experiment, one obtains the probability distribution $\{p(j|x, y)\}$.

3. For any dimension d , the maximum cardinality of this set in quantum theory is $n_{\text{max}}(\mathcal{S}_{\text{ext},d}) = d^2$ [43].

Similarly, one can also characterize free operations δ_d based on the above procedure. Equivalently, we express the operation δ_d using its Kraus decomposition as $\delta_d \equiv \{K_{j,\delta_d}^\dagger K_{j,\delta_d}\}$ which might not be unique. The operator $K_{j,\delta_d}^\dagger K_{j,\delta_d}$ corresponds to the positive operator-valued measure (POVM) element of the operation δ_d . Let us denote the set of these free operations as \mathcal{O}_d which is composed of all possible POVM elements of every free operation acting on \mathbb{C}^d . Let us now consider a set of linearly independent positive operators $|\Delta_{d,i}\rangle\langle\Delta_{d,i}| \in \mathcal{O}_d$ such that they form a basis of \mathcal{O}_d . Let us denote the set of these linearly independent operators as $\mathcal{O}_{\text{ext},d} = \{|\Delta_{d,i}\rangle\langle\Delta_{d,i}|\}_i$. The cardinality of $\mathcal{O}_{\text{ext},d}$ is denoted as $n(\mathcal{O}_{\text{ext},d})$. Any operator in \mathcal{O}_d can be expressed as a linear combination of elements in $\mathcal{O}_{\text{ext},d}$. For instance, in coherence resource theory $n(\mathcal{O}_{\text{ext},d}) = d$ [5], for the resource theory of asymmetry $n(\mathcal{O}_{\text{ext},d}) \leq (d-1)^2 + 1$ [10], and for imaginarity in dimension two [7], $n(\mathcal{O}_{\text{ext},2}) = 3$. Again, in quantum theory for any dimension d : $n_{\text{max}}(\mathcal{O}_{\text{ext},d}) = d^2$. Any resource theory defined on a d -dimensional Hilbert space, from now on will be denoted as $\{\mathcal{S}_d, \mathcal{O}_d\}$.

Prepare-and-measure scenario. We consider here a PM scenario with two parties, producer and observer in two different spatially separated labs who can communicate only via a quantum channel that can support a quantum system bounded by the dimension d . The producer can freely choose y inputs based on which it prepares quantum states ρ_y and sends it to the observer. The observer now freely chooses x inputs based on which quantum operations (generalised quantum instruments) are applied on the incoming quantum state to obtain the outcomes j . This is illustrated in Fig. 1.

From the above setup, one obtains the probability distribution $\vec{p} = \{p(j|x, y)\}$ where $p(j|x, y)$ denotes the probability of obtaining outcome j by the operation box given the inputs

x, y . In quantum theory, the inputs y generate a qudit state ρ_y acts on \mathbb{C}^d and the operation box does the operation denoted by the POVM elements as $\mathbb{O}_x = \{K_{j,x}^\dagger K_{j,x}\}$ such that they act on \mathbb{C}^d . Now, the probabilities $p(j|x, y)$ are given by

$$p(j|x, y) = \text{Tr}(K_{j,x}^\dagger K_{j,x} \rho_y). \quad (1)$$

Notice that in the above description, we treat operations in the same way as measurements in quantum theory. In general, it may not be possible to obtain these probabilities in a practical setting. However, we show below that treating operations similar to measurement allows us to detect the presence of a quantum resource in a semi-device-independent way. To detect resources in an experiment, one can restrict to measurements and all below analysis follows straightaway. Furthermore, the correlations \vec{p} in general are not convex due to the dimension constraint. Consequently, finding general results for semi-device-independent schemes is highly nontrivial.

Results. Using the above framework, let us now establish two no-go theorems that apply to a large class of quantum resource theories. The first one concerns the case when the set of linearly independent free states is strictly less than d^2 .

Theorem 1. Consider a resource theory $\{\mathcal{S}_d, \mathcal{O}_d\}$ such that the cardinality of the $\mathcal{S}_{\text{ext},d}$ is N with $N < d^2$. Consider now the PM scenario described above with $y = 0, 1, \dots, N$ and $x = 0, 1, \dots, d^2 - 1$ and any $j \geq 2$ such that one obtains $\vec{p} = \{p(j|x, y)\}$. Then, there exists at least one probability distribution \vec{p}_Q in quantum theory that cannot be realized by free states in \mathcal{S}_d .

The proof of the above theorem can be found in Appendix A of Ref. [44]. Let us extend the above theorem for resource theories with the number of linearly independent free operations strictly less than d^2 .

