Letter

Fermionic anyons: Entanglement and quantum computation from a resource-theoretic perspective

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Quantum computational models can be approached via the lens of resources needed to perform computational tasks, where a computational advantage is achieved by consuming specific forms of quantum resources, or, conversely, resource-free computations are classically simulable. Can we similarly identify quantum computational resources in the setting of more general quasi-particle statistics? In this work, we develop a framework to characterize the separability of a specific type of one-dimensional quasiparticle known as a fermionic anyon. As we evince, the usual notion of partial trace fails in this scenario, so we build the notion of separability through a fractional Jordan-Wigner transformation, leading to an entanglement description of fermionic-anyon states. We apply this notion of fermionic-anyon separability and the unitary operations that preserve it, mapping it to the free resources of matchgate circuits. We also identify how entanglement between two qubits encoded in a dual-rail manner, as standard for matchgate circuits, corresponds to the notion of entanglement between fermionic anyons.

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Introduction. Over the last five decades, our notion of identical particles in nature has expanded beyond fermions and bosons. Many two-dimensional systems were shown to contain anyonic excitations [1,2], which are quasi-particles characterized by the nontrivial phases their wave functions acquire under particle exchange. These include fractional quantum Hall states [3,4], topological spin liquids [5,6], and semiconductor nanowire arrays [7,8]. These systems are seen as possible platforms for fault-tolerant quantum computing [1,8], given their inherent error-correcting properties [9–11] and recent experimental evidence of their existence and predicted properties [12].

Although anyons are most commonly associated with two-dimensional systems, they can also be defined in one dimension. Some notable examples are anyons obtained by dimensional reduction [13,14], or appearing as a free-particle description of one-dimensional systems with two-body interactions [15–19]. The one-dimensional anyons considered here are motivated by their role in solving many-body systems with three-body interactions [20–26] and have been investigated in optical lattice implementations [27–30]. Although they lack the topological properties of their two-dimensional counterparts [31], their relation to standard fermionic and bosonic systems via generalized Jordan-Wigner transformations [32–34] makes them a good case study for generalizations of quantum computing with bosonic and fermionic linear optics.

In this Letter, we develop a framework to define and investigate the separability of fermionic anyons. Since this is well understood for fermions, the naive approach is to directly repurpose definitions of fermionic entanglement to their anyonic counterparts [35–38]. However, this sometimes leads to nonsensical results. For example, we notice that single-particle transformations on a manifestly unentangled pure state can result in states with a nonzero entanglement entropy.

Within subspaces of fixed particle number, we circumvent these problems by a well-motivated approach for single-particle entanglement, and revise the definition of Schmidt coefficients of a composite fermionic-anyon state based on a noncanonical transformation over the anyonic states. Specifically, we map the anyonic algebra to another system that satisfies an anticommutative algebra, and prove that the Schmidt coefficients of the resulting mapped state coincide with those of the original anyonic state.

We showcase our approach by investigating the connection between separability and classical simulability in these systems. Free-fermionic quantum circuits [39] and matchgate computing [40,41] are quantum computing settings where separability and computational power are tightly connected. Nearest-neighbor matchgate circuits can be mapped to free-fermion dynamics, and both are known to be classically simulable. However, it is known that supplementing these systems with any nonmatchgate operation (in the fermionic picture, adding an interaction between particles) or any nonmatchgate generated state (respectively, any non-Gaussian

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fermionic state) is enough to allow for universal quantum computation [42]. Here, we leverage this connection to make a similar statement for fermionic anyons. In particular, we notice that both nonseparable states or transformations, as per our definition, can be seen as computational resources. Moreover, specific values of the fermionic-anyon phase φ recover well-known results. Fermionic-anyon dynamics reduces to fermionic linear optics when $\varphi = 0$, and to "qubit linear optics" (or matchgate quantum computing) when $\varphi = \pi$.

