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Letter

Faithful geometric measures for genuine tripartite entanglement

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We present a faithful geometric picture for genuine tripartite entanglement of discrete, continuous, and hybrid quantum systems. We first find that the triangle relation $\mathcal{E}^{\alpha}_{i|jk} \leqslant \mathcal{E}^{\alpha}_{j|ik} + \mathcal{E}^{\alpha}_{k|ij}$ holds for all subadditive bipartite entanglement measure $\mathcal{E}$, all permutations under parties i, j, k, all $\alpha \in [0, 1]$, and all pure tripartite states. Then, we rigorously prove that the nonobtuse triangle area, enclosed by side $\mathcal{E}^{\alpha}$ with $0 < \alpha \leqslant 1/2$, is a measure for genuine tripartite entanglement. Finally, it is significantly strengthened for qubits that given a set of subadditive and nonsubadditive measures, some state is always found to violate the triangle relation for any $\alpha > 1$, and the triangle area is not a measure for any $\alpha > 1/2$. Our results pave the way to study discrete and continuous multipartite entanglement within a unified framework.

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Introduction. Entanglement is the most puzzling feature of quantum theory and plays an indispensable role in various quantum processing tasks, such as device-independent cryptography [1] and teleportation [2]. It is thus of fundamental and practical interest to investigate the characterization and quantification of entanglement. During past decades, extensive research has been devoted to studying entanglement for bipartite quantum systems and, correspondingly, numerous methods have been developed to quantify bipartite entanglement [3–7]. However, much less progress has been achieved for multipartite entanglement, mainly due to the complicated structures of multipartite states and operations [8]. Besides, it still lacks a unified approach to quantifying entanglement of both discrete and continuous quantum systems, except for using a well-chosen entanglement witness [4] or Fisher information matrix [9] to certify the presence of entanglement.

In this work, we mainly study the entanglement properties of discrete, continuous, and even hybrid tripartite systems. In particular, using the well-explored measures for bipartite entanglement, we present a unified geometric approach to tackle the problems of (1) how to characterize tripartite entanglement and (2) how to quantify genuine tripartite entanglement via *proper* measures that are able to detect all genuinely entangled states that are useful in multiparty information tasks.

Our first main result is a geometric characterization of tripartite entanglement based on a class of triangle relations. Specifically, given any tripartite state $|\psi\rangle_{ABC}$, the degree of entanglement between one party and the rest, quantified by a subadditive measure $\mathcal{E}$, satisfies

$$\mathcal{E}^{\alpha}_{i|jk} \leqslant \mathcal{E}^{\alpha}_{j|ik} + \mathcal{E}^{\alpha}_{k|ij}, \tag{1}$$

and its permutations under three parties, $i, j, k = A, B, C$, for power $\alpha \in [0, 1]$. This relation suggests that entanglement owned by one party is no larger than the sum of entanglement by the other two, complementary to monogamy of entanglement [10–12] and closely related to the entanglement polytopes [13,14] and the quantum marginal problem [15,16]. Furthermore, Tsallis entropy [17] is chosen as the measure $\mathcal{E}$ to exemplify that it is valid for all discrete, all Gaussian, and all discrete-discrete-continuous pure tripartite states, significantly generalizing previous results for qubits [18–21]. When it comes to the qubit, Eq. (1) is obtained for subadditive measures, such as von Neumann entropy [22], Tsallis entropy [17,23], squared concurrence [24–26], squared negativity [27], and nonsubadditive ones, including Schmidt weight [28,29] and Rényi-2 entropy [30].

As illustrated in Fig. 1, the triangle relation (1) and its permutations provide us with a nice geometric picture for tripartite entanglement, in the sense that its bipartition entanglement, measured by $\mathcal{E}^{\alpha}$, can be interpreted as the side of a triangle. It is further proven to be *faithful* for genuine tripartite entanglement that the induced triangle with $\alpha \in (0, 1)$ is nondegenerate or, equivalently, has nonzero area, if and only if the tripartite state, either pure or mixed, is genuinely entangled.

