Letter

Detecting beyond-quantum nonlocality using standard local quantum observables

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We study the detection of beyond-quantum nonlocal states that can exist in a theoretical model whose local systems are standard quantum theory in the framework of general probabilistic theories (GPTs). We find that device-dependent detections are possible for beyond-quantum nonlocal states in GPTs even though device-independent detections are not valid. We give a device-dependent detection based on local observables to distinguish any beyond-quantum nonlocal state from all standard quantum states. In particular, we give a way to detect any beyond-quantum nonlocal state of the two-qubit system by observing only spin observables on local systems. Our results will help in the experimental detection of beyond-quantum nonlocality or justification of standard quantum theory.

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Introduction. Bell's inequality [1] (or CHSH inequality [2]) is one of the important ways to detect quantum nonlocality in our physical systems. Bell-CHSH inequality (hereinafter CHSH inequality) consists of bipartite players and their local operations. It is especially important that the protocol of CHSH inequality can be implemented by local observables. In other words, by implementing the protocol of CHSH inequality as a bipartite communication task, we can experimentally detect quantum nonlocality of our physical systems when Bell-CHSH inequality is violated. Actually, the violation of CHSH inequality is confirmed in physical experiments [3–9]. Moreover, CHSH inequality can be implemented without certification of measurement devices. Such detection without certification of measurement devices is called *device-independent* (DI) detection [10–19]. These remarkable results played an important role in the early studies of quantum physics and quantum information theory to ensure that our physical systems truly possess quantum nonlocality.

However, it is not sufficient for the strict verification of quantum theory to detect standard quantum nonlocality because there are many other theoretical models with nonlocality than quantum systems. Such models can be described as general probabilistic theories (GPTs) [21–44,47]. GPTs are a framework for general theoretical models with states and measurements, including classical and quantum theories. The *PR box* [21–23] is a typical example of nonlocal models with beyond-quantum nonlocality. In the PR box, the CHSH value attains four even though the bound in quantum theory is given as $2\sqrt{2}$, known as Tirelson's bound [9]. In other words, the CHSH inequality can detect the beyond-quantum nonlocality in PR box.

In contrast to models that can be detected by CHSH inequality, there are models that cannot be detected by CHSH inequality even though their local systems are completely equivalent to standard quantum systems. Such models are called *entanglement structures* (ESs) with local quantum systems [34,39–42,47], including many models other than the standard entanglement structure (SES), i.e., the standard quantum model defined by the tensor product. Some ESs have fewer nonlocal states than the SES [39,41], and some ESs have beyond-quantum nonlocal states, i.e., nonlocal states that do not belong to the SES [40,42,44,47]. In order to ensure that our physical systems obey truly standard quantum theory, it is also necessary to verify whether beyond-quantum nonlocal states exist or not. However, preceding studies [45–47] have revealed that all ESs satisfy Tirelson's bound, i.e., CHSH inequality cannot distinguish the SES from any beyond-quantum nonlocal state in ESs.

What was worse, as we and some preceding studies [18,19] mention, is that not only CHSH inequality but also any DI detection cannot distinguish any beyond-quantum nonlocal state from the SES. References [18,19] clarify that any DI detection is impossible for any beyond quantum states in bipartite systems. Besides this, we simply point out that there exist some beyond-quantum states that cannot be distinguished by any DI detection even in multipartite cases (Theorem 1).

Therefore, to detect beyond-quantum states, we need to consider *device-dependent* (DD) detection, which is based on certified local measurements. Recently, pioneering works [48,49] verified the mathematical dimension of standard

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TABLE I. Necessary number of certified measurement settings for detection of beyond-quantum states. As known with the advantage of DI protocol over DD protocol, it costs many physical resources to prepare certified objects. Our DD detection protocol in the d-dimensional and Λ-parties setting needs $\prod_{\lambda\in\Lambda}(d_\lambda^2-1)$ number of measurement settings for certified observables, which costs many resources. However, in the two-qubit case, we give an efficient protocol that works with a smaller number of measurement settings than general cases.

