# Extending temporal-coupled-mode theory to the antiresonant regime

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(Received 10 April 2024; accepted 31 July 2024; published 5 September 2024)

The antiresonant effects, which are opposite to the resonant effects, describe the destructive interferences that occur in optical resonators. There is a growing trend in the recent development of photonic devices that explore antiresonant effects such as topology structures and photonic molecules with asymmetrical geometries. For the design of coupled optical resonators, temporal-coupled-mode theory (TCMT) has been widely applied, which provides a simple yet effective physical picture of such systems. However, TCMT is valid for optical frequencies that resonate well with the cavities. It becomes powerless to handle mode coupling at optical frequencies that regimes for investigating mode coupling between antiresonances and resonances in coupled-optical-cavity systems via taking the analogy of the mode splitting. Our theory agrees well with the transfer matrix method as well as the experimental results. A concrete example is given to show that our work offers an opportunity to extend the application of TCMT in the design of advanced coupled-cavity structures.

DOI: 10.1103/PhysRevA.110.033510

## I. INTRODUCTION

Temporal-coupled-mode theory (TCMT) has been widely applied as a mathematical tool to analyze the spectral response [1,2], describe the Hamiltonian [3-7], and calculate the S matrix [6], which is developed for resonance structures. It provides a clear physical picture depicting the interaction between different resonators. TCMT is suitable for linear optics and can be extended to analyze nonlinear effects with simple modifications [8,9]. With the help of TCMT and the progress of photonics integrated circuit manufacturing technology, various novel coupled-cavity structures have been realized and provided a fertile ground for many applications [10–14]. Besides, many interesting phenomena have been demonstrated in coupled-cavity systems where destructive interference plays an important role such as microresonator array for topology applications [15] and parity-time systems for nonlinear optical signal processing [16]. However, TCMT is only effective around the resonant frequency. It becomes powerless to handle antiresonance situations (i.e., destructive interference occurs in the resonator at the incident frequency). On the other hand, the transfer matrix method (TMM) can provide rigorous solutions, but it fails to reflect a clear physical connotation [17]. An alternative method applied to solve antiresonant issues entails that the involved structures are regarded as a waveguide, while TCMT is employed to describe other resonant elements [15]. Nonetheless, the response of the antiresonant elements is hard to predict.

In this article, we propose an antiresonant TCMT (ATCMT) to capture the physics when antiresonances are

involved in coupled-cavity systems. The key idea is to model the antiresonances by taking the analogy of the mode splitting produced by two coupled resonant modes. The ATCMT is then applied to deal with a situation in which antiresonant elements couple with resonant elements. The results of the theory are consistent with the transfer matrix method around the antiresonant frequency over a wide range of coupling parameters. Furthermore, the transmission spectrum measured from our fabricated device validates our theoretical predictions. Our work provides a path to extend the TCMT in the design of advanced coupled-cavity structures.

#### **II. THEORETICAL MODEL OF THE ATCMT**

Firstly, we consider the power enhancement in an all-pass microring resonator (MRR), as sketched in Fig. 1(a). The external excitation is launched into the bus waveguide and then couples into the MRR. Using the TMM, it can be described as

$$\begin{pmatrix} s_{\text{out}} \\ A_1 \end{pmatrix} = \begin{pmatrix} r_0 & -j\kappa_0 \\ -j\kappa_0 & r_0 \end{pmatrix} \begin{pmatrix} s_{\text{in}} \\ A_2 \end{pmatrix},$$

$$A_2 = t e^{j\phi} A_1,$$
(1)

where  $s_{in}$  and  $s_{out}$  are the input and output complex amplitude,  $A_{i=1,2}$  is the complex amplitude in the MRR as shown in Fig. 1(a), t is the MRR's round-trip field attenuation factor, and  $\phi$  is the round-trip phase. The field transmission and coupling coefficients between the MRR and the bus waveguide are  $r_0$  and  $\kappa_0$ . The coupler is assumed to be lossless, so that  $r_0^2 + \kappa_0^2 = 1$ . The transmission T and the power enhancement F, defined as the ratio of the power inside MRR to the input

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FIG. 1. Schematic of the microring resonator (MRR) (a) and two coupled MRRs (c). The two systems are coupled to a bus waveguide. The power spectral responses in MRR are shown in panels (b) and (d). Panel (b) is the intracavity power spectral response corresponding to panel (a). Panel (d) is the power spectral response in MRR1 shown in panel (c). The red star indicates antiresonance.

power, can be easily calculated as

$$T = \left| \frac{t - r_0 e^{j\phi}}{1 - tr_0 e^{j\phi}} \right|^2,$$
  

$$F = \left| \frac{-j\kappa_0}{1 - tr_0 e^{j\phi}} \right|^2.$$
 (2)