Our second main result is a class of faithful measures for genuine tripartite entanglement. Particularly, the triangle area, induced by the above relation (1),

$$\mathcal{A}(|\psi\rangle_{ijk}) = \sqrt{Q(Q-\mathcal{E}^{\alpha}_{i|jk})(Q-\mathcal{E}^{\alpha}_{j|ik})(Q-\mathcal{E}^{\alpha}_{k|ij})}, \tag{2}$$

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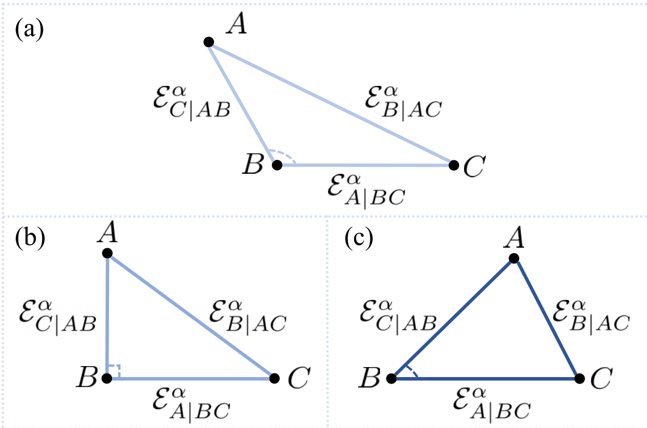

FIG. 1. The faithful geometric interpretation for genuine tripartite entanglement of discrete, continuous, and hybrid quantum systems. Following the triangle relation (1) for any pure state $|\psi\rangle_{ijk}$ shared by three parties, $i, j, k = A, B, C$, three bipartition entanglement, measured by $\mathcal{E}^{\alpha}_{i|jk}$ with $\alpha \in [0, 1]$, can be interpreted as the side of a triangle. The faithfulness is obtained in Proposition 2 for generic tripartite states and Theorem 2 for three-qubit states that the triangle has nonzero area if and only if the state is genuinely entangled. Depending on whether the condition (9) is satisfied, the triangle can be categorized into (a) obtuse, e.g., Tsallis entropy (3) with $\alpha > 1/2$ for some state; (b) right angled, e.g., Schmidt weight with $\alpha = 1/2$ for a class of W-class states; and (c) acute, e.g., all subadditive measures with $\alpha \in [0, 1/2)$ for all pure states. In the last two cases, the triangle area (2) satisfies LOCC monotonicity as per Theorems 1 and 3.

with the semiperimeter $Q = (\mathcal{E}^{\alpha}_{i|jk} + \mathcal{E}^{\alpha}_{j|ik} + \mathcal{E}^{\alpha}_{k|ij})/2$, is a natural quantifier for genuine tripartite entanglement. We analytically prove that for any subadditive measure $\mathcal{E}$ with $\alpha \in (0, 1/2]$, the area is monotonic under local operations and classical communication (LOCC), thus being a reliable entanglement measure. Importantly, since the proof of LOCC monotonicity is independent of measure and state, it is widely applicable to the discrete and/or continuous systems. Useful lower and upper bounds are derived for these geometric measures, related to the well-known multipartite entanglement measures, such as genuinely multipartite concurrence [31] and global entanglement measure [32,33].

Finally, our results are significantly strengthened for qubits, in the sense that given a set of entanglement measures, some state is always found to violate the triangle relation (1) with any $\alpha > 1$ and to violate the LOCC monotonicity of triangle area (2) with any $\alpha > 1/2$. As a byproduct, our results confirm the concurrence fill [34–36] and the ergotropic fill [37] as feasible entanglement measures, and overcome an incompleteness in the proof in [36] to show the LOCC monotonicity of the concurrence area. Generalizations of our results are also discussed.

Entanglement measures. The resource theory of entanglement [38,39] is first briefly recapped. Within this framework, nonentangled states correspond to free states, while entangled ones can be recognized as essential resources to accomplish impossible tasks in the classical realm. In order to quantify these resources, an entanglement measure $\mathcal{E}$ is typically introduced as some function which maps any quantum state to a non-negative number, i.e., $\mathcal{E}(\rho) \geqslant 0$ for state ρ. Further, it needs to meet the following extra requirements [3,4,40,41]: (1) faithfulness, i.e., $\mathcal{E}(\rho) = 0$ if and only if ρ is separable or nonentangled; (2) LOCC monotonicity, i.e., $\mathcal{E}(\rho) \geqslant \sum_m p_m \mathcal{E}(\rho_m)$ for any ρ and its LOCC ensemble $\{p_m, \rho_m\}$, requiring that entanglement never increases under free operations of LOCC; and (3) symmetry, i.e., given a pure bipartite state $|\psi\rangle_{ij}$, global entanglement is determined by the local state, i.e., $\mathcal{E}(|\psi\rangle_{ij}) \equiv E(\rho_i) = E(\rho_j)$, where E is properly defined on states $\rho_{i(j)} = \mathrm{Tr}_{j(i)}(|\psi\rangle_{ij}\langle\psi|)$.

