Informational nonequilibrium concentration

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Informational contributions to thermodynamics can be studied in isolation by considering systems with fully degenerate Hamiltonians. In this regime, being in nonequilibrium (termed *informational nonequilibrium*) provides thermodynamic resources, such as extractable work, *solely* from the information content. The usefulness of informational nonequilibrium creates an incentive to obtain more of it, motivating the question of how to *concentrate it*: can we increase the local informational nonequilibrium of a product state $\rho \otimes \rho$ under a global closed system (unitary) evolution? We fully solve this problem analytically, showing that it is *impossible* for two-qubits, and it is always possible to find states achieving this in higher dimensions. Specifically for two-qutrits, we find that there is a single unitary achieving optimal concentration for *every* state, for which we uncover a *Mpemba-like effect*. We further discuss the notion of *bound resources* in this framework, initial global correlations' ability to *activate* concentration, and applications to concentrating purity and intrinsic randomness.

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I. INTRODUCTION

Information is central to our modern understanding of thermodynamics [1]. To model a system's thermodynamic behaviors, one must consider both its energy and information contents [2]. In fact, control over one allows influence over the other: by manipulating a system's information content, one can cool the system down via algorithmic cooling [3], convert bits into work via Szilard engine [4], or transmit energy [5]. Alternately, by consuming energy, one can manipulate encoded information, e.g., by erasing information via Landauer's principle [6] or performing computation [7–9].

The informational contributions to thermodynamics can be isolated from the energetic ones (allowing them to be independently studied and quantified) by considering fully degenerate Hamiltonians [10]. In the absence of energy gaps, thermodynamic transformations must arise from information processing. In this regime, thermal equilibrium is described by the maximally mixed state, and all other states are considered to be in *informational nonequilibrium*. This notion coincides with purity when considering a fixed system size [10], allowing purity also to be studied within this framework. By understanding this special case of thermodynamics, insights can be gained into the general case, where both energy and information are considered.

Given that informational nonequilibrium (and hence purity) is a resource in thermodynamics [10–15], it is natural to want to increase the amount one has. Such questions were previously considered via *resource distillation* [11], where several copies of a less resourceful state are converted into fewer copies of a more resourceful state, with the help of an arbitrary (possibly infinite) supply of free states. Additionally, these protocols may also require global operations that act simultaneously on many copies of the system and free states. Practically implementing resource distillation can, therefore, be highly expensive in labs, well beyond the ability of existing quantum technological platforms (e.g., Refs. [16–18]). As being able to concentrate multiple noisy objects into a single, more resourceful object is crucial for quantum technologies, we here focus on the *smallest* (and hence *most practically feasible*) setting for which resource enhancement can be studied. We coin the term *resource concentration* to describe this paradigm: given two copies of a state, $\rho_A \otimes \rho_B$, can we enhance the informational nonequilibrium in A via a global unitary? The aim is thus to concentrate as much of the resource as possible locally, using only globally resource preserving operations. Note, we define the task without access to any free states: it is a closed system dynamics, which will also allow us to keep a complete accounting of the information changes. This differs from resource distillation protocols that only concern the input and output states, ignoring any "junk" produced in the process.

Here, we fully solve the resource concentration problem for informational nonequilibrium and purity and fur-

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ther investigate the concentration of intrinsic randomness [19]. Unexpectedly, our framework uncovers a phenomenon in resource concentration that is similar to the Mpemba effect [20–24].

II. PURITY AND INFORMATIONAL NONEQUILIBRIUM

Consider a quantum system with dimension $d < \infty$. Qualitatively, *purity* of this system is a physical property about whether it is in a pure state. However, *informational nonequilibrium* of this system is another physical property about whether it is *not* in the maximally mixed state $\mathbb{I}^{(d)}/d$, where $\mathbb{I}^{(d)}$ is the identity operator [the superscript "(d)" denotes the dimension dependence whenever needed]. Hence, purity is *independent* of the actual physical dimension d, while informational nonequilibrium is, by definition, *dependent* on d. It is thus clear that informational nonequilibrium and purity are two *different* properties. In a more quantitative language, if the system is in a state ρ, then its purity captures how close ρ is to some pure state, while its informational nonequilibrium quantifies how distant ρ is from $\mathbb{I}^{(d)}/d$. As an example, as is well known, most practically realisable "qubits" are actually two levels of a multilevel system. The state $\mathbb{I}^{(2)}/2$ of those two levels is not a resource if one stays in that subspace, but becomes a resource if one starts accessing other levels. Its purity is, of course, the same. Since our aim is to study how the informational nonequilibrium can be increased, we have to steer clear of the trivial way that consists in just redefining the dimension. In all that follows, the dimension is fixed, and we aim at increasing the resource by quantum operations on states.

Quantifying informational nonequilibrium

Before stating our central question, we need to *quantify* informational nonequilibrium. To this end, for a d-dimensional state ρ, we adopt the following figure-of-merit:

$$\mathcal{P}(\rho) := D_{\max}(\rho \,\|\, \mathbb{I}^{(d)}/d). \tag{1}$$

Here, $D_{\max}(\rho \,\|\, \sigma) := \log_2 \min\{\lambda \geqslant 0 \,|\, \rho \leqslant \lambda\sigma\}$ is the *max-relative entropy* [25], widely used for its numerical feasibility and operational relevance [26–29]. Explicitly,

$$\mathcal{P}(\rho) = \log_2 d\|\rho\|_\infty. \tag{2}$$

Hence, $\mathcal{P}$ quantifies informational nonequilibrium by checking ρ's most "nonmaximally mixed" eigenvalue. In general, $\mathcal{P}$ can also act as a dimension-*dependent* measure of purity. When the system is a qubit, up to a unitary, any state reads $\rho = \|\rho\|_\infty|0\rangle\langle 0| + (1 - \|\rho\|_\infty)|1\rangle\langle 1|$ with $\|\rho\|_\infty \geqslant 1/2$. Thus, any nondecreasing function of $\|\rho\|_\infty$ can quantify purity.