Protocol	No. certified measurement settings
DI detection	Impossible
DD detection (Λ-partite d_λ-dimensional quantum systems)	$\prod_{\lambda\in\Lambda}(d_\lambda^2-1)$
DD detection for two qubits ($\Lambda=\{A,B\}, d_A=d_B=2$)	$3<9=\prod_{\lambda\in\Lambda}(d_\lambda^2-1)$

quantum theory out of other models in GPTs based on DD detection by simple full tomography from the experimental data in quantum physics. However, the results cannot be applied to ESs because they deal with only low-dimensional models and any ES has the same dimension as standard quantum theory. That is, the high cost of the full tomography made them restrict their model in this way. Hence, this paper deals with a more efficient DD detection of an arbitrary beyond-quantum nonlocal state in ESs by an experimental protocol. First, we give a DD detection separating an arbitrary given beyond-quantum state from all standard quantum states as an inequality defined by local observables even in multipartite cases (Theorem 2). Next, we give a protocol to implement the above detection. Our protocol consists of local operations by local players (for example, Alice and Bob in bipartite cases) and classical communication by them. In the protocol, local players detect whether a target state is beyond-quantum or not. If the target state is truly beyond-quantum, they conclude that the target state is beyond-quantum with high probability.

Our criteria and protocol are implemented by a complicated sequence of certified local observables for each beyond-quantum states in general. Because DD detections need certified observables, this protocol takes large costs even without full tomography. However, in the two-qubit case, i.e., in the bipartite 2×2-dimensional case, we give a more efficient detection of a beyond-quantum nonlocal state by observing Pauli's spin observables in a specific order with an uncertified unitary operation. It is known that maximally entangled states are detected when Alice and Bob observe Pauli's spin observable $\sigma_x, \sigma_y, \sigma_z$ in the same order with the sequence of coefficients $(1,-1,1)$ [55,56]. Similar to this result, we clarify that the sequence of coefficients (1,1,1) detects any beyond-quantum "pure" state (Theorem 3) by applying an uncertified unitary operation. As a result, we give a detection for beyond-quantum pure states as an inequality based on fixed certified observables and an uncertified unitary operation in the two-qubit case. Comparing the necessary resources of the two-qubit case with general cases, our protocol in the two-qubit case needs a small number of certified objects (Table I). This efficiency in the two-qubit case is convenient for the future direction of actual experimental verification like [48,49].

The setting of GPTs and entanglement structures. The central postulate of GPTs is the condition that any state ρ in the state space $\mathcal{S}$ and any measurement $\{M_i\}$ in the measurement space $\mathcal{M}$ satisfy $\Pr(\rho, M_i)\geqslant 0$, where $\Pr(\rho, M_i)$ denotes the probability to get an outcome i by $\{M_i\}$ with ρ. This paper deals with the detection of global states by local measurements. For this aim, we consider a composite model of local standard quantum systems given by the Hilbert spaces $\mathcal{H}_\lambda$ for $\lambda\in\Lambda$, where Λ is the set of all labels of local systems. In this case the probability is given as $\Pr(\rho, M_i)=\mathrm{Tr}\rho M_i$.

In the setting of GPTs, a model of composite system is not uniquely determined even if the local systems are fixed, and a model of composite system is given by each measurement space. Such a model is called an *entanglement structure* [34,39–42,47], which is also introduced by operational postulates [34]. We give a more detailed explanation in the Supplemental Material [50].

For the Bell scenario, this paper considers the case that the measurement space $\mathcal{M}$ contains the product of any local POVMs. Under this constraint of measurement space, the largest state space $\mathcal{S}_{\max}(\Lambda)$ consistent with the concept of GPTs is given as

$$\mathcal{S}_{\max}(\Lambda) := \left\{\rho\in\mathcal{L}_{\mathrm{H}}\left(\bigotimes_{\lambda\in\Lambda}\mathcal{H}_\lambda\right)\middle|\mathrm{Tr}\rho=1,\right.$$
$$\left.\times\ \mathrm{Tr}\rho M\geqslant 0 \text{ for any } M\in\mathrm{SEP}(\Lambda)\right\}. \quad (1)$$

Here the set $\mathrm{SEP}(\Lambda)$ is defined as

$$\mathrm{SEP}(\Lambda) := \mathrm{Conv}\left\{\bigotimes_{\lambda\in\Lambda}x_\lambda\middle| x_\lambda\in\mathcal{L}_{\mathrm{H}}^{+}(\mathcal{H}_\lambda)\right\}, \quad (2)$$

where $\mathrm{Conv}(X)$ denotes the convex hull of a set X. In other words, in the scenario of GPTs and the detection of global states by local measurements, any state in $\mathcal{S}_{\max}$ is available. Due to the definition of $\mathcal{S}_{\max}(\Lambda)$, there exist states in $\mathcal{S}_{\max}(\Lambda)$ that do not belong to the standard quantum state space, i.e., the set of density matrices denoted by $\mathcal{S}_{\mathrm{sta}}(\Lambda)$. In this paper we call such a state in $\mathcal{S}_{\max}(\Lambda)\setminus\mathcal{S}_{\mathrm{sta}}(\Lambda)$ a *beyond-quantum state*, and we denote the set of all beyond-quantum states as $\mathrm{BQS}(\Lambda)$.