One notable example satisfying the above conditions is Tsallis entropy [17],

$$\mathcal{T}(|\psi\rangle_{ij}) = \mathcal{T}(\rho_i) \equiv \frac{1 - \mathrm{Tr}(\rho_i^q)}{q-1}, \quad q \geqslant 1, \tag{3}$$

for any pure bipartite state $|\psi\rangle_{ij}$. It recovers von Neumann entropy in the limit $q \to 1$ and reduces to linear entropy or generalized concurrence [25,26] by $q = 2$, both of which also admit the subadditivity (see [42–44], including the Supplemental Material),

$$E(\rho_{ij}) \leqslant E(\rho_i) + E(\rho_j), \tag{4}$$

for any ρ_{ij} in discrete and continuous systems. Indeed, whether all these requirements can be satisfied depends on both the state space and the measure, and there exist measures without subadditivity [45]. For example, the measure of Rényi-2 entropy is not subadditive for qudits [46], while it is for the Gaussian [47]. In the following, we discuss how to use bipartite entanglement measures to study multipartite entanglement.

Triangle relations and geometric picture for tripartite entanglement. Any pure tripartite state $|\psi\rangle_{ijk}$ admits three bipartition among parties i, j, k, of which bipartite entanglement is quantified by $\mathcal{E}_{i|jk}, \mathcal{E}_{j|ik}, \mathcal{E}_{k|ij}$, respectively, with a bipartite entanglement measure $\mathcal{E}$. We can obtain the following result.

Proposition 1. For any subadditive measure $\mathcal{E}$, the triangle relation (1) holds for all pure tripartite states, all permutations under three parties, and all $\alpha \in [0, 1]$.

The proof is as follows. First, note from the symmetric property that $\mathcal{E}_{k|ij} \equiv E(\rho_{ij}) = E(\rho_k)$ holds, with $\rho_{ij(k)} = \mathrm{Tr}_{k(ij)}(|\psi\rangle_{ijk}\langle\psi|)$. It then follows from the subadditivity that $\mathcal{E}_{k|ij} = E(\rho_{ij}) \leqslant E(\rho_i) + E(\rho_j) = \mathcal{E}_{i|kj} + \mathcal{E}_{j|ki}$, proving Eq. (1) with $\alpha = 1$. Finally, we have

$$\mathcal{E}^{\alpha}_{i|jk} \leqslant (\mathcal{E}_{j|ik} + \mathcal{E}_{k|ij})^{\alpha} \leqslant \mathcal{E}^{\alpha}_{j|ik} + \mathcal{E}^{\alpha}_{k|ij}, \quad \forall \alpha \in [0, 1). \tag{5}$$

The first inequality follows from $x^r \leqslant y^r$ for non-negative $x \leqslant y$ and $r < 1$, and the second from $(x + y)^r \leqslant x^r + y^r$ for non-negative x, y and $r < 1$. Thus, the triangle relation (1) holds for all pure states and all $\alpha \in [0, 1]$, and its permutations can be obtained similarly. We further prove in the Supplemental Material [44] that in terms of the Tsallis entropy as per (3), Eq. (1) is valid for all discrete, all Gaussian, and all discrete-discrete-continuous pure tripartite states, and also generalized to a polygon relation for pure discrete and Gaussian multipartite states, significantly generalizing previous results derived for qubits [18–20].

As illustrated in Fig. 1, the triangle relation (1) yields a geometric description of $|\psi\rangle_{ijk}$ that bipartite entanglement corresponds to the side of a triangle. If the state is biseparable, i.e., at least one zero $\mathcal{E}^{\alpha}_{i|jk}$, then its triangle degenerates to a line or a point. Next, we show the converse is also true.

Proposition 2. For any subadditive measure $\mathcal{E}$, the triangle area (2) enclosed by the relation (1) with $\alpha \in (0, 1)$ is nonzero if and only if the pure tripartite state is genuinely entangled.