One-dimensional fermionic anyons. Given a one-dimensional set of m sites (or modes), we define a family of operator algebras $\{\mathcal{A}_m^\varphi | \varphi \in [0, 2\pi)\}$ over $\mathbb{C}$, generated by operators $\{a_{\varphi,i} | i = 1, ..., m\}$, satisfying

$$\begin{aligned} a_{\varphi,i}a_{\varphi,j}^\dagger + e^{-i\varphi\epsilon_{ij}} a_{\varphi,j}^\dagger a_{\varphi,i} &= \delta_{ij}, \\ a_{\varphi,i}a_{\varphi,j} + e^{i\varphi\epsilon_{ij}} a_{\varphi,j}a_{\varphi,i} &= 0, \end{aligned} \tag{1}$$

with ϵ_{ij} given by

$$\epsilon_{ij} = \begin{cases} 1, & \text{if } i < j \\ 0, & \text{if } i = j \\ -1, & \text{if } i > j. \end{cases}$$

The variable φ is called the *statistical parameter* and determines the kind of particle described by the algebra. If $\varphi = 0$ we identify $f_i := a_{0,i}$, and $\mathcal{A}_m^0 \equiv \mathcal{F}_m$, where $\mathcal{F}_m$ is the algebra of m-mode fermionic operators. If $\varphi = \pi$, then for all i, j we have $[a_{\pi,i}, a_{\pi,j}] = 0$ as well as $\{a_{\pi,i}, a_{\pi,j}^\dagger\} = 0$, and we identify $\mathcal{A}_m^\pi \equiv \mathcal{Q}_m$, where $\mathcal{Q}_m$ is the algebra of operators for m-mode hardcore bosons, or qubits [43]. For any other value of φ, the algebra $\mathcal{A}_m^\varphi$ describes particles with exotic exchange statistics called *fermionic anyons*.

In the Supplemental Material [44] we show that, for all φ, the algebras $\mathcal{A}_m^\varphi$ have a well-defined Fock-space representation with number operators of the form $a_{\varphi,i}^\dagger a_{\varphi,i}$. Therefore, a general pure state of N fermionic anyons has the form

$$|\psi\rangle = \sum_{I_N} w_{I_N} a_{\varphi,i_1}^\dagger ... a_{\varphi,i_N}^\dagger |\text{vac}\rangle, \tag{2}$$

where $|\text{vac}\rangle$ is the vacuum state, $I_N = \{i_1 < \cdots < i_N\}$ is a shorthand for the list of particle indices, and $\sum_{I_N} |w_{I_N}|^2 = 1$.

Separability for fermionic anyons. For a quantum system with two sets of degrees of freedom, a standard quantifier of correlations for pure states is the entanglement entropy [45],

$$E(|\psi\rangle) = S(\rho_{\text{red}}). \tag{3}$$

Here, $S(\rho)$ is the von Neumann entropy of ρ, and ρ_{red} is the reduced state obtained by tracing out one of the subsystems. However, naively applying a particle partial trace on systems of fermionic anyons and computing its entanglement according to Eq. (3) can lead to nonsensical results. To illustrate this, consider the state

$$|\psi_\theta\rangle = \frac{1}{\sqrt{2}}(a_{\varphi,1}^\dagger a_{\varphi,2} + \cos\theta a_{\varphi,1}^\dagger a_{\varphi,4}^\dagger + i \sin\theta a_{\varphi,2}^\dagger a_{\varphi,4}^\dagger)|\text{vac}\rangle.$$

It can be obtained by applying a fermionic-anyon *single-particle operation* on a manifestly separable state $|\psi\rangle = \frac{1}{\sqrt{2}} a_{\varphi,1}^\dagger (a_{\varphi,2}^\dagger + a_{\varphi,4}^\dagger)|\text{vac}\rangle$, i.e.,

$$|\psi_\theta\rangle = \exp[i\theta(a_{\varphi,1}^\dagger a_{\varphi,2} + a_{\varphi,1}^\dagger a_{\varphi,2})]|\psi\rangle. \tag{4}$$

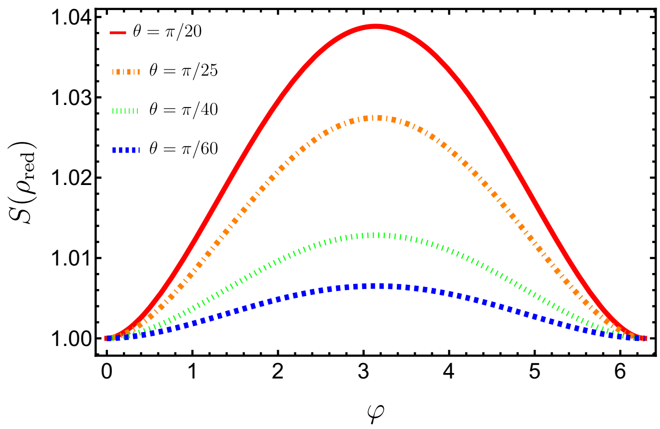

FIG. 1. Fermionic anyon entanglement as defined in Eq. (3) of the state $|\psi_\theta\rangle$. The plot shows how the von Neumann entropy of the single-particle state varies as a function of φ, even though $|\psi_\theta\rangle$ should not have any particle entanglement.