Theorem 2. Consider a resource theory $\{\mathcal{S}_d, \mathcal{O}_d\}$ such that the cardinality of the $\mathcal{O}_{\text{ext},d}$ is N such that $N < d^2$. Consider now the PM scenario described above with $y = 0, 1, \dots, d^2 - 1$ and $x = 0, 1, \dots, N$ and any $j \geq 2$ such that one obtains $\vec{p} = \{p(j|x, y)\}$. Then, there exists at least one probability distribution \vec{p}_Q in quantum theory that can not be realized by free operations in \mathcal{O}_d .

We refer to Appendix A of Ref. [44] for the proof of the aforementioned theorem. Theorem 1 shows that one can always detect the presence of a resource state corresponding to the resource theory $\{\mathcal{S}_d, \mathcal{O}_d\}$ in the PM scenario if $n(\mathcal{S}_{\text{ext},d}) < d^2$. Similarly, Theorem 2 shows that one can always detect the presence of a resource operation corresponding to the resource theory $\{\mathcal{S}_d, \mathcal{O}_d\}$ in the PM scenario if $n(\mathcal{O}_{\text{ext},d}) < d^2$. One can straightaway infer from the above results that if one wants to detect the presence of a resource state as well as operation simultaneously, then one needs to consider the PM scenario with $y, x = 0, 1, \dots, d^2 - 1$. Thus, we obtain the following corollary of Theorems 1 and 2.

Corollary 1. Consider a resource theory $\{\mathcal{S}_d, \mathcal{O}_d\}$ such that the cardinality of the $\mathcal{S}_{\text{ext},d}$ and $\mathcal{O}_{\text{ext},d}$ is less than d^2 . Consider now the PM scenario described above with $x, y = 0, 1, \dots, d^2 - 1$ and any $j \geq 2$ such that one obtains $\vec{p} = \{p(j|x, y)\}$. Then, there exists at least one probability distribution \vec{p}_Q in quantum theory that cannot be realized by either the free states in \mathcal{S}_d or operations in \mathcal{O}_d . Thus, observing \vec{p}_Q allows one to conclude that at least one of the preparations and operations is resourceful.

Let us remark here that the number of inputs x, y in the above corollary might not be optimal. As we will see below for particular resource theories, there might also exist probability distributions with a lower number of inputs x, y such that at least one of the preparations and operations is resourceful. For this purpose, we construct witnesses which is a function of the probability distribution \vec{p} of the form

$$W_{\mathcal{S}_d, \mathcal{O}_d} = f(\vec{p}) \leq \beta_{\mathcal{S}_d, \mathcal{O}_d}, \quad (2)$$

where $\beta_{\mathcal{S}_d, \mathcal{O}_d}$ is the maximum value attainable using the free states and operations in the resource theory $\{\mathcal{S}_d, \mathcal{O}_d\}$. Consequently, the presence of a resource can be verified if the above bound is violated. Furthermore, from a practical perspective it is usually beneficial to construct witnesses as one can never exactly obtain the theorised correlations in an experiment and witnesses provide a gap that can tolerate such errors.

Witnesses. Let us begin by giving a generic construction of witnesses that follows from Corollary 1. For this purpose, we consider the probability distribution \vec{p}_Q which cannot be attained by free states and operations. As $x, y = 0, 1, \dots, d^2 - 1$, the witness is given by

$$W_{\mathcal{S}_d, \mathcal{O}_d} = - \sum_{x,y=0}^{d^2-1} |p(0|x, y) - p_Q(0|x, y)| \leq \beta_{\mathcal{S}_d, \mathcal{O}_d}, \quad (3)$$

where $\beta_{\mathcal{S}_d, \mathcal{O}_d} = -\varepsilon$ and $\varepsilon \rightarrow 0^+$. The value of the above functional, when at least one state and operation is resourceful, is 0. Consequently, the gap between the values obtainable using free states and operations and quantum theory is close to 0 and thus highly susceptible to experimental errors.