We prove Proposition 2 by contradiction. Indeed, it equates to proving that the triangle relation (1) with $\alpha \in (0, 1)$ can never be saturated by genuinely entangled states with three nonzero $\mathcal{E}^{\alpha}_{i|jk}$. If $\mathcal{E}^{\alpha}_{i|jk} = \mathcal{E}^{\alpha}_{j|ik} + \mathcal{E}^{\alpha}_{k|ij}$ holds for some $\alpha \in (0, 1)$ and nonzero $\mathcal{E}^{\alpha}$, then

$$\begin{aligned}\mathcal{E}_{i|jk} &= \left(\mathcal{E}^{\alpha}_{i|jk}\right)^{1/\alpha} = \left(\mathcal{E}^{\alpha}_{j|ik} + \mathcal{E}^{\alpha}_{k|ij}\right)^{1/\alpha} \\ &> \left(\mathcal{E}^{\alpha}_{j|ik}\right)^{1/\alpha} + \left(\mathcal{E}^{\alpha}_{k|ij}\right)^{1/\alpha} = \mathcal{E}_{j|ik} + \mathcal{E}_{k|ij}. \end{aligned} \qquad (6)$$

The first inequality follows from $(x + y)^r > x^r + y^r$ for positive x, y and $r = 1/\alpha > 1$. It is obvious that Eq. (6) contradicts the triangle relation (1). Thus, we complete the proof of Proposition 2, which provides a faithful geometric picture for genuinely entangled states.

It is remarked that whether the triangle inequality (1) with $\alpha = 1$ can be saturated depends on the state space and the measure. For example, there is $\mathcal{S}_{A|BC} = 2$ and $\mathcal{S}_{B|AC} = \mathcal{S}_{C|AB} = 1$ for genuinely entangled state $(|000\rangle + |101\rangle + |210\rangle + |311\rangle)/2$ and von Neumann entropy $\mathcal{S}$, while equality can never be achieved by genuinely entangled three-qubit states and Tsallis entropy (3) with $q > 1$ (see the Supplemental Material [44]).

Proposition 2 indicates that the triangle area (2) is a natural quantifier for pure tripartite entanglement. For a general state ρ, using the convex-roof construction,

$$\mathcal{A}(\rho) := \inf_{\{p_m, |\psi_m\rangle\}} \sum_m p_m\, \mathcal{A}(|\psi_m\rangle), \qquad (7)$$

where the infimum is over all pure decompositions $\rho = \sum_m p_m |\psi_m\rangle\langle\psi_m|$, one can show that $\mathcal{A}(\rho) = 0$ if and only if ρ is biseparable, admitting a decomposition of which all pure states are biseparable. This implies that the triangle area is a faithful quantifier of genuine tripartite entanglement for both pure and mixed states.

Triangle area as an entanglement measure. We continue to derive a stronger result that the triangle area is a measure for genuine tripartite entanglement.

Theorem 1. For any subadditive measure $\mathcal{E}$, the triangle area (2) with $\alpha \in (0, 1/2]$ admits LOCC monotonicity and hence is a reliable entanglement measure.

Before proceeding to prove Theorem 1, we first introduce the parametrized vector $\mathbf{x} = (x_1, x_2, x_3)^{\top} = (\mathcal{E}^{2\alpha}_{i|jk}, \mathcal{E}^{2\alpha}_{j|ik}, \mathcal{E}^{2\alpha}_{k|ij})^{\top}$. Correspondingly, the area (2) can be rewritten as (see the Supplemental Material [44])

$$f(\mathbf{x}) = \tfrac{1}{4}\sqrt{-x_1^2 + 2x_1(x_2 + x_3) - (x_2 - x_3)^2}\,. \qquad (8)$$

Evidently, the function f is continuous and permutation invariant under parameters x_i. Moreover, we have

Lemma 1. For $\alpha \in [0, 1/2]$, Eq. (8) is nondecreasing and concave as a function of $(x_1, x_2, x_3)^{\top} = (\mathcal{E}^{2\alpha}_{i|jk}, \mathcal{E}^{2\alpha}_{j|ik}, \mathcal{E}^{2\alpha}_{k|ij})^{\top}$.