III. INFORMATIONAL NONEQUILIBRIUM CONCENTRATION PROBLEMS

A. Defining the concentration problems

Now, we can define *informational nonequilibrium concentration problems* (INCPs). For a d-dimensional state ρ, its INCP asks: *is there a two-qudit unitary U_{AB} achieving $\mathcal{P}(\sigma_A^{(U)}) > \mathcal{P}(\rho_A)$, where $\sigma_A^{(U)} := \mathrm{tr}_B[U_{AB}(\rho_A \otimes \rho_B)U_{AB}^\dagger]$?* See also Fig. 1. Namely, can one use a closed system operation in the two-qudit system to concentrate informational nonequilibrium into the first system (A)? When this is possible, we say the unitary U is a solution to the state-dependent INCP of the state ρ. Note, in this work, subscripts denote the subsystems the operators live.

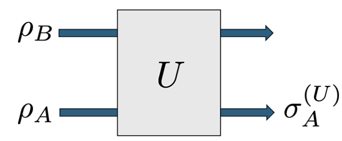

FIG. 1. Informational nonequilibrium concentration problems. For two copies of a state ρ, we study whether one can use a global unitary U_{AB} to enhance the informational nonequilibrium locally in A, in the sense that $\mathcal{P}(\sigma_A^{(U)}) > \mathcal{P}(\rho_A)$ [$\mathcal{P}$ is defined in Eq. (1)].

Here, we only consider closed system dynamics (i.e., unitary), rather than channels (i.e., completely positive trace-preserving linear maps [30]). In addition to allowing for detailed accounting of information changes in the system, this prevents situations in which a channel could discard a state and replace it with a pure one (i.e., using the environment as a "purity bank"). Moreover, we only assess the ability to concentrate informational nonequilibrium when given two copies of the *same* state. This is the simplest instance of an INCP, and relaxation of this restriction is left for future work. As a remark, INCPs are related to (but different from) algorithmic cooling [3]. More precisely, INCPs can be considered as a specific form of algorithmic cooling in which both the target system and machine are initially in the same state and only unitary evolution is allowed to achieve the cooling. Applying these restrictions allows for a complete analytical solution to the optimal algorithmic cooling protocol to be found when considering the figure-of-merit defined in Eq. (1).

To solve INCPs, we first present a result including INCPs as special cases. With a given bipartite system AB with (not necessarily equal) local dimensions d_A, d_B, we define the map

$$\mathcal{E}_{(U_{AB},\eta_B)}(\rho_A) := \mathrm{tr}_B[U_{AB}(\rho_A \otimes \eta_B)U_{AB}^\dagger], \tag{3}$$

where ρ_A (η_B) is with dimension d_A (d_B), and U_{AB} is a unitary acting on AB. Then, in Appendix A, we show that

Result 1. Given d_A, d_B, then, for every U_{AB}, ρ_A, η_B, we have

$$\max_{U_{AB}} 2^{\mathcal{P}[\mathcal{E}_{(U_{AB},\eta_B)}(\rho_A)]} = \max_{\Pi_{AB}^{(d_B)}} d_A \mathrm{tr}\big[\Pi_{AB}^{(d_B)}(\rho_A \otimes \eta_B)\big]. \tag{4}$$

Here "$\max_{\Pi_{AB}^{(d_B)}}$" maximizes over all rank-d_B projector in AB.

B. Solving informational nonequilibrium concentration problems

Result 1 fully quantifies the optimal performance of relocating informational nonequilibrium from B to A. We can solve an INCP by computing the following difference:

$$\begin{aligned}\Delta\mathcal{P}(\rho) &:= \max_{U_{AB}} \mathcal{P}[\mathcal{E}_{(U_{AB},\rho_B)}(\rho_A)] - \mathcal{P}(\rho_A) \\ &= \max_{\Pi_{AB}^{(d_B)}} \log_2\big(\mathrm{tr}\big[\Pi_{AB}^{(d_B)}(\rho_A \otimes \rho_B)\big]/\|\rho_A\|_\infty\big),\end{aligned} \tag{5}$$

where we set $\eta = \rho$ in Result 1. By solving the above optimization, once the optimal value is positive, informational nonequilibrium can be concentrated in A with the initial state $\rho_A\otimes\rho_B$; namely, ρ's INCP has a solution. To further solve this, let us write $\rho=\sum_{i=0}^{d-1}a_i^{\downarrow}|i\rangle\langle i|$, where $d=d_A=d_B$ and $a_i^{\downarrow}\geqslant a_{i+1}^{\downarrow}$ for every i. Then we have

$$\max_{\Pi_{AB}^{(d)}}\mathrm{tr}\big[\Pi_{AB}^{(d)}(\rho_A\otimes\rho_B)\big]=\max_{\Pi_{AB}^{(d)}}\sum_{ij}a_i^{\downarrow}a_j^{\downarrow}\langle ij|\Pi_{AB}^{(d)}|ij\rangle. \quad (6)$$