Since particle entanglement is invariant under single-particle operations, the entanglement entropy of $|\psi_\theta\rangle$ should also be invariant, and hence zero by construction. As shown in Fig. 1, however, that is not the case.

For standard fermions, single-particle operations must act as changes of basis over single-particle systems, implying they have the second-quantized form

$$f_i^\dagger \to U f_i^\dagger U^\dagger = \sum_{j=1}^m U_{ij} f_j^\dagger, \tag{5}$$

where U_{ij} are elements of an $m \times m$ unitary matrix. This map is well defined for fermions because it is canonical, i.e., does not change particle commutation relations. However, defining single-particle operations for fermionic anyons by analogy with Eq. 5 (i.e., replacing f with a_φ) does not produce a canonical transformation. To properly define these operations for fermionic anyons, we must find an appropriate definition for their *canonical transformations*.

As shown in Refs. [46,47], creation and annihilation operators for fermionic anyons (for any φ) can be identified with operators in the usual fermionic algebra via the relation

$$J_\varphi(a_{\varphi,j}) = f_j e^{-i\varphi \sum_{k=1}^{j-1} f_k^\dagger f_k}, \tag{6}$$

known as the *fractional Jordan-Wigner transform* (JWT). It follows that $a_{\varphi,i}^\dagger a_{\varphi,i} = f_i^\dagger f_i$, from which we obtain the inverse relationship

$$J_\varphi^{-1}(f_j) = a_{\varphi,j} e^{i\varphi \sum_{k=1}^{j-1} a_{\varphi,k}^\dagger a_{\varphi,k}}, \tag{7}$$

Thus, we define a map J_φ over operators in $\mathcal{F}_m$ that is linear, invertible, and preserves operator products and conjugation (see the Supplemental Material [44]), and use it to shift between the fermionic and anyonic forms of any operator. In other words, given $O \in \mathcal{F}_m$, we define the mapped operators via JWT by the following:

$$O =: [O]_0 = J_\varphi([O]_\varphi), \quad [O]_\varphi = J_\varphi^{-1}([O]_0). \tag{8}$$

Thus, if $[U]_0$ is a single-particle change of basis over fermionic states, then $[U]_\varphi$ must have the same action in terms of fermionic-anyon states. For fermionic systems, an N-particle state $|\phi\rangle$ is *separable*, i.e., it has no particle entanglement, if there is an N-particle state in the Fock basis and a single-particle operator U such that [48,49]

$$|\phi\rangle = [U]_0(f_{i_1}^\dagger \dots f_{i_N}^\dagger)|\text{vac}\rangle, \tag{9}$$

where we can assume that $i_1 < \cdots < i_m$. These states are described by a single Slater determinant, with only exchange correlations due to symmetry [50,51]. We now extend this notion for general fermionic anyons, leading to the central definition in this work:

Definition 1 (Separable fermionic anyon states). A pure N-particle fermionic-anyon state $|\phi\rangle$ is separable if and only if there is a single-particle fermionic operator $[U]_0$ such that

$$|\phi\rangle = [U]_\varphi(a_{\varphi,i_1}^\dagger \dots a_{\varphi,i_N}^\dagger)|\text{vac}\rangle, \tag{10}$$

where $[U]_\varphi = J_\varphi^{-1}([U]_0)$.

Having defined a separability criterion, we investigate entanglement in fermionic anyons by adapting corresponding concepts for fermions. For instance, for fermionic systems, Ref. [52] shows that single-particle entanglement can be quantified through the minimization over all possible mode representations of

$$E_{SP}(|\psi\rangle) = \min_f \sum_i H(\langle f_i^\dagger f_i\rangle, \langle f_i f_i^\dagger\rangle), \tag{11}$$

where $H(p, 1-p) = -p\log p - (1-p)\log(1-p)$ is the Shannon binary entropy, and f_i are the fermionic operators transformed according to the Bogoliubov transformation in Eq. (5). For fermionic anyons, the entanglement can be obtained by mapping the anyonic state into a fermionic form, calculating the minimization of Eq. (11), and translating the new state back into anyonic form. Since J_φ is a *-algebra endomorphism, we obtain the following theorems (see the Supplemental Material [44] for proof).

Theorem 1 (Single-particle entanglement for fermionic anyons). For a fermionic-anyon state with fixed particle number, there exists a mode representation such that its single-particle reduced state has the same eigenvalues as the corresponding fermionic single-particle state, which minimizes Eq. (11).