The periodic peaks in the power enhancement spectrum manifest the resonant longitudinal modes. The antiresonant frequencies exist in the middle of adjacent resonant longitudinal modes, where the power enhancements become a minimum, as marked by a red star in Fig. 1(b). Meanwhile, we notice that the mode splitting that occurs in the strongcoupling regime quenches resonance at the original frequency, leading to a destructive feature [18]. Figure 1(c) shows a system with two coupled identical MRRs, where the bus waveguide only couples to one of the resonators. The power enhancement spectrum of the MRR1 presents two resonant modes due to the mode splitting. The power enhancement spectrum of MRR1 presents two resonant modes due to the mode splitting. A red star marker in Fig. 1(d) represents the antiresonance point, which occurs between the two split resonant modes. From the TMM point of view, this is due to the fact that MRR2 introduces a  $\pi$ -phase shift to MRR1 at the resonant frequency, leading to a transformation from constructive to destructive interference. From the TCMT point of view, it is known that the TCMT of a single resonator yields the same outcome as the first-order Taylor expansion of the TMM at the resonant frequency [17]. Similarly, one can expand the transfer matrix results at the antiresonant frequency to study the antiresonance. However, the Lorentzian function, which inevitably emerges from the first-order expansion, does not align with the trend in the variation of the frequency response curve around the antiresonant frequency. Therefore,



FIG. 2. Comparison of through-port transmission (a) and inner power enhancement (b) between the ATCMT and the TMM. The blue lines indicate the results obtained by Eq. (2), and the red lines were obtained based on Eqs. (3) and (6).

retaining second-order expansion of the TCMT is necessary. Since the TCMT of two coupled modes yields second-order Taylor expansion terms, the mathematical foundation of the antiresonance model has thus been laid coherently.

Following this idea, we proceed to describe the response near the antiresonant frequency for an all-pass microring shown in Fig. 1(a) using two coupled modes, i.e.,

$$\frac{da_1}{dt} = \left(-j\omega_0 - \gamma_0 - \frac{\mu_0^2}{2}\right)a_1 - jg_e b_1 - j\mu_0 s_{\rm in},$$
  
$$\frac{db_1}{dt} = (-j\omega_0 - \gamma_1)b_1 - jg_e a_1,$$
 (3)

where  $|a_1|^2 = |A_1|^2 T_R$  is the stored energy in the MRR,  $b_1$  is an imaginary mode.  $|s_{in}|^2$  represents the power flowing in the input waveguide,  $\omega_0$  is the antiresonant frequency,  $\gamma_0$  denotes the intrinsic loss rate of the MRR, and  $\mu_0$  is the energy coupling coefficient between the MRR and the bus waveguide.  $g_e$  and  $\gamma_1$  are both equivalent parameters, and  $g_e$  represents the coupling coefficient between the two modes and  $\gamma_1$  is the mode loss of  $b_1$ . By comparing the results of the transfer matrix method, the following relationship is obtained:

$$g_e = \sqrt{\frac{4}{T_R^2} - \frac{4\gamma_0}{T_R} - \frac{2\mu_0^2}{T_R}},$$
 (4)

$$\gamma_1 = \frac{2}{T_R},\tag{5}$$

where  $T_R = n_g L/c$  is the round-trip group delay. The output at the through port is

$$s_{\rm out} = s_{\rm in} - j\mu_0 a_1. \tag{6}$$

Figure 2 compares the results obtained by the TMM and our model. The through-port transmittance is shown in Fig. 2(a), and the power enhancement in MRR1 is shown in Fig. 2(b). The red lines, calculated using the ATCMT, match the blue lines, the results from the transfer matrix method. Around the antiresonant frequency, the power enhancement comes to its minimum value, which means that all the power of the input light at the antiresonant frequency is directed to the transmission port.



FIG. 3. Schematic of two differently sized coupling MRRs (a), in which the perimeter of the large MRR2 is four times that of the small MRR1. The response of the two individual MRRs is shown in panel (b). There are five MRR2 resonant peaks between MRR1 adjacent resonant peaks, and the red star indicates the coupling between the antiresonance and the resonance. The equivalent model is shown in panel (c); the antiresonant MRR1 can be regarded as two coupled resonant MRRs.

# III. ATCMT FOR MODE COUPLING BETWEEN ANTIRESONANCES AND RESONANCES

In this section, we apply the ATCMT to study the coupling between antiresonances of one cavity and resonances of another cavity. We consider a system consisting of two directly coupled MRRs, as sketched in Fig. 3(a).The perimeters of the two MRRs are in the ratio of 1:4, which means that within one free spectral range (FSR) of MRR1, there are three frequencies which are resonant in MRR2 but not in MRR1, as illustrated in Fig. 3(b). We investigate the coupling behavior of a specific frequency between MRR1 and MRR2, as indicated by the red stars in Fig. 3(b), which is resonant in MRR2 but antiresonant in MRR1. Using the TMM, the transmission and power enhancements in the two MRRs can be calculated as