Here we remark that the above setting corresponds to the well-considered setting of GPTs. In the terminology of general settings of GPTs, the state space $\mathcal{S}_{\max}(\Lambda)$ corresponds to the state space given by the maximal tensor product of positive cones [34]. The detailed explanation is given in the Supplemental Material [50].

A beyond-quantum state σ can be regarded as an entanglement witness because any element $\sigma\in\mathrm{BQS}(\Lambda)$ satisfies that $\mathrm{Tr}\sigma\rho\geqslant 0$ for any separable states $\rho\in\mathcal{S}(\mathrm{SEP}(\Lambda))$ and $\mathrm{Tr}\sigma\rho<0$ for a certain entangled standard quantum state $\rho\in\mathcal{S}_{\mathrm{sta}}(\Lambda)$. Conversely, any linear witness $f:\mathcal{L}_{\mathrm{H}}(\bigotimes_{\lambda\in\Lambda}\mathcal{H}_\lambda)\to\mathbb{R}$ of an entangled state corresponds to a Hermitian matrix σ satisfying $f(\rho)=\mathrm{Tr}\rho\sigma$ because $f(\rho)$ must be negative for an entangled state ρ, which implies that σ must be an element $\mathrm{BQS}(\Lambda)$ by normalization. We do not give concrete

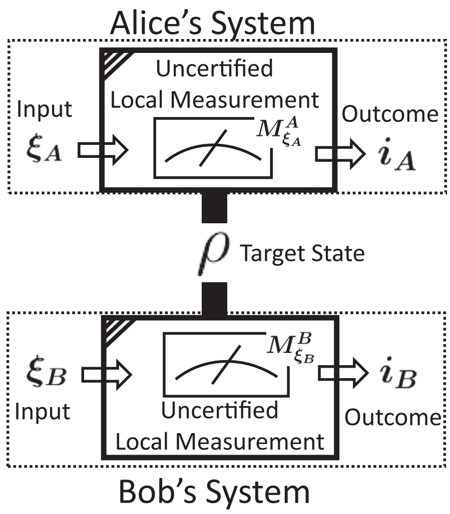

FIG. 1. DI detection in bipartite case $\Lambda = \{A, B\}$. Alice and Bob apply uncertified local measurements $\boldsymbol{M}^A_{\xi_A} := \{M^A_{\xi_A;i_A}\}_{i_A \in I_A}$ and $\boldsymbol{M}^B_{\xi_B} := \{M^B_{\xi_B;i_B}\}_{i_B \in I_B}$ to the bipartite system prepared in a given nonlocal state ρ. Then Alice and Bob determine whether ρ is beyond-quantum by the probability $\mathrm{Tr}\rho(M^A_{\xi_A;i_A} \otimes M^B_{\xi_B;i_B})$.

examples here, but we give an concrete example of beyond-quantum states as (23) in the Supplemental Material [50]. Our interest is how we detect beyond-quantum states if they exist.

Impossibility of device-independent detection for beyond-quantum states. As we will see later, our detection is based on *device-dependent* (DD) setting, which is different from the setting of *device-independent* (DI) detection considered as well. In order to emphasize the necessity of DD setting, we see the impossibility of the DI detection of a beyond-quantum state (Fig. 1). In the DI detection, we have no certificate of measurement devices.

In the following discussion, every local quantum system $\mathcal{H}_\lambda$ is controlled by the player labeled by $\lambda \in \Lambda$. When the composite system is prepared in the target global state ρ, depending on an input $\xi_\lambda \in \Xi_\lambda$, the player $\lambda \in \Lambda$ applies the local POVM $\boldsymbol{M}^\lambda_{\xi_\lambda} := \{M^\lambda_{\xi_\lambda;i_\lambda}\}_{i_\lambda \in I_\lambda}$ on the local quantum system $\mathcal{H}_\lambda$, where I_λ is the set of measurement outcomes of $\boldsymbol{M}^\lambda_{\xi_\lambda}$, and i_λ expresses a measurement outcome of $\boldsymbol{M}^\lambda_{\xi_\lambda}$. Here we choose the set I_λ independently of the input $\xi_\lambda \in \Xi_\lambda$.