Let us order the sequence $\{a_i^{\downarrow}a_j^{\downarrow}\}_{i,j=0}^{d-1}$ again in a nonincreasing way, and let us call the reordered sequence $\{c_k^{\downarrow}(\rho)\}_{k=0}^{d^2-1}$; namely, for every k, we have $c_k^{\downarrow}(\rho)=a_i^{\downarrow}a_j^{\downarrow}$ for some i,j such that each pair (i,j) appears exactly once, and $c_k^{\downarrow}(\rho)\geqslant c_{k+1}^{\downarrow}(\rho)$. Physically, $\{c_k^{\downarrow}(\rho)\}_{k=0}^{d^2-1}$ is the set of ordered eigenvalues of $\rho\otimes\rho$. Finally, for a normal operator M, its *Ky Fan K-norm* [31], $\|M\|_{K\text{-KF}}$, is defined as the sum of its K largest eigenvalues. With this notion, we obtain

$$\max_{\Pi_{AB}^{(d)}}\mathrm{tr}\big[\Pi_{AB}^{(d)}(\rho_A\otimes\rho_B)\big]=\sum_{k=0}^{d-1}c_k^{\downarrow}(\rho)=\|\rho\otimes\rho\|_{d\text{-KF}}; \quad (7)$$

i.e., it is the Ky Fan d-norm of $\rho\otimes\rho$. Then, we arrive at the following analytical expression, serving as the complete solution to any finite-dimensional INCP:

Result 2. For a d-dimensional state ρ, we have

$$\Delta\mathcal{P}(\rho)=\log_2\left(\|\rho\otimes\rho\|_{d\text{-KF}}/\|\rho\|_\infty\right). \quad (8)$$

ρ's INCP has a solution if and only if $\|\rho\otimes\rho\|_{d\text{-KF}}>\|\rho\|_\infty$.

The Ky Fan norm has previously been used to bound the ability of thermal operations [32] to cool systems [33]. Result 2 now provides it with a novel operational meaning: it quantifies the optimal amount of informational nonequilibrium (and also purity) that can be concentrated given two copies of a state via unitary dynamics. Moreover, as well as providing an analytical necessary and sufficient condition for the existence of INCPs' solutions, Result 2 also tells us the fundamental limitation of purity concentration; i.e., $\Delta\mathcal{P}(\rho)$ is the highest concentratable amount with a fixed dimension.

C. No two-qubit concentration of informational nonequilibrium and purity

It is rather surprising to know that we (only) cannot concentrate informational nonequilibrium and purity in the simplest case: two-qubits. Before stating the result, we recall that, as we argued before, for a qubit state ρ, increasing purity *is equivalent to* enhancing $\|\rho\|_\infty$. Then, in Appendix B, we prove the following no-go result.

Result 3. INCPs of qubit states have no solution. Moreover, this conclusion is independent of the choice of purity measure.

Hence, for two qubits, the structure of quantum theory forbids any possible concentration of informational nonequilibrium and purity. Moreover, this fundamental limitation is truly *independent* of the measure that we use.

D. Informational nonequilibrium concentration beyond qubits is possible

It turns out that informational nonequilibrium concentration is a *generic* phenomenon existing beyond qubits. This is because the necessary and sufficient condition for INCP's solutions to exist (Result 2) can always be satisfied by some ρ when the local dimension d is strictly greater than 2. To better illustrate this, let us consider a simple example, which is an *effective qubit* in a qudit: $\rho=p|0\rangle\langle 0|+(1-p)|1\rangle\langle 1|$ with $1/2\leqslant p\leqslant 1$ in a d-dimensional system with $d>2$. As long as $p<1$, we have

$$\|\rho\otimes\rho\|_{d\text{-KF}}\geqslant p^2+2p(1-p)>\|\rho\|_\infty, \quad (9)$$

which implies concentration due to Result 2. This means that concentration of informational nonequilibrium and purity can indeed happen. In fact, the following state-*independent* unitary can do the job:

$$U_{AB}^{\text{simple}}:|10\rangle_{AB}\leftrightarrow|02\rangle_{AB}. \quad (10)$$

Hence, when the local system is beyond a single qubit, INCPs, in general, can have solutions. Finally, we note a simple upper bound

$$\Delta\mathcal{P}(\rho)\leqslant\mathcal{P}(\rho), \quad (11)$$

which means that the *initial* informational nonequilibrium limits the optimal concentration. See Appendix C for proof.

E. Optimal two-qutrit purity concentration must generate global correlations

Now, we know concentrating purity is possible via INCPs. A natural question is the following: *Can we concentrate local purity without generating global correlation?* In the two-qutrit case, we show that, surprisingly, it is *impossible* due to the special structure of qutrits. More precisely, the three largest eigenvalues of a two-qutrit state $\rho_A\otimes\rho_B$ are $a_0^{\downarrow}a_0^{\downarrow}$, $a_0^{\downarrow}a_1^{\downarrow}$, $a_1^{\downarrow}a_0^{\downarrow}$. Then, Result 2 implies that

$$\Delta\mathcal{P}(\rho)=\log_2(a_0^{\downarrow}+2a_1^{\downarrow}). \quad (12)$$

For a better understanding, we plot the analytical result Eq. (12) in Fig. 2. Also, one can check that the unitary U_{AB}^{simple} defined in Eq. (10) achieves Eq. (12); i.e., U_{AB}^{simple} is *optimal*. Let $\sigma_{AB}^{\text{opt}}(\rho):=U_{AB}^{\text{simple}}(\rho_A\otimes\rho_B)U_{AB}^{\text{simple},\dagger}$ be U_{AB}^{simple}'s global output. To quantify the global output's correlation, we use the quantum mutual information, a widely used correlation measure. Formally, for a bipartite state η_{AB}, its *quantum mutual information* [30] is $I(A:B)_{\eta_{AB}}:=S(\eta_A)+S(\eta_B)-S(\eta_{AB})$,[1] where $S(\eta):=-\mathrm{tr}(\eta\log_2\eta)$ is the *von Neumann entropy* [30]. Then, in Appendix D, we show that following result.