Theorem 1 implies that there exists a fermionic-anyon mode representation that reflects the particle separability, even though the von Neumann entropy of the single-particle reduced state, obtained through the partial trace on another basis, does not characterize the separability. Such a representation ensures that the reduced state is diagonal and independent of the statistical parameter φ. Consequently, the entropy of the reduced state accurately reflects the entanglement of a single particle to the $N-1$ fermionic anyons. It is possible to generalize the single-particle entanglement for mixed states $\rho = \sum_x p_x|\psi_x\rangle\langle\psi_x|$,

$$E(\rho) = \inf_{\{p_x,|\psi_x\rangle\}} \sum_x p_x E_{SP}(|\psi_x\rangle), \tag{12}$$

where the $|\psi_x\rangle$ are given in Eq. (9). Therefore, for a one-dimensional system with N fermionic anyons, the entanglement between two particles at modes i and j can be computed by taking the partial trace concerning the $N-2$ modes in the minimal entropic basis. This basis is obtained by applying the JWT to the fermionic space. However, when the two particles are in a pure state, it is possible to derive the analog of a *Schmidt decomposition* for fermionic anyons. This involves mapping the state using JWT and calculating the well-known Schliemann decomposition in the fermionic space [53].

Theorem 2 (Schmidt decomposition for fermionic anyons). Any pure state of two fermionic anyons with a fixed number of modes has a Schmidt decomposition with the same expansion coefficients as its Schliemann fermionic state counterpart.

The decomposition is obtained by a dressed unitary transformation $[U_{SD}]_\varphi$ that maps a fermionic anyon state $|\psi\rangle = \sum_{m<n} w_{m,n}a_m^\dagger a_n^\dagger|\text{vac}\rangle$, written in a given basis, onto its Schmidt decomposition $|\psi'\rangle$ as

$$|\psi'\rangle = [U_{SD}]_\varphi|\psi\rangle = \sum_\mu \omega_\mu \alpha_{2\mu}^{(1)\dagger}\alpha_{2\mu-1}^{(2)\dagger}|\text{vac}\rangle, \tag{13}$$

where $[U_{SD}]_\varphi = J_\varphi^{-1}[U_{SD}]_0$, and $[U_{SD}]_0$ is a single-particle fermionic unitary operator that maps the fermionic state $J_\varphi(|\psi\rangle)$, written in a given basis, in its Schliemann decomposition with coefficients given by ω_μ.

Fermionic linear optics and fermionic anyons. To showcase what insights can be drawn from an entanglement theory for fermionic anyons, we apply the formal framework we proposed to particle-based quantum computing. Specifically, we define a family of computational models based on two-mode "linear-optical elements," which reduce to well-known fermionic linear optics when $\varphi = 0$, and show how our notion of separability closely tracks the regime of classical simulability of these models.

Let PS_i, $BS_{i,j}$, and $PA_{i,j}$ be of the form

$$PS_i(\theta) = \exp[i\theta(a_{\varphi,i}^\dagger a_{\varphi,i})]$$
$$BS_{i,j}(\theta) = \exp[i\theta(a_{\varphi,i}^\dagger a_{\varphi,j} + a_{\varphi,j}^\dagger a_{\varphi,i})]$$
$$PA_{i,j}(\theta) = \exp[i\theta(a_{\varphi,i}^\dagger a_{\varphi,j}^\dagger + a_{\varphi,j} a_{\varphi,i})].$$

We refer to these unitaries as *Gaussian optical elements* or, by analogy with linear optics, phase shifters (PS_i), beam splitters ($BS_{i,j}$), and parametric amplifiers ($PA_{i,j}$). A product of Gaussian optical elements is called an optical circuit. When $\varphi = 0$, this set of transformations acting on Fock states and followed by single-mode number detectors defines a computational model called *fermionic linear optics* (FLO). When $\varphi = \pi$, they are called *matchgates* [41], which we refer to here as *qubit linear optics* (QLO). For any other value of φ, we refer to quantum computing with optical circuits by *fermionic-anyon linear optics* (ΦLO). What operations are analogous to matchgates for fermionic anyons?

Let us use J_φ to translate known FLO results into results for ΦLO and QLO. First, we look at how fermionic optical elements transform under J_φ^{-1}. We are interested in invariant operations under J_φ^{-1}, i.e., that have the same operator decomposition in all particle systems. Since phase shifters are generated by Hamiltonians proportional to $f_i^\dagger f_i$, they must be invariant under the action of J_φ^{-1}—as must, in fact, be any operator whose fermionic form contains only products of number operators.