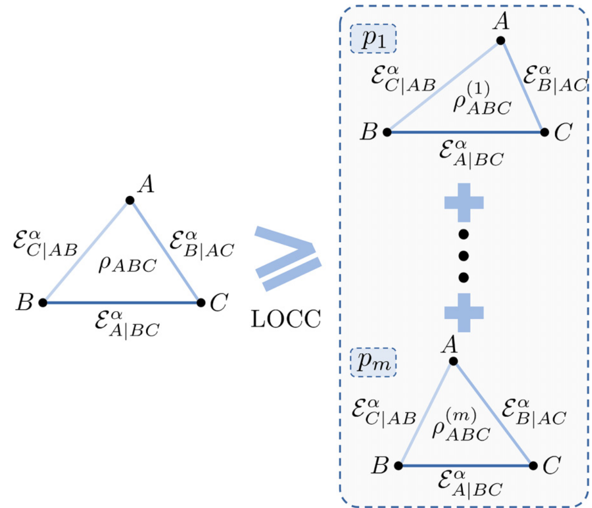

FIG. 2. LOCC monotonicity of the triangle area (2) with $\alpha \in (0, 1/2]$, which is proven in Theorem 1. It indicates that the triangle area is a measure for genuine tripartite entanglement. This is further strengthened for qubits in Theorem 3 that given a set of measures, the LOCC monotonicity can be violated by three-qubit states for $\alpha > 1/2$.

The nondecreasing tendency of f over each x_i is determined by its non-negative first derivatives,

$$\frac{\partial f}{\partial x_i} = \frac{\partial \mathcal{A}}{\partial \mathcal{E}^{2\alpha}_{i|jk}} = \frac{\mathcal{E}^{2\alpha}_{j|ik} + \mathcal{E}^{2\alpha}_{k|ij} - \mathcal{E}^{2\alpha}_{i|jk}}{16\mathcal{A}} \geqslant 0. \qquad (9)$$

The inequality follows from Proposition 1 that $x_1 \leqslant x_2 + x_3$, $x_2 \leqslant x_1 + x_3$, and $x_3 \leqslant x_1 + x_2$ for $\alpha \leqslant 1/2$. It immediately yields that the triangle enclosed by (1) with $\alpha \in [0, 1/2]$ is nonobtuse, as its interior angles obey $\cos\theta_i = (\mathcal{E}^{2\alpha}_{j|ik} + \mathcal{E}^{2\alpha}_{k|ij} - \mathcal{E}^{2\alpha}_{i|jk})/2\mathcal{E}^{\alpha}_{j|ik}\mathcal{E}^{\alpha}_{k|ij} \geqslant 0$. The concavity of f is determined by its Hessian matrix, which is nonpositive definite (see the Supplemental Material [44]).

We then use Lemma 1 to obtain the proof of LOCC monotonicity of the triangle area, as displayed in Fig. 2. Being restricted to the pure state $|\psi\rangle$ and any pure LOCC ensemble $\{p_m, |\psi_m\rangle\}$, we have

$$\begin{aligned}\sum_m p_m\, \mathcal{A}(|\psi_m\rangle) &= \sum_m p_m\, f(\mathbf{x}^m) \leqslant f\Big(\sum_m p_m\, \mathbf{x}^m\Big) \\ &= f\Big(\sum_m p_m\, x_1^m, \sum_m p_m\, x_2^m, \sum_m p_m\, x_3^m\Big) \\ &\leqslant f(x_1, x_2, x_3) = f(\mathbf{x}) = \mathcal{A}(|\psi\rangle). \end{aligned} \qquad (10)$$

All equalities follow directly from Eq. (8) by associating each pure state with a parametrized vector $\mathbf{x}^m$, the first inequality from Lemma 1, and the second from Lemma 1 and the fact that $\mathcal{E}^{2\alpha}_i$ is a measure of bipartite entanglement for $\alpha \leqslant 1/2$, i.e., $x_i \geqslant \sum_m p_m\, x_i^m$ for $i = 1, 2, 3$. For a general state and its general LOCC ensemble, using the convex-roof rule (7) and thus convexity of the area leads to a similar proof of LOCC monotonicity. Thus, we complete the proof of Theorem 1. It is worth noting that as the proof is independent of both state and measure, Theorem 1 is widely applicable to the discrete, continuous, and even hybrid quantum systems.

TABLE I. Genuine tripartite entanglement measured by triangle areas and GMC. $\mathcal{A}_1$ describes the triangle area of Tsallis entropy with $q=1$ and $\mathcal{A}_2$ the area of Tsallis entropy with $q=2$, with $\alpha=1/2$. The concurrence is used in GMC. A normalization coefficient $(16/3)^{1/2}$ is applied to the triangle area to guarantee $\mathcal{A}_1, \mathcal{A}_2 \leqslant 1$ for all three-qubit states.