Result 4. Two-qutrit optimal purity concentration must generate global correlation: $I(A:B)_{\sigma_{AB}^{\text{opt}}(\rho)}>0$ if $\Delta\mathcal{P}(\rho)>0$.

Hence, counterintuitively, one *must* increase global correlation and local purity simultaneously. Since a pure state cannot be correlated with any other system, this result means that *it is impossible to map a nonpure qutrit state to a perfect pure state* in the current setting. This finding further uncovers a trade-off relation between making local states purer and generating global correlation (and makes local states less pure).

[1]Here, $\eta_A:=\mathrm{tr}_B(\eta_{AB})$ and $\eta_B:=\mathrm{tr}_A(\eta_{AB})$.

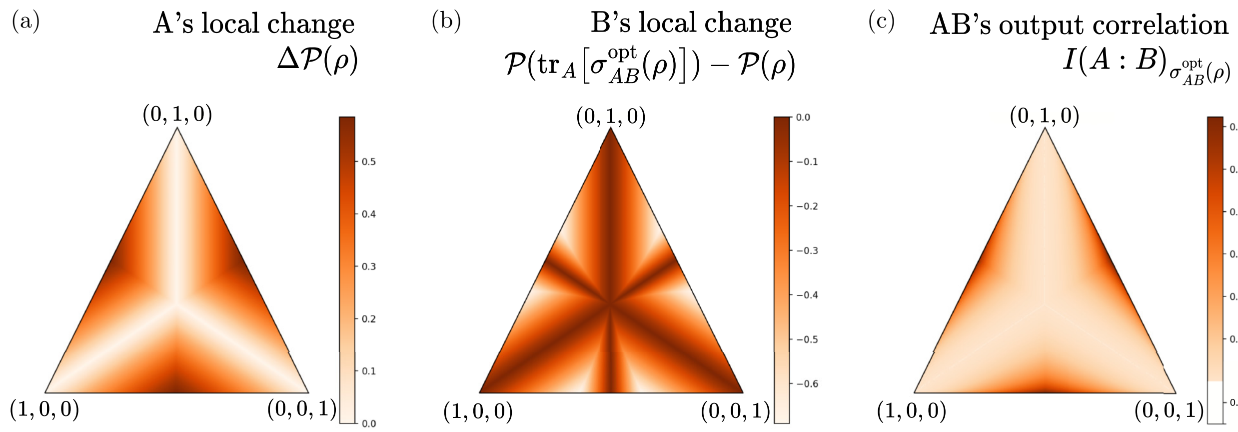

FIG. 2. Graphical depictions of two-qutrit cases. Here, we plot the analytical result Eq. (12). Each point in the triangle, (a_0, a_1, a_2), represents the eigenvalues of the qutrit state, ρ, with the color giving the change of informational nonequilibrium. (a) Optimal increment in A according to Eq. (12). States with bound resources are those on the white lines running from the corners to the center of the triangle. (b) Change in B when A achieves the optimal increment $\Delta\mathcal{P}(\rho)$. (c) The mutual information between A and B after the optimal concentration. One can then see that $\Delta\mathcal{P} > 0$ is accompanied with nonvanishing correlation, as claimed in Result 4. States for which no correlations are created have been explicitly highlighted in white, and can be seen to coincide with the states possessing bound purity.

IV. PHYSICAL IMPLICATIONS

A. "Mpemba-like" effect for purity concentration

Equation (12) implies that, when we optimally concentrate purity in A, B's local purity, as measured by $\mathcal{P}$, can be *invariant*. To see this, consider the one-parameter family of qutrit states

$$\rho^{(p)} = p|0\rangle\langle 0| + (1-p)|1\rangle\langle 1|, \tag{13}$$

with $1/2 \leqslant p \leqslant 1$. Letting $\rho^{(p)} \otimes \rho^{(p)}$ evolve under the optimal unitary U_{AB}^{simple} [Eq. (10)], B's local output reads

$$\sigma_B^{\mathrm{opt}}(\rho^{(p)}) = p^2|0\rangle\langle 0| + (1-p)|1\rangle\langle 1| + p(1-p)|2\rangle\langle 2|, \tag{14}$$

meaning that B has output purity

$$\mathcal{P}\big[\sigma_B^{\mathrm{opt}}(\rho^{(p)})\big] = \log_2(3\max\{p^2; 1-p\}). \tag{15}$$

Now, by setting $p = 1/2$, Eq. (12) says that the optimal purity increment in A is $\Delta\mathcal{P}(\rho^{(1/2)}) = \log_2(3/2) > 0$. Meanwhile, locally in B, we have

$$\mathcal{P}\big[\sigma_B^{\mathrm{opt}}(\rho^{(1/2)})\big] = \log_2(3/2) = \mathcal{P}(\rho^{(1/2)}) \tag{16}$$

[see also Fig. 2 and Eq. (D2) in Appendix D]. Hence, purity, as measured by $\mathcal{P}$, does not change in B when we optimally increase it in A.[2] In other words, the sum of local resources is not conserved.