Definition 1. A tuple of a beyond-quantum state ρ_0 and a family of local POVMs $((\boldsymbol{M}^\lambda_{\xi_\lambda})_{\xi_\lambda \in \Xi_\lambda})_{\lambda \in \Lambda}$ is called standard quantum simulable when the following condition holds: There exists a tuple of a standard quantum state $\rho_1 \in \mathcal{S}_{\mathrm{sta}}(\Lambda)$ and a family of local POVMs $((\boldsymbol{M}'^\lambda_{\xi_\lambda})_{\xi_\lambda \in \Xi_\lambda})_{\lambda \in \Lambda}$ such that the relation

$$\mathrm{Tr}\rho_0 \bigotimes_{\lambda \in \Lambda} M^\lambda_{\xi_\lambda;i_\lambda} = \mathrm{Tr}\rho_1 \bigotimes_{\lambda \in \Lambda} M'^\lambda_{\xi_\lambda;i_\lambda} \tag{3}$$

holds for any $\xi_\lambda \in \Xi_\lambda, i_\lambda \in I_\lambda$.

Then a beyond-quantum state $\rho_0 \in \mathcal{S}(\mathrm{SEP}^*(\Lambda))$ is distinguished device independently by a family of local POVMs $((\boldsymbol{M}^\lambda_{\xi_\lambda})_{\xi_\lambda \in \Xi_\lambda})_{\lambda \in \Lambda}$ from all standard quantum states if and only if the tuple of ρ_0 and $((\boldsymbol{M}^\lambda_{\xi_\lambda})_{\xi_\lambda \in \Xi_\lambda})_{\lambda \in \Lambda}$ never satisfies standard quantum simulability. In other words, the possibility of DI detection is equivalent to the impossibility of the simulation by a pair of a standard quantum state and a family of local POVMs.

However, the following theorem implies the impossibility of DI detections of beyond-quantum states.

Theorem 1. For any Λ, there exists a beyond-quantum state ρ such that the tuple of the state ρ and any local POVMs satisfies standard quantum simulability. Especially, in the bipartite case, i.e., $\Lambda = \{A, B\}$, any tuple of any beyond-quantum state $\rho_0 \in \mathrm{BQS}(\Lambda)$ and a family of local POVMs $((\boldsymbol{M}^A_{\xi_A})_{\xi_A \in \Xi_A}, (\boldsymbol{M}^B_{\xi_B})_{\xi_B \in \Xi_B})$ satisfies standard quantum simulability.

Here we remark that similar results to Theorem 1 are shown by Refs. [18,19]. Although Refs. [18,19] proved a similar statement, it does not formulate the problem with GPTs. Besides, Refs. [18,19] focused on the impossibility for any beyond-quantum states in bipartite cases, but they did not focus on the impossibility in multipartite cases. Further, while the proof in [18] has a problem caused by an inverse of a key operator, our proof does not have such a problem because our proof is straightforward and different from that of Ref. [18], as shown in the Supplemental Material [50].

Due to Theorem 1, it is impossible to distinguish a beyond-quantum state from all standard quantum states. To resolve this problem, instead of measurement devices without certification, we need to employ measurement devices that are identified with certifications. This problem setting is called DD detection.

Device-dependent detection of beyond-quantum state and its implementation. Now we discuss a DD detection of an arbitrary given beyond-quantum state in ESs. In the following analysis, instead of the joint distribution, as a simple indicator, we focus on the sum of an expectation of a function $f((\xi_\lambda)_\lambda, (i_\lambda)_\lambda)$, i.e., $\sum_{(\xi_\lambda)_\lambda,(i_\lambda)_\lambda} f((\xi_\lambda)_\lambda, (i_\lambda)_\lambda)\mathrm{Tr}\rho \bigotimes_{\lambda \in \Lambda} M^\lambda_{\xi_\lambda;i_\lambda}$ so that the magnitude relationship of this indicator makes the required discrimination. For our simple analysis, we assume $f((\xi_\lambda)_\lambda, (i_\lambda)_\lambda) = \Pi_{\lambda \in \Lambda} f(\xi_\lambda, i_\lambda)$. Then this value can be rewritten as