Fermionic beam splitters are generated by Hamiltonians proportional to $f_i^\dagger f_j + f_j^\dagger f_i$. Those are transformed by J_φ^{-1} into $a_{\varphi,i}^\dagger e^{i\varphi \sum_{k=i+1}^{j-1} n_k} a_{\varphi,j} + a_{\varphi,j}^\dagger e^{-i\varphi \sum_{k=i+1}^{j-1} n_k} a_{\varphi,i}$. This form implies that J_φ^{-1} leaves only nearest-neighbor beam splitters invariant. It is known that all single-particle fermionic operators can be decomposed as products of phase shifters and nearest-neighbor beam splitters by using the fSWAP gate, given by

$$f\mathrm{SWAP}_{i,i+1} = \exp\left[i\frac{\pi}{2}(f_i^\dagger - f_{i+1}^\dagger)(f_i - f_{i+1})\right], \tag{14}$$

which is itself expressible as a product of nearest-neighbor beam splitters and phase shifters [54]. Therefore, we conclude that, even if fermionic-anyon single-particle operators are complicated, they can always be decomposed in fermionic-anyon nearest-neighbor optical elements.

Fermionic parametric amplifiers are generated by Hamiltonians proportional to $f_i^\dagger f_j^\dagger + f_j f_i$. Their J_φ^{-1} transforms are given by $e^{-i2\varphi \sum_{k=1}^{i-1} n_k} a_i^\dagger e^{-i\varphi \sum_{k=i+1}^{j-1} n_k} a_j^\dagger + e^{i2\varphi \sum_{k=1}^{i-1} n_k} a_i e^{i\varphi \sum_{k=i+1}^{j-1} n_k} a_j$. The only case where a parametric amplifier is invariant under J_φ^{-1} is when $\{i, j\} = \{1, 2\}$. Nevertheless, we also show in the Supplemental Material [44] that an arbitrary fermionic $PA_{i,j}$ can be decomposed in terms of $PA_{1,2}$ and fSWAP, implying a similar decomposition for their anyonic counterparts.

In Ref. [55], it was shown that FLO circuits are easy to simulate classically in the sense that if $[U]_0$ is an FLO circuit, there is a polynomial-time classical algorithm that computes the matrix elements of $[U]_0$ in the Fock basis. Now, since the Fock-basis elements of $[U]_\varphi$ are, by construction, the same as those of $[U]_0$, the same algorithm can efficiently compute the matrix elements of $[U]_\varphi$ for states in the fermionic-anyon Fock space. Therefore, any ΦLO circuit composed only of $PA_{1,2}$ and nearest-neighbor beam splitters must be easy to simulate classically in the same sense. For the special case of QLO, this recovers well-known simulability results for circuits of nearest-neighbor matchgates [56].

Given that all FLO circuits are easy to simulate classically represented either in fermionic or anyonic form, we might ask if all ΦLO circuits are also easy to simulate. The answer, however, is no, for the following reasons. A fermionic-anyon beam splitter is generated by a Hamiltonian proportional to $a_{\varphi,i}^\dagger a_{\varphi,j} + a_{\varphi,j}^\dagger a_{\varphi,i}$. Under J_φ, this gets transformed into $f_i^\dagger e^{-i\varphi \sum_{k=i+1}^{j-1} n_k} f_j + f_j^\dagger e^{i\varphi \sum_{k=i+1}^{j-1} n_k} f_i$, which only generates a fermionic Bogoliubov transformation if $j = i + 1$ and, therefore, is not generally an FLO circuit. It was shown in Ref. [57] that non-nearest-neighbor beam splitters allow for universal quantum computation with fermionic anyons for all $\varphi \neq 0$, which also reduces to a known result for matchgates when $\varphi = \pi$ [41]. Furthermore, it is known that almost all fermionic non-Gaussian operations (i.e., gates outside of FLO) can extend it to universality [58], from which follows the analogous statement for ΦLO for any $\varphi \neq 0$.

To summarize, the set of FLO circuits is strictly smaller than the set of ΦLO (or QLO) circuits. Furthermore, the J_φ^{-1} map sends FLO circuits into a small subset of ΦLO circuits, which is particularly easy to simulate classically (when acting on the Fock basis). What about the computational power of these models with input states not on the Fock basis?