More intriguingly, our calculation uncovers a phenomenon similar to the *Mpemba effect* [20–24]. Loosely speaking, the Mpemba effect describes the phenomenon that, evolving under a given dynamics for a fixed amount of time, an initially colder system may reach a higher final temperature than an initially hotter system (and vice versa). Here, under the dynamics U_{AB}^{simple}, we observe that in system B a *purer* initial state can be mapped to a *more mixed* final state. As an example of this, consider $p = p_+ := (\sqrt{5}-1)/2$. Clearly $\rho^{(p_+)}$ is *purer* than $\rho^{(1/2)}$. Now, when $\rho^{(p_+)} \otimes \rho^{(p_+)}$ undergoes U_{AB}^{simple}, B's local output $\sigma_B^{\mathrm{opt}}(\rho^{(p_+)})$ satisfies

$$\mathcal{P}\big[\sigma_B^{\mathrm{opt}}(\rho^{(p_+)})\big] < \mathcal{P}\big[\sigma_B^{\mathrm{opt}}(\rho^{(1/2)})\big]. \tag{17}$$

So a purer environment interacting with a purer state is left in a more mixed state than an initially more mixed environment interacting with a more mixed state. Intuitively, one might expect that if the environment (B) is initially purer, then optimal concentration in A would leave the environment less mixed. However, here we see the *opposite*, with purer initial states leading to more mixed environments, which is a phenomenon that captures a similar flavor to the Mpemba effect. We leave further explorations for future projects.

B. Notion of "bound" informational nonequilibrium

From Result 3, if a nonpure qubit state is not maximally mixed, it carries nonvanishing resources that are *not yet* the highest but *cannot* be concentrated further. We coin the term *bound informational nonequilibrium* for such states, and we briefly discuss their properties here beyond qubit. First, in qutrits, Eq. (12) implies that $\Delta\mathcal{P}(\rho) = 0$ *if and only if* $a_1^{\downarrow} = a_2^{\downarrow}$. Namely, *all nonpure qutrit states with exactly two-fold degeneracy in their smaller eigenvalue have bound informational nonequilibrium.* Notably, in qutrits, small perturbations are enough to remove bound informational nonequilibrium by breaking the equality $a_1^{\downarrow} = a_2^{\downarrow}$. Meanwhile, in qubits, no perturbation can do so. Hence, interestingly, depending on the physical system's dimension, bound informational nonequilibrium could be either very robust (when $d = 2$) or very fragile (when $d = 3$) against noises. Now, generally, for a d-dimensional state ρ, Result 2 implies that $\Delta\mathcal{P}(\rho) = 0$ *if*

[2] Physically, this is because $\mathcal{P}$ only focuses on the "purest" occupation (i.e., the maximal eigenvalue). Manipulating less pure occupations cannot change $\mathcal{P}$'s value.

and only if $\|\rho\otimes\rho\|_{d\text{-KF}} = \|\rho\|_\infty$. This thus implies *all nonpure qudit states with exactly* $(d-1)$*-fold degeneracy in their smaller eigenvalue have bound informational nonequilibrium*. This is because all such states are of the form $\rho(p,|\psi\rangle) := p|\psi\rangle\langle\psi| + (1-p)\mathbb{I}/d$ for some pure state $|\psi\rangle$ and $0 < p < 1$, and one can check that $\|\rho(p,|\psi\rangle)\otimes\rho(p,|\psi\rangle)\|_{d\text{-KF}} = \|\rho(p,|\psi\rangle)\|_\infty$. Physically, this means that dephasing process $(\cdot)\mapsto p\mathcal{I}(\cdot) + (1-p)\mathrm{tr}(\cdot)\mathbb{I}/d$ on pure states produces bound informational nonequilibrium as long as $0 < p < 1$. Namely, dephasing processes are strong enough to negate the possibility of concentration.

C. Initial correlations can activate informational nonequilibrium concentration

Importantly, by allowing initial correlation, even an almost-vanishing amount, can make informational nonequilibrium concentration possible. To see this, suppose one has the two-qudit *isotropic state* [34] $p|\Phi^+\rangle\langle\Phi^+|_{AB} + (1-p)\mathbb{I}_{AB}/d_{AB}$, where $|\Phi^+\rangle_{AB} := \sum_{i=0}^{d-1}|ii\rangle_{AB}$ is maximally entangled and $0\leqslant p\leqslant 1$. Locally, both systems are maximally mixed, a state for which no informational nonequilibrium can be concentrated. However, by considering the two-qudit unitary that maps $|\Phi^+\rangle\leftrightarrow|00\rangle$, one can obtain nonmaximally mixed marginal, resulting in informational nonequilibrium concentration. The physics is that one can consume the global correlation (even a classical, nonentangled one) to generate local purity. Namely, we can relocate the genuinely global purity into local systems. This also shows that the two-qubit no-go result (Result 3) is not robust to practical noise and experimental error bars and one can consume global correlation to break it. Notably, the same argument works for *arbitrary* $\rho = \sum_i a_i|i\rangle\langle i|$ by considering $p|\rho\rangle\langle\rho|_{AB} + (1-p)\rho_A\otimes\rho_B$, where $|\rho\rangle_{AB} := \sum_i\sqrt{a_i}|ii\rangle_{AB}$ is ρ's purification. Hence, global correlations are useful resources for activating local concentrations of informational nonequilibrium and purity.

At this point, one may wonder the following: *To what extent can global entanglement enhance the concentration?* This is, again, captured by the Ky Fan norm. To see this, if two copies of ρ are entangled via $|\rho\rangle_{AB}$, a global unitary mapping as $|\rho\rangle_{AB}\leftrightarrow|00\rangle_{AB}$ can achieve concentration in A with the increment $\Delta\mathcal{P}_{\rm corr}(\rho) := \log_2 d - \log_2 d\|\rho\|_\infty = -\log_2\|\rho\|_\infty$. Using Result 2, the optimal concentration without any global correlation is $\Delta\mathcal{P}(\rho) = \log_2(\|\rho\otimes\rho\|_{d\text{-KF}}/\|\rho\|_\infty)$. Consuming $|\rho\rangle_{AB}$'s entanglement leads to the additional concentration

$$\Delta\mathcal{P}_{\rm corr}(\rho) - \Delta\mathcal{P}(\rho) = -\log\|\rho\otimes\rho\|_{d\text{-KF}}. \quad (18)$$

Thus, the Ky Fan norm not only characterises INCPs' solutions, it is also the *entanglement advantage* in INCPs.