	$\|\psi_1\rangle$	$\|\psi_2\rangle$	$\|\psi_3\rangle$
GMC	0.5878	0.7071	0.7071
$\mathcal{A}_1$	0.7329	0.6009	0.8251
$\mathcal{A}_2$	0.6487	0.5	0.7638

For $\alpha > 1/2$, neither the nonobtuse condition (9) nor concavity of f can always be satisfied, signaling the possibility of violating LOCC monotonicity and thus not being a measure. This is confirmed in the Supplemental Material [44] that with Tsallis entropy, the triangle area with $\alpha > 1/2$ can be increased by LOCC on a family of W-class states.

Upper and lower bounds. Following again from Lemma 1 that the area is nondecreasing under each $\mathcal{E}^{\alpha}_{i|jk}$, we can obtain a lower bound,

$$\mathcal{A}(|\psi\rangle_{ijk}) \geqslant \min \frac{\sqrt{3}}{4}\{\mathcal{E}^{2\alpha}_{i|jk}, \mathcal{E}^{2\alpha}_{j|ik}, \mathcal{E}^{2\alpha}_{k|ij}\}, \tag{11}$$

which can be interpreted as proportional to the squared smallest side of the triangle. This bound is also a measure for genuine tripartite entanglement. In particular, if $\mathcal{E}^{\alpha}$ is Tsallis-2 entropy with $\alpha = 1/2$, then it recovers the well-known genuinely multipartite concurrence (GMC) [31]. Additionally, the triangle area is upper bounded by

$$\mathcal{A}(|\psi\rangle_{ijk}) \leqslant \sqrt{Q\left(\frac{3Q-2Q}{3}\right)^3} \leqslant \frac{\mathcal{E}^{2\alpha}_{i|jk} + \mathcal{E}^{2\alpha}_{j|ik} + \mathcal{E}^{2\alpha}_{k|ij}}{4\sqrt{3}}, \tag{12}$$

proportional to the average of the squared sides. The first inequality follows from $xyz \leqslant (\frac{x+y+z}{3})^3$ and the second from $(x+y+z)^2 \leqslant 3(x^2+y^2+z^2)$ for non-negative x, y, z. In terms of Tsallis-2 entropy, it reduces to the global entanglement measure for $\alpha = 1/2$ [32,33], which, however, can be nonzero even if the state is biseparable.

We exemplify the main differences between the geometric measures and GMC for genuine tripartite entanglement. Denote $\mathcal{A}_1$ by the triangle area of Tsallis entropy with $q=1$, equivalent to von Neumann entropy, and $\mathcal{A}_2$ by the area of Tsallis entropy with $q=2$, with $\alpha = 1/2$. The bipartite measure in GMC refers to concurrence. We, in particular, consider three states $|\psi_1\rangle = (\sin\frac{\pi}{5}|000\rangle + \cos\frac{\pi}{5}|100\rangle + |111\rangle)/\sqrt{2}$, $|\psi_2\rangle = \cos\frac{\pi}{8}|000\rangle + \sin\frac{\pi}{8}|111\rangle$, $|\psi_3\rangle = \frac{1}{2}|000\rangle + \frac{1}{2}|100\rangle + \frac{1}{\sqrt{2}}|111\rangle$. It is shown in the rows of Table I that two triangle areas have different entanglement orderings of three states, in comparison to GMC, while different columns of Table I indicate that these three measures lead to different entanglement orderings of three states.

Strengthened results for qubits. When it is restricted to three-qubit states, all of the above results can be significantly strengthened. Here we consider the subadditive measures such as von Neumann entropy $\mathcal{S}$, Tsallis entropy $\mathcal{T}$, squared concurrence $\mathcal{C}^2$, and squared negativity $\mathcal{N}^2$, and nonsubadditive ones, including Schmidt weight $\mathcal{W}$ and Rényi-2 entropy $\mathcal{R}$. They can be unified via some function on the smallest eigenvalue λ of the reduced state for any pure two-qubit state (see the Supplemental Material [44]),

$$\begin{aligned}
&\mathcal{S}(\lambda) = -\lambda\log_2\lambda - (1-\lambda)\log_2(1-\lambda),\\
&\mathcal{T}(\lambda) = \frac{1-\lambda^q-(1-\lambda)^q}{q-1}, \quad \mathcal{C}^2(\lambda) = \mathcal{N}^2(\lambda) = 4\lambda(1-\lambda),\\
&\mathcal{W}(\lambda) = 2\lambda, \quad \mathcal{R}(\lambda) = -\log_2[\lambda^2 + (1-\lambda)^2].
\end{aligned} \tag{13}$$

Consequently, we can derive the following results, for which the proofs are deferred to the Supplemental Material [44].