$$T = \left| \frac{r_0 - t_1 \frac{r_1 - t_2 e^{j\phi_2}}{1 - t_1 r_0 \frac{r_1 - t_2 e^{j\phi_2}}{1 - t_2 r_1 e^{j\phi_2}} e^{j\phi_1}}}{1 - t_1 r_0 \frac{r_1 - t_2 e^{j\phi_2}}{1 - t_2 r_1 e^{j\phi_2}} e^{j\phi_1}} \right|^2,$$

$$F_1 = \left| \frac{-j\kappa_0}{1 - t_1 r_0 \frac{r_1 - t_2 e^{j\phi_2}}{1 - t_2 r_1 e^{j\phi_2}} e^{j\phi_1}} \right|^2,$$

$$F_2 = \left| \frac{-j\kappa_1 \sqrt{t_1}}{1 - t_2 r_2 e^{j\phi_2}} \right|^2 F_1.$$
(7)

According to the analysis given above, the antiresonant cavity, i.e., MRR1, is replaced by two coupled resonant cavities, as sketched in Fig. 3(c). Then, the whole system can be described by the ATCMT as

$$\frac{da_1}{dt} = \left(-j\omega_0 - \gamma_0 - \frac{\mu_0^2}{2}\right)a_1 - jg_e b_1 - j\mu_e a_2 - j\mu_0 s_{\rm in}, 
\frac{db_1}{dt} = (-j\omega_0 - \gamma_1)b_1 - jg_e a_1, 
\frac{da_2}{dt} = (-j\omega_0 - \gamma_2)a_2 - j\mu_e a_1,$$
(8)

where  $|a_{i=1,2}|^2$  is the energy in MRR1 and MRR2;  $\gamma_2 = \gamma_0 + \gamma_1$  $\mu_1^2 T_{R1}/2$  is the mode decay rate of MRR2;  $\mu_1$  is the energy coupling coefficient between MRR1 and MRR2, which is related to the field coupling coefficient by  $\mu_1 = \kappa_1 / \sqrt{T_{R1} T_{R2}}$ ; and  $\mu_e = j\mu_1 \sqrt{1 - \mu_0^2 T_{R1}/2}$  is the coupling coefficient between MRR1 and MRR2. All the coefficients are derived by comparing the ATCMT with the TMM expanded to secondorder terms. There are several interesting features in Eq. (8). First, the loss  $\gamma_2$  includes both the intrinsic loss rate of MRR2 and the coupling loss, which is introduced by the antiresonance in MRR1. Physically, it means that the energy stored in MRR2 would be directly coupled to the output channel. Second, the coupling between the resonant mode in MRR2 and MRR1, i.e.,  $\mu_e$ , is a pure imaginary number, which is different from conventional TCMT where the coupling is real. It causes a correction for the loss of  $a_2$ . This is because the energy that couples from the resonant mode in MRR2 to MRR1 does not solely couple to the output channel; a portion of energy returns.

In Fig. 4, the transmission at the through port and the power enhancement in the two resonators are displayed for two coupled MRRs. The through transmission measured by the experiment is represented by the solid gray line in Fig. 4(a). The blue line is obtained based on the TMM, and the solid red line is obtained by Eqs. (6) and (8). The ATCMT agrees well with the experimental result and the TMM, as shown in Fig. 4(b), which is a zoomed-in view of Fig. 4(a). The power enhancement spectra of MRR1 and MRR2 obtained by the ATCMT is consistent with the TMM, as shown in Figs. 4(c) and 4(d), respectively. Note that the power enhancement of MRR1 has been found by ATCMT with good agreement with the TMM results. It should be noted that for the structure shown in Fig. 3(a), the antiresonant cavity MRR1 is commonly regarded as a waveguide [15]. As a result, it is equivalent to an all-pass MRR, as sketched in Fig. 4(e). In this case, the antiresonant element is reduced to a coupling coefficient. The power enhancements at the points L and R marked by the stars, shown in Fig. 4(e), can be derived using conventional TCMT. As shown in Fig. 4(f), the results cannot describe the power enhancement in MRR1 correctly.

To characterize the feasibility of our theory over wide parameter space, we further scan the coupling coefficients of the structure shown in Fig. 3(a) and use the ATCMT to calculate the transmission in the special frequency marked by red stars in Fig. 3(b). According to the relationship between the field and the energy coupling coefficient,  $\mu_{0max} = 1/\sqrt{T_{R1}}$ and  $\mu_{1max} = 1/\sqrt{T_{R1}T_{R2}}$ , which corresponds to  $\kappa_0 = \kappa_1 = 1$ .



FIG. 4. The through-port transmittance and the power enhancement in the two resonators. The through-port transmittance measured by the experiment is the solid gray line in panel (a), and panel (b) corresponds to the box in panel (a). The power enhancement in MRR1 and MRR2 is shown in panels (c) and