D. Application to concentrating intrinsic randomness

Finally, as Result 2's application, we show that informational non-equilibrium concentration implies the ability to concentrate *intrinsic randomness*. The intrinsic randomness of a state ρ is loosely speaking defined by choosing the measurement, such that even a powerful adversary has difficulty in guessing its outcomes. We refer to Ref. [19] for all the exact definitions, and just use the result of their optimization: the intrinsic randomness of ρ is given by $-\log P^*_{\rm guess}(\rho)$, with the guessing probability

$$P^*_{\rm guess}(\rho) = (\mathrm{tr}\sqrt{\rho})^2/d. \quad (19)$$

One can see that a smaller $P^*_{\rm guess}$ means a higher purity. In particular, given a pure state, there exist measurements whose outcomes can be maximally unpredictable [$P^*_{\rm guess}(\rho) = 1/d$]; while the maximally mixed state has no intrinsic randomness since $P^*_{\rm guess}(\rho) = 1$.

Despite being an alternative way to measure purity, we note that $\mathcal{P}$ *and* $P^*_{\rm guess}$ *do not define the same order on states.* That is, $\mathcal{P}(\sigma) > \mathcal{P}(\rho)$ does not necessarily imply $P^*_{\rm guess}(\sigma) < P^*_{\rm guess}(\rho)$ (see Appendix E for the explicit example). Hence, an increase in $\mathcal{P}$ does not automatically guarantee an increase in intrinsic randomness. Nonetheless, we show that whenever informational nonequilibrium can be concentrated (i.e., $\Delta\mathcal{P} > 0$), it is always possible to increase intrinsic randomness as well (i.e., decreasing $P^*_{\rm guess}$).

Result 5. When $\Delta\mathcal{P}(\rho) > 0$, there exists a pairwise permutation unitary $V : |i,j\rangle\leftrightarrow|0,k\rangle$ for some i,j,k achieving

$$\mathcal{P}\big(\sigma_A^{(V)}\big) > \mathcal{P}(\rho_A) \quad\text{and}\quad P^*_{\rm guess}\big(\sigma_A^{(V)}\big) < P^*_{\rm guess}(\rho_A), \quad (20)$$

where $\sigma_A^{(V)} := \mathrm{tr}_B[V_{AB}(\rho_A\otimes\rho_B)V_{AB}^\dagger]$.

The proof is given in Appendix F, leading to an explicit formula [Eq. (F9)] for the possible enhancement of $P^*_{\rm guess}$.

E. Experimental practicality

Finally, we comment on INCPs' practical feasibility. INCPs' formulation allows them to be studied in *nitrogen vacancy* (NV) center spin systems, considering the effect of partially nondegenerate qubit or qudit energy levels and finite difference between NV center spin systems. In Fig. 1, ρ_A and ρ_B can be two closely populated NV centres where U_{AB} can be realized by the dipole-dipole interaction between NVs [16,17]. The system (and thus the dimension of qudit) can be selected from the electron spins, ^{14}N (^{15}N) nuclear spins, and ^{13}C nuclear spins subsystems [17,18]. Further experimental explorations are beyond the scope of this work and are left for future research.

V. DISCUSSIONS

Thermodynamically, solving INCPs is calculating the highest locally extractable work given access to $\rho\otimes\rho$ and joint unitary operations. By performing optimal unitaries, a local system is maximally driven from equilibrium, hence becoming maximally thermodynamically resourceful, and one can extract work from it via, e.g., Szilard engine [4]. As important follow-ups, our resource concentration framework applies to the more general thermodynamic setting with nondegenerate Hamiltonians via concentration of athermality [35] as well as other resources, such as unspeakable coherence [36] or entanglement. Finally, the Mpemba-like effect and the notion of bound purity are both worth exploring.

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APPENDIX A: PROOF OF RESULT 1

Proof. Using Eq. (2), we analyze

$$\max_{U_{AB}} \|\mathcal{E}_{(U_{AB},\eta_B)}(\rho_A)\|_\infty = \max_{U_{AB},|\phi\rangle_A} \mathrm{tr}[U_{AB}^\dagger(|\phi\rangle\langle\phi|_A\otimes\mathbb{I}_B)U_{AB}(\rho_A\otimes\eta_B)] \leqslant \max_{\Pi_{AB}^{(d_B)}} \mathrm{tr}\big[\Pi_{AB}^{(d_B)}(\rho_A\otimes\eta_B)\big], \quad (A1)$$

where $U_{AB}^\dagger(|\phi\rangle\langle\phi|_A\otimes\mathbb{I}_B)U_{AB}$ is a rank-d_B projector in AB and results in the last inequality. Now, we note that, for an arbitrarily given rank-d_B projector $\Pi_{AB}^{(d_B)}$, we can write

$$\Pi_{AB}^{(d_B)} = \sum_{n=1}^{d_B} |\kappa_n\rangle\langle\kappa_n|_{AB}, \quad (A2)$$

where $\{|\kappa_n\rangle_{AB}\}_{n=1}^{d_B}$ is an orthonormal set with d_B many pure states. By considering the unitary $\widetilde{U}_{AB}^\dagger$ mapping as

$$|0\rangle_A\otimes|n\rangle_B \leftrightarrow |\kappa_n\rangle_{AB}\ \forall n, \quad (A3)$$

and keeping all other basis states untouched, we obtain

$$\widetilde{U}_{AB}^\dagger(|0\rangle\langle 0|_A\otimes\mathbb{I}_B)\widetilde{U}_{AB} = \Pi_{AB}^{(d_B)}. \quad (A4)$$