$$\sum_{(\xi_\lambda)_\lambda,(i_\lambda)_\lambda} f((\xi_\lambda)_\lambda, (i_\lambda)_\lambda)\mathrm{Tr}\rho \bigotimes_{\lambda \in \Lambda} M^\lambda_{\xi_\lambda;i_\lambda} = \sum_{(\xi_\lambda)_\lambda} \mathrm{Tr}\rho \bigotimes_{\lambda \in \Lambda} \mathcal{O}^\lambda_{\xi_\lambda}, \tag{4}$$

where $\mathcal{O}^\lambda_{\xi_\lambda} := \sum_{i_\lambda} f(\xi_\lambda, i_\lambda) M^\lambda_{\xi_\lambda;i_\lambda}$. The Hermitian matrices $\mathcal{O}^\lambda_{\xi_\lambda}$ can be regarded as standard quantum observables with the POVMs $\boldsymbol{M}^\lambda_{\xi_\lambda}$ and outcomes $f(\xi_\lambda, i_\lambda)$, respectively. Therefore, the value $\mathrm{Tr}\rho \bigotimes_{\lambda \in \Lambda} \mathcal{O}^\lambda_{\xi_\lambda}$ corresponds to the expectation value of the standard quantum observable $\mathcal{O}^\lambda_{\xi_\lambda}$ with the state ρ. Hereinafter, we abbreviate the pair of POVMs and outcomes in the left-hand side of (4) to the right-hand side of (4) by using observables, according to this correspondence.

Based on the sum of the expectation of standard quantum local observables, the following theorem gives a DD detection of any beyond-quantum state from all standard quantum states.

Theorem 2. Given an arbitrary beyond-quantum state $\rho_0 \in \mathrm{BQS}(\Lambda)$, there exist families of local observables $\{\mathcal{O}^\lambda_k\}_{k=1}^m$ and a real number α satisfying the following two properties:

(1) $\mathrm{Tr}\rho_0 \sum_{k=1}^{m} \bigotimes_{\lambda\in\Lambda} \mathcal{O}_k^\lambda > \alpha.$

(2) $\sup_{\rho_1\in\mathcal{S}_{\mathrm{sta}}(\Lambda)} \mathrm{Tr}\rho_1 \sum_{k=1}^{m} \bigotimes_{\lambda\in\Lambda} \mathcal{O}_k^\lambda \leqslant \alpha.$

The proof of Theorem 2 is written in the Supplemental Material, but we remark that we can find $\{\mathcal{O}_k^\lambda\}_{k=1}^m$ and α as follows. The tuple of all generalized Pauli's observables on $\mathcal{H}_\lambda$ with multiplying certain real numbers works as the operators $\{\mathcal{O}_k^\lambda\}_{k=1}^m$ in Theorem 2. When the dimension of $\mathcal{H}_\lambda$ is d_λ, we have the identity matrix on $\mathcal{H}_\lambda$ and $d_\lambda^2 - 1$ number of generalized Pauli's observables σ_i^λ $(i = 0, \dots, d_\lambda^2 - 1)$. In general, the set $\{\mathcal{O}_k^\lambda\}_{k=1}^m$ is given as the tuple of all products of σ_i^λ with certain scalar times, and the number of observables m in Theorem 2 is given as $m = \prod_{\lambda\in\Lambda} d_\lambda^2$. However, when the local observables composing the product observable contain a number of the identity matrices, the systems having the identity matrix are not required to be measured. That is, a local system $\mathcal{H}_\lambda$ is required to be measured by $d_\lambda^2 - 1$ number of generalized Pauli's observables so that the actual total number of necessary measurement settings is given as $\prod_{\lambda\in\Lambda}(d_\lambda^2 - 1)$.