Let us now construct witnesses tailored to specific resource theories that can detect the presence of concerned quantum resources that provide a considerable gap between the maximum value obtainable using free states and operations to quantum theory. In the first example, we construct a witness for the resource theory of coherence [4,5] in the PM scenario when the channel is limited to transmit qubits. In this theory free states and free operations are known as incoherent states and incoherent operations, respectively. Here, we consider the PM scenario with $x = 0, 1$ and $y = 0, 1, 2$. In this scenario, the witness to observe coherence can be constructed as

$$W_C = p(0|0, 0) + p(0|0, 1) + p(0|1, 0) + p(1|1, 1) + p(1|0, 2) \leq \beta_C, \quad (4)$$

where β_C can be evaluated by considering incoherent states and operations [5]. The Kraus operator of any incoherent operation $\{K_j\}$ is given by $K_j = \sum_{i=0}^{d-1} c_{i,j} |f(i)\rangle \langle i|$ where $f(i)$ is a function with domain and range being the labels of the incoherent basis vectors and thus the corresponding POVM is given by $K_j^\dagger K_j = \sum_{i=0}^{d-1} |c_{i,j}|^2 |i\rangle \langle i|$. Consequently, restricting to $d = 2$ we find that $\beta_C = 4$. Any violation of this bound would imply that at least one of the states and operations is coherent. Now, consider the following states:

$$\rho_0 = |0\rangle \langle 0|, \quad \rho_1 = |+\rangle \langle +|, \quad \rho_2 = |\bar{1}\rangle \langle \bar{1}|, \quad (5)$$

and measurements,

$$K_{0,0} = |\bar{0}\rangle \langle \bar{0}|, \quad K_{1,0} = |\bar{1}\rangle \langle \bar{1}|, \\ K_{0,1} = |\bar{+}\rangle \langle \bar{+}|, \quad K_{1,0} = |\bar{-}\rangle \langle \bar{-}|, \quad (6)$$

where $\{|\bar{0}\rangle, |\bar{1}\rangle\}$ are the eigenvectors of $\frac{\sigma_x + \sigma_z}{\sqrt{2}}$ and $\{|\bar{+}\rangle, |\bar{-}\rangle\}$ are the eigenvectors of $\frac{\sigma_x - \sigma_z}{\sqrt{2}}$. Here, σ_x and σ_z denote respective Pauli matrices. Then, we obtain that $W_C = 1 + 4 \cos^2(\pi/8) = 3 + \sqrt{2}$, which violates the upper bound, i.e., $\beta_C = 4$. Since the resource theories of asymmetry [10] and athermality [42,45] are equivalent to the coherence theory [4,5] for $d = 2$, the same finding can be readily applied to both theories. Moreover, we identify that the above witness W_C in Eq. (4) can be generalized to the case where the channel can transmit qudits. For this purpose, we consider PM scenario in Fig. 1 such that the number of inputs in the preparation box is d^2 and the operation box is two and is given as $y \equiv y_0 y_1$ where $y_0, y_1 = 0, 1, \dots, d-1$ and $x = 0, 1$. Based on the quantum random access codes (QRAC) for arbitrary dimensions introduced in Ref. [35], we represent the witness of coherence in arbitrary dimensions as

$$W_{C,d} = \sum_{x=0,1} \sum_{y_0, y_1=0}^{d-1} p(j = y_x | x, y_0 y_1) \leq \beta_{C,d}, \quad (7)$$

where $\beta_{C,d}$ is the maximum value of $W_{C,d}$ attainable using incoherent states and operations. In the Supplemental Material [44], we show that $\beta_{C,d} = d^2 + d$. Consider now the following observables, $Z_d = \sum_{i=0}^{d-1} \omega^i |i\rangle\langle i|$, $X_d = \sum_{i=0}^{d-1} |i\rangle\langle i+1|$ which are d -dimensional generalisation of the Pauli Z, X observables, respectively, with $\omega = e^{2\pi i/d}$. Then, putting the states $|\psi_{y_0 y_1}\rangle = X_d^{y_0} Z_d^{y_1} |\psi_{00}\rangle$ where $|\psi_{00}\rangle = \frac{1}{\sqrt{2\sqrt{d}(1+\sqrt{d})}}((\sqrt{d}+1)|0\rangle + \sum_{i=1}^{d-1} |i\rangle)$ along with the operations Z_d, X_d corresponding to input $x = 0, 1$, gives us $W_{C,d} = d^2 + d\sqrt{d}$. This further shows us that the resource that gives the quantum advantage in QRAC is quantum coherence.