Hence, the inequality in Eq. (A1) is achieved, and the desired result follows. ■

APPENDIX B: PROOF OF RESULT 3

Proof. Write $\rho = p|0\rangle\langle 0| + (1-p)|1\rangle\langle 1|$ with $1/2\leqslant p\leqslant 1$. Using Result 2, it suffices to check

$$c_0^\downarrow(\rho)+c_1^\downarrow(\rho) = p^2 + p(1-p) = p = \|\rho\|_\infty. \quad (B1)$$

Hence, we can never have the strict inequality ">." Result 2 implies that it is impossible to increase $\|\rho\|_\infty$. Importantly, in a qubit, this further means that increasing the difference between two eigenvalues is impossible. Hence, two-qubit purity cannot be concentrated, *independent* of the choice of measures. ■

APPENDIX C: PROOF OF EQ. (11)

Proof. A direct computation shows that

$$\begin{aligned}\Delta\mathcal{P}(\rho) &= \mathcal{P}\big(\sigma_A^{(U)}\big) - \mathcal{P}(\rho_A) \\ &= D_{\max}[\mathrm{tr}_B(U_{AB}(\rho_A\otimes\rho_B)U_{AB}^\dagger)\,\|\,\mathbb{I}_A/d] \\ &\quad - D_{\max}(\rho\,\|\,\mathbb{I}/d) \\ &\leqslant D_{\max}[\rho_A\otimes\rho_B\,\|\,(\mathbb{I}_A\otimes\mathbb{I}_B)/d^2] - D_{\max}(\rho\,\|\,\mathbb{I}/d) \\ &= D_{\max}(\rho\,\|\,\mathbb{I}/d) = \mathcal{P}(\rho),\end{aligned} \quad (C1)$$

where we use the data-processing inequality under the channel $\mathrm{tr}_B(U_{AB}(\cdot)U_{AB}^\dagger)$, and the fact that $D_{\max}[\rho_A\otimes\rho_B\,\|\,(\mathbb{I}_A\otimes\mathbb{I}_B)/d^2] = 2D_{\max}(\rho\,\|\,\mathbb{I}/d)$. ■

Interestingly, by applying this bound to both A and B, we conclude that the sum of local changes of informational nonequilibrium in A and B is upper bounded by $2\mathcal{P}(\rho)$.

APPENDIX D: PROOF OF RESULT 4

Proof. First, we have

$$\begin{aligned}\sigma_A^{\mathrm{opt}}(\rho) &:= \mathrm{tr}_B\big[\sigma_{AB}^{\mathrm{opt}}(\rho)\big] \\ &= a_0^\downarrow(a_0^\downarrow+2a_1^\downarrow)|0\rangle\langle 0| + [a_1^\downarrow a_1^\downarrow + (1-a_2^\downarrow)a_2^\downarrow]|1\rangle\langle 1| \\ &\quad + a_2^\downarrow|2\rangle\langle 2|;\end{aligned} \quad (D1)$$

$$\begin{aligned}\sigma_B^{\mathrm{opt}}(\rho) &:= \mathrm{tr}_A\big[\sigma_{AB}^{\mathrm{opt}}(\rho)\big] \\ &= a_0^\downarrow(a_0^\downarrow+2a_2^\downarrow)|0\rangle\langle 0| + a_1^\downarrow|1\rangle\langle 1| \\ &\quad + [a_2^\downarrow a_2^\downarrow + (1-a_1^\downarrow)a_1^\downarrow]|2\rangle\langle 2|.\end{aligned} \quad (D2)$$

Also, since $S[\sigma_{AB}^{\mathrm{opt}}(\rho)] = S(\rho\otimes\rho) = 2S(\rho)$, we have

$$I(A:B)_{\sigma_{AB}^{\mathrm{opt}}(\rho)} = S\big[\sigma_A^{\mathrm{opt}}(\rho)\big] + S\big[\sigma_B^{\mathrm{opt}}(\rho)\big] - 2S(\rho), \quad (D3)$$

which is strictly positive if $\Delta\mathcal{P}(\rho) = \log_2(a_0^\downarrow + 2a_1^\downarrow) > 0$, as shown in Fig. 2 (which provides further illustrations). ■

APPENDIX E: $\mathcal{P}$ AND P_{guess}^* DO NOT DEFINE THE SAME ORDER ON STATES

To see a counterexample, in a five-level system, consider states

$$\sigma = |0\rangle\langle 0|/2 + (|1\rangle\langle 1| + |2\rangle\langle 2| + |3\rangle\langle 3| + |4\rangle\langle 4|)/8, \quad (E1)$$

and

$$\rho = (|0\rangle\langle 0| + |1\rangle\langle 1| + |2\rangle\langle 2|)/3. \quad (E2)$$

Then we have

$$\mathcal{P}(\sigma) = \log_2(5/2) > \log_2(5/3) = \mathcal{P}(\rho). \quad (E3)$$

At the same time, we also have

$$P_{\mathrm{guess}}^*(\sigma) = (1/\sqrt{2}+\sqrt{2})^2/5 > 3/5 = P_{\mathrm{guess}}^*(\rho). \quad (E4)$$

Hence, $\mathcal{P}(\sigma) > \mathcal{P}(\rho)$ does not necessarily imply $P_{\mathrm{guess}}^*(\sigma) < P_{\mathrm{guess}}^*(\rho)$.