Theorem 2 guarantees that the joint distribution with ρ_0 cannot be simulated by the joint distribution with any standard quantum state ρ_1 under the common local measurements. The above discussion can be understand in terms of the semidefinite programming (SDP) with the target function $\max_{\rho_1\in\mathcal{L}_{\mathrm{H}}(\bigotimes_{\lambda\in\Lambda}\mathcal{H}_\lambda)} \mathrm{Tr}\rho_1 \sum_{k=1}^m \bigotimes_{\lambda\in\Lambda} \mathcal{O}_k^\lambda$ and the conditions $\mathrm{Tr}\rho_1 = 1$ and $\rho_1 \geqslant 0$. The second relation in Theorem 2 shows that the solution of the SDP is upper bounded by α. The first relation in Theorem 2 states that ρ_0 attains a strictly larger value than the solution, and therefore, ρ_0 is not positive semidefinite, i.e., beyond-quantum. In the setting of DD detection, we focus the set of all pairs of a beyond-quantum state and local POVMs. This set is more complicated than the set of quantum correlations even though they are equivalent in DI setting.

Next, we see that the detection given by Theorem 2 is implemented as the following DD detection protocol on the bipartite scenario (Fig. 2):

Aim and Strategy

(1) Local players aim to determine whether a given target global state ρ is beyond-quantum or not.

(2) Players choose $\{\bigotimes_{\lambda\in\Lambda} \mathcal{O}_k^\lambda\}_{k=1}^m$ and α given in Theorem 2 based on their prediction that the target state ρ is close to a beyond-quantum state ρ_0.

(3) Players repeat the following protocol by nm-times for sufficiently large n.

The Whole Protocol

(1) **Setup**: We assume that a generator can prepare the composite system with the same target state ρ, repetitively.

(2) ***l*th Round**: The generator prepares the composite system with the target state ρ. They measure their local observables $\mathcal{O}_k^\lambda$ with $l = qn + k$ $(1 \leqslant k \leqslant m$, q is the integer part of the quotient $l/n)$. As a result, they get outcomes o_l^λ, respectively.

(3) **Determination**: They share their outcomes with classical communication. They then calculate the value $\frac{1}{n}\sum_{l=1}^{nm} \Pi_{\lambda\in\Lambda} o_l^\lambda$. If the inequality $\frac{1}{n}\sum_{l=1}^{nm} \Pi_{\lambda\in\Lambda} o_l^\lambda > \alpha$ holds, they conclude that the target state ρ is beyond-quantum.

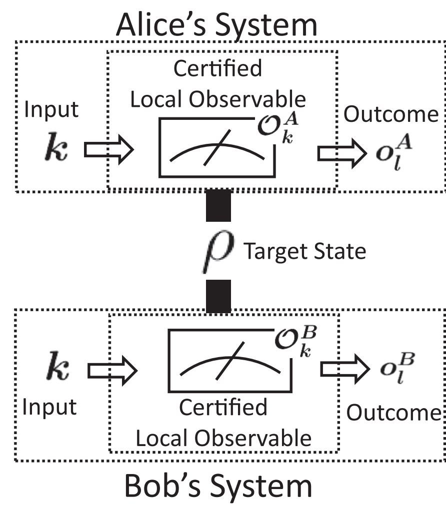

FIG. 2. The lth round of the detection protocol of the criterion given in Theorem 2 in the case of $\Lambda = \{A, B\}$. Local players Alice and Bob aim to detect whether ρ is beyond-quantum. Alice and Bob prepare only their certified local observables with a certain order given in Theorem 2, and they estimate the expectation value in Theorem 2 as the average of all outcomes gotten in nm rounds of observation.

Justification

(1) On the limit $n \to \infty$, the value $\frac{1}{n}\sum_{l=1}^{nm} \Pi_{\lambda\in\Lambda} o_l^\lambda$ approximates the expectation value $\sum_{k=1}^m \mathrm{Tr}\rho \bigotimes_{\lambda\in\Lambda} \mathcal{O}_k^\lambda$.

(2) If n is sufficiently larger and the inequality $\frac{1}{n}\sum_{l=1}^{nm} \Pi_{\lambda\in\Lambda} o_l^\lambda > \alpha$ holds, Theorem 2 ensures that the target state ρ is beyond-quantum with sufficiently large probability.

(3) If $\frac{1}{n}\sum_{l=1}^{nm} \Pi_{\lambda\in\Lambda} o_l^\lambda \leqslant \alpha$ holds for sufficiently large n, their prediction ρ_0 is sufficiently different from the target state ρ.

In this way, any beyond-quantum state can be detected by a finite number of certified local quantum observables with large probability. In general cases, this detection require large costs because we need to certify a number of local quantum observables dependent on a target state. However, in the 2×2 dimensional bipartite case, we can reduce the number of certified devices for detection of all pure beyond-quantum states.