In Ref. [42], the authors show a magic-state injection protocol that uses only nearest-neighbor QLO operations to perform universal quantum computation. Furthermore, they also show that *any* fermionic non-Gaussian state is a magic state for the same protocol. Since the transformations themselves are nonuniversal, we can identify a computational *resource*, necessary for a quantum speedup, in the magic states and identify the set of fermionic Gaussian states as resource free. This dichotomy matches that defined by the notion of separability: (pure) Gaussian fermionic states are also the free states if one views entanglement as the resource, as done in Ref. [59] based on a definition of one-body entanglement entropy.

The proposed methods allow us to repurpose these previous results and draw similar conclusions for fermionic anyons. By writing the magic state injection protocol in terms of J_φ^{-1} invariant optical elements, and subsequently applying $J_\varphi^{-1} \circ J_\pi$ to the corresponding circuit, the same injection protocol can use fermionic-anyon magic states to induce a non-FLO operation. Following the approach developed in Ref. [59], our results imply that the notions of free states for both types of resources (computational power and entanglement) match for fermionic anyons as they do for fermions.

Our formalism can also be used to understand previous results about matchgate circuits (i.e., QLO). References [41,60,61], for example, consider supplementing circuits of nearest-neighbor matchgates with other resources. The authors use a dual-rail encoding, where we can encode a logical 0 (resp. 1) qubit state as the $|01\rangle$ (resp. $|10\rangle$) state of two physical qubits. In that case, Ref. [54] shows that the fSWAP gate cannot be used to generate entanglement between the two logical qubits, whereas the SWAP gate can—a curious role reversal, given that the fSWAP is a maximally entangling two-qubit gate and the SWAP is not entangling. Our formalism provides an alternative interpretation that resolves this conundrum neatly: there is a notion of entanglement between logical qubits in a matchgate circuit, corresponding to the one we proposed, when one views the state of a physical qubit as the occupation number of a fermionic-anyon mode (at $\varphi = \pi$). This notion of entanglement would naturally differ from the standard definition of entanglement between the physical qubits, but would be more relevant to the computational complexity of matchgate circuits with non-Gaussian elements—for instance, it is an interesting question for future work whether this alternative notion of entanglement translates into a quantitative measure of the complexity of classical simulation of matchgate circuits.

Conclusions. In summary, we have introduced a resource-theoretic framework for investigating the separability of fermionic anyons and their connection to quantum computing. We characterized the entanglement of fermionic anyons, and showed that the concept of fermionic-anyon separability can be mapped to the free resources of matchgate circuits. Our framework was applied to particle-based quantum computing, revealing that fermionic-anyon linear-optical circuits can be expressed using nearest-neighbor beam splitters, phase shifters, and mode swaps. Additionally, we showed that universal quantum computation with fermionic anyons can be achieved by introducing nonseparable states, similar to the magic-state injection protocol presented in Ref. [42].

Finally, we translated the statistical parameter φ of the fermionic-anyon commutation relation into two well-established forms of universal quantum computation: fermionic and qubit-based. When $\varphi = 0$, universal anyonic quantum computation reduces to fermionic linear optics; similarly, qubit linear optics can be obtained by interpreting a physical qubit as the occupation of a fermionic-anyon mode at $\varphi = \pi$. This approach creates a matchgate scheme where magic states are entangled as per our definition rather than the traditional notion for qubits. These notions are not equivalent, and our definition is instead a type of particle entanglement when one interprets qubits as occupation numbers of exotic particles [43]. Nonetheless, we consider it to have already helped to reinterpret the results of Ref. [41] in a clearer manner. We leave it as a direction for future research to investigate further consequences of viewing qubit circuits via the lens of our definition of fermionic-anyon entanglement, as well as the possible resourceful limitations and costs of such a quantum computational model. Naturally, one could ask: what quantum computational models exist for intermediate parameters $\varphi \in (0, \pi)$? We leave this question unanswered and also propose a potential generalization model for two-dimensional fermionic anyons, inviting further investigation.

To achieve a comprehensive quantum computation framework, we need to establish a measurement method specific to fermionic anyons. One can explore the measurement disturbance model presented in Ref. [62] and adapt it for fermionic anyons employing JWT. Considering the techniques outlined in Ref. [59], one could fully describe the theoretical resources available for fermionic anyons computation.

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[44] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevA.110.L010404 for we give further details of the analytical results which have been omitted from the main text. Specifically, we provide an expanded and mathematically rigorous discussion of the fermionic-anyon algebra, their Jordan-Wigner transformation and subsequent connections to fermionic linear optics and classical simulability.

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