In the next example, we consider resource theory of imaginarity [7,8] and construct a witness for the detection of imaginarity. Again, we take the PM scenario with $x = 0, 1, 2$ and $y = 0, 1, 2, 3$ and $d = 2$, and the witness to observe imaginarity can be designed as

$$W_I = W_C + p(0|2, 3) - |p(0|2, 0) - p(1|2, 0)| - |p(0|2, 1) - p(1|2, 1)| \leq \beta_I, \quad (8)$$

where β_I is the maximal value attainable considering real states and operations [7]. Consequently, to evaluate β_I , we recall that any state $\rho \in \mathbb{C}^2$ can be expressed as $\rho = (\mathbb{1} + r_x \sigma_x + r_y \sigma_y + r_z \sigma_z)/2$, where σ_i are Pauli matrices and r_i 's are components of the Bloch vector \vec{r} . Since in the resource theory of imaginarity free states satisfy $\rho = \rho^*$, one can readily obtain that any free state can be expressed as $\rho = 1/2(\mathbb{1} + r_z \sigma_z + r_x \sigma_x)$. Similarly, for the POVM elements, we have $K_j^\dagger K_j = m_0 \mathbb{1} + m_z \sigma_z + m_x \sigma_x$. Using these representations we evaluate that $\beta_I \leq 5$. Any violation of this bound would imply that at least one of the states and measurements must be complex. Now, consider the following states:

$$\begin{aligned} \rho_0 &= |0\rangle\langle 0|, & \rho_1 &= |+\rangle\langle +|, \\ \rho_2 &= |\bar{0}\rangle\langle \bar{0}|, & \rho_3 &= |+_y\rangle\langle +_y|, \end{aligned} \quad (9)$$

and measurements

$$\begin{aligned} K_{0,0} &= |\bar{0}\rangle\langle \bar{0}|, & K_{1,0} &= |\bar{1}\rangle\langle \bar{1}|, & K_{0,1} &= |\bar{+}\rangle\langle \bar{+}|, \\ K_{1,0} &= |\bar{-}\rangle\langle \bar{-}|, & K_{0,2} &= |+_y\rangle\langle +_y|, & K_{1,2} &= |-_y\rangle\langle -_y|, \end{aligned} \quad (10)$$

where $|\pm_y\rangle = 1/\sqrt{2}(|0\rangle \pm i|1\rangle)$. Using the above states and operations one can obtain $W_I = 4 + \sqrt{2}$, which violates the upper bound given in Eq. (8). In Appendix B of Ref. [44], we also find specific witnesses for resource theory of purity and magic.

Discussions. In the above work, we considered the prepare-and-measure scenario where the measurement box can perform generalized quantum instruments. By limiting the quantum channel to transmit d -dimensional states, we showed that for any resource theory, be it convex or nonconvex, with fewer than d^2 linearly independent free states or operations, it is always possible to detect if the preparation box produces at least one resource state and if the operation box performs at least one resourceful operation. While our focus in this letter has been on single-partite systems, it is worth noting that our method can be readily extended to multipartite systems, thus encompassing the resource theory of entanglement.

In subchannel discrimination task [26], it was shown that for every convex resource theory, the advantage obtained is directly related to the robustness measure of the state, which followed from the fact that all the quantum resources share the same linear witness. In our case, such a general relation cannot be obtained due to the simple fact that one might not always obtain a linear witness to observe the advantage in the PM scenario [For instance, see the witness of imaginarity in Eq. (8)]. Moreover, to detect resources of unknown states or operations in the PM scenario, one would require more than one state, operations and witnesses to detect resources which in general will be different for different resource theories. Furthermore, the optimization again has to be carried over the states (unlike subchannel discrimination), so the advantage cannot be related to the robustness measure of some states. However, it remains an interesting open problem whether the advantage obtained is related to some operational quantity. This would further address the problem of finding sufficient conditions to violate a witness in the PM scenario.

A practical application of our results can be illustrated in the following scenario: Imagine an experimental laboratory intending to purchase a preparation box guaranteed to produce at least one resource state, and an operation box that performs at least one resourceful operation. Suppose the experimenter has access to a quantum channel capable of supporting at max qudit systems. In that case, they can verify the capabilities of these boxes without needing to trust them or have any prior knowledge about their workings.

Our work opens up various interesting questions in the PM scenario. As previously mentioned in Fig. 1, our objective was to confirm the presence of resources within the two boxes. Now, consider a situation in which the boxes are intended to generate resource states and operations only. It would be interesting to verify if the PM scenario can confirm whether the boxes are functioning as intended, i.e., they are not producing any free state or operation. The following is another intriguing follow-up question to consider: Consider now a theory in dimension d where free states are given by the convex combination of d^2 linearly independent vectors. An instance of such linearly independent vectors is provided in Ref. [43]. Let us for the time being assume that it is a valid resource theory. Even if such a set of free states would not

include every quantum state in dimension d , it is not clear whether such a resource would be detected using the above results in the PM scenario.

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