APPENDIX F: PROOF OF RESULT 5

Proof. Using Result 2, $\Delta\mathcal{P}(\rho) > 0$ implies

$$\sum_{k=0}^{d-1} c_k^{\downarrow}(\rho) > \|\rho\|_\infty = \sum_{i=0}^{d-1} \|\rho\|_\infty a_i^{\downarrow}, \tag{F1}$$

where we recall that $\rho = \sum_{i=0}^{d-1} a_i^{\downarrow} |i\rangle\langle i|$ and $a_i^{\downarrow} \geqslant a_{i+1}^{\downarrow}\ \forall\, i$. By construction, we must have $c_0^{\downarrow}(\rho) = \|\rho\|_\infty^2$ and $a_0^{\downarrow} = \|\rho\|_\infty$. Consequently, we have

$$\sum_{k=1}^{d-1} (c_k^{\downarrow}(\rho) - \|\rho\|_\infty a_k^{\downarrow}) > 0. \tag{F2}$$

This means there exists at least one k value, say k_*, achieving

$$c_{k_*}^{\downarrow}(\rho) > \|\rho\|_\infty a_{k_*}^{\downarrow}. \tag{F3}$$

Let us write $c_{k_*}^{\downarrow}(\rho) = a_{i_*}^{\downarrow} a_{j_*}^{\downarrow}$ for some indices i_*, j_*. Then the inequality $c_{k_*}^{\downarrow}(\rho) > \|\rho\|_\infty a_{k_*}^{\downarrow}$ can be translated into

$$a_{i_*}^{\downarrow} a_{j_*}^{\downarrow} > a_0^{\downarrow} a_{k_*}^{\downarrow}. \tag{F4}$$

Now consider the pairwise permutation unitary

$$V_{AB} : |i_*, j_*\rangle \leftrightarrow |0, k_*\rangle. \tag{F5}$$

Define $\delta_* := a_{i_*}^{\downarrow} a_{j_*}^{\downarrow} - a_0^{\downarrow} a_{k_*}^{\downarrow} > 0$. Then, one can check that

$$\begin{aligned} \sigma_A^{(V)} &:= \mathrm{tr}_B[V_{AB}(\rho_A \otimes \rho_B) V_{AB}^{\dagger}] \\ &= \rho_A + \delta_*(|0\rangle\langle 0|_A - |i_*\rangle\langle i_*|_A). \end{aligned} \tag{F6}$$

This means that $\mathcal{P}(\sigma_A^{(V)}) > \mathcal{P}(\rho_A)$ since the occupation of $|0\rangle$ is increased by δ_*. The final step is to argue that this unitary is able to decrease the guessing probability. Since $P_{\rm guess}^*(\rho_A) = (\mathrm{tr}\sqrt{\rho_A})^2/d$ [19], decreasing $P_{\rm guess}^*$ is equivalent to decreasing $\mathrm{tr}\sqrt{\rho_A}$; namely, it suffices to show that $\mathrm{tr}\sqrt{\rho_A} > \mathrm{tr}\sqrt{\sigma_A^{(V)}}$. Then a direct computation shows that (remember that $a_0^{\downarrow}$ is the largest one among all $a_i^{\downarrow}$'s)

$$\begin{aligned} &\left(\sqrt{a_{i_*}^{\downarrow} - \delta_*} + \sqrt{a_{i_*}^{\downarrow}}\right)\left(\sqrt{a_0^{\downarrow} + \delta_*} - \sqrt{a_0^{\downarrow}}\right) \\ &\quad < \left(\sqrt{a_0^{\downarrow} + \delta_*} + \sqrt{a_0^{\downarrow}}\right)\left(\sqrt{a_0^{\downarrow} + \delta_*} - \sqrt{a_0^{\downarrow}}\right) = \delta_* \\ &\quad = \left(\sqrt{a_{i_*}^{\downarrow} - \delta_*} + \sqrt{a_{i_*}^{\downarrow}}\right)\left(\sqrt{a_{i_*}^{\downarrow}} - \sqrt{a_{i_*}^{\downarrow} - \delta_*}\right). \end{aligned} \tag{F7}$$

Note that we have the above strict inequality because $\delta_* > 0$ and $\sqrt{a_{i_*}^{\downarrow} - \delta_*} + \sqrt{a_{i_*}^{\downarrow}} > 0$ (it cannot be zero, otherwise we cannot have $\delta_* > 0$). Hence, we conclude that

$$\sqrt{a_0^{\downarrow} + \delta_*} - \sqrt{a_0^{\downarrow}} < \sqrt{a_{i_*}^{\downarrow}} - \sqrt{a_{i_*}^{\downarrow} - \delta_*}. \tag{F8}$$

Finally, we note that $\sqrt{\rho_A}$ and $\sqrt{\sigma_A^{(V)}}$ are different only in the subspace spanned by $|i_*\rangle$ and $|0\rangle$. This can be explicitly seen by Eq. (F6). Consequently, one can check that

$$\begin{aligned} \mathrm{tr}\sqrt{\rho_A} - \mathrm{tr}\sqrt{\sigma_A^{(V)}} = \sqrt{a_0^{\downarrow}} + \sqrt{a_{i_*}^{\downarrow}} - \sqrt{a_0^{\downarrow} + \delta_*} \\ - \sqrt{a_{i_*}^{\downarrow} - \delta_*} > 0, \end{aligned} \tag{F9}$$

which concludes the proof. ∎

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