Let us consider the case $\Lambda = \{A, B\}$ with $\dim(\mathcal{H}_A) = \dim(\mathcal{H}_B) = 2$. First, we define the following function $\mathbb{A}_{\mathrm{Pauli}}(\cdot; U_A, U_B)$ using Pauli's spin matrices and unitary operations:

$$\mathbb{A}_{\mathrm{Pauli}}(\rho; U_A, U_B) := \sum_{c=x,y,z} \mathrm{Tr}(U_A \otimes U_B)\rho(U_A^\dagger \otimes U_B^\dagger)\sigma_c \otimes \sigma_c, \tag{5}$$

where U_A, U_B and $\sigma_x, \sigma_y, \sigma_z$ denote unitary matrices on $\mathcal{H}_A, \mathcal{H}_B$, and Pauli's spin observables are defined as

$$\sigma_x := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{6}$$

respectively.

The function $\mathbb{A}_{\text{Pauli}}(\cdot; U_A, U_B)$ satisfies the following properties.

Theorem 3. The following two properties hold:

(1) For any beyond-quantum pure state $\rho_0 \in \text{BQS}(A; B)$, there exist unitary matrices U_A, U_B such that $\mathbb{A}_{\text{Pauli}}(\rho_0; U_A, U_B) > 1$.

(2) $\sup_{\substack{U_A, U_B:\ \text{unitary} \\ \rho_1 \in \mathcal{S}_{\text{sta}}(A;B)}} \mathbb{A}_{\text{Pauli}}(\rho_1; U_A, U_B) \leqslant 1$.

The proof of Theorem 3 is given in the Supplemental Material [50]. Theorem 3 implies that in 2×2-dimensional case, if a target state ρ_0 is beyond-quantum pure, there exists a pair of unitary matrices U_A and U_B such that the function $\mathbb{A}_{\text{Pauli}}(\cdot; U_A, U_B)$ detects the beyond-quantum pure state ρ_0 from all standard quantum states ρ_1. If we can apply the unitary operations in the whole protocol, it is not necessary to certify the description of unitary matrices. As a result, we only need to prepare the measurement settings to certify the observables $\sigma_x, \sigma_y, \sigma_z$ in the tuple of $\sigma_k \otimes \sigma_k$ for $k = x, y, z$ without the tuple of $\sigma_i \otimes \sigma_j$ for $i \neq j$. The criterion given in Theorem 3 is approximately implemented by a reiteration of the protocol in Fig. 2 without the certification of unitary operations. As a result, this protocol takes less costs than that of Theorem 2 because we need only three types of measurement settings of certified observables instead of $\prod_{\lambda \in \Lambda}(d_\lambda^2 - 1) = 9$.

Conclusion. In this paper we have discussed the detection of a beyond-quantum state in ESs of GPTs. Even though local systems of ESs are equivalent to standard quantum systems, we have shown that any DI detection, including CHSH inequality, cannot separate any beyond-quantum state in ESs from the standard quantum states even in multipartite cases. In contrast to DI detection, we have given a DD detection of an arbitrary given beyond-quantum state in any ES from all states in the SES based on local standard quantum observables. Also, we have given an experimental implementation of the detection as a protocol with local players. The detection needs a large number of observables in general. However, in the two-qubit case, we have given a more efficient detection based on Pauli's spin observables. The detection can be implemented only by a smaller number of certified Pauli's spin observables and uncertified unitary operations.

An interesting remaining study is actual verification of beyond-quantum states based on our detection by experimental datum similar to Refs. [48,49]. The detection in Refs. [48,49] is based on the full tomography, which restricts their model due to the high cost. Our detection is a more efficient way than full tomography, which implies that our detection can extend the studies of [48,49]. Therefore, our detection enables us to design better experiments than that of preceding studies [48,49]. Another remaining study is a strict estimation of the error probability of the detection protocol. In order to discuss how surely our models contain beyond-quantum nonlocal states, we need to discuss the protocol in the context of hypothesis testing. This study is also helpful for the above actual verification. As well, from the viewpoint of foundations, it is also interesting to generalize our detection to general composite models in GPTs. The difficulty in determinIng composite systems in GPTs is a well-considered open problem, and the above generalization has the potential to give a new perspective on it.

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