Theorem 2. For the measure set $\{\mathcal{S}, \mathcal{T}, \mathcal{C}^2, \mathcal{N}^2, \mathcal{W}, \mathcal{R}\}$, the triangle relation (1) holds for any $\alpha \in [0, 1]$ on all pure three-qubit states, and can be violated by some state for any $\alpha > 1$. Moreover, the triangle area (2) is nonzero if and only if the three-qubit state is genuinely entangled, except for Schmidt weight with $\alpha = 1$.

Theorem 2 immediately yields that the nonsubadditive measures, such as Schmidt weight and Rényi-2 entropy, are subadditive on all two-qubit states with rank no larger than 2. It also strengthens Proposition 2, in the sense that $\alpha = 1$ optimally upper bounds the triangle relation (1) for three-qubit states, and the faithful geometric picture is extended to the bound $\alpha = 1$ for Tsallis entropy. Additionally, the triangle relation with $\alpha = 1$ recovers the ones already obtained in [18,19], and reduces to the entanglement polytopes [13,14] in the context of Schmidt weight.

Theorem 3. For the measure set $\{\mathcal{W}, \mathcal{C}^2, \mathcal{N}^2, \mathcal{S}, \mathcal{T}, \mathcal{R}\}$, the triangle area (2) with $\alpha \in (0, 1/2]$ is an entanglement measure for three-qubit states, while it is not for $\alpha > 1/2$.

We note that violating the LOCC monotonicity by the area induced by subadditive measures with $\alpha > 1/2$ naturally implies the same violation for generic tripartite systems in Theorem 1. Moreover, Theorem 3 rigorously confirms the concurrence fill [35,36] and the ergotropic fill [37] as feasible entanglement measures. It is also found in the Supplemental Material [44] that the proof in [36] is incomplete to guarantee the LOCC monotonicity of the concurrence area.

Discussion. We have presented a unified geometric picture suitable to characterize tripartite entanglement of discrete, continuous, and even hybrid quantum systems, and then proposed using the triangle area as a faithful measure for genuine tripartite entanglement. We have also obtained useful lower and upper bounds for these geometric measures, and explored their connections and differences with the well-known measures for multipartite entanglement. In particular, our results are significantly strengthened for qubits, which also generalize previous results and solve open questions left in previous works.

Generalizations of our results are given as follows. With regard to LOCC monotonicity, it follows from the convexity that the triangle area (2) with $\alpha \in (0, 1/2]$ also admits a weaker monotonicity in the form of $\mathcal{A}(\rho) \geqslant \sum_m \mathcal{A}(\sum_m p_m\rho_m)$. It is thus interesting to investigate whether our results can be applied to the measures only satisfying this weaker LOCC monotonicity, i.e., $\mathcal{E}(\rho) \geqslant \sum_m \mathcal{E}(\sum_m p_m\rho_m)$. If the measure is nonfaithful, i.e., $\mathcal{E}(\rho)=0$ for some entangled state ρ, the corresponding triangle area can still be a measure for tripartite entanglement, but may no longer be faithful. For any nonsubadditive measure $\mathcal{E}$, it has been shown in [48] that it always satisfies the triangle relation (1) for some $0<\beta<+\infty$, implying $\mathcal{E}^{\beta}$ is subadditive. It follows from Lemma 1 that $\mathcal{E}^{\alpha}$ with $\alpha\in(0,\beta/2]$ satisfies the nonobtuse condition (9) and the enclosing area is a measure for genuine tripartite entanglement. However, it could be challenging to obtain a proper β for the nonsubadditive measure.

Finally, we point out that the triangle relation (1) can be generalized to a polygon relation for both discrete and continuous multipartite states [44]. Hence, we expect our results to aid significant progress in studying entanglement of multipartite systems [49]. Furthermore, on the basis of our faithful measure for genuine tripartite entanglement in Theorem 1, it is interesting to study the classification of multipartite entangled states. We also hope our results find applications in studying other multipartite quantum resources, such as genuine nonlocality [50] and steering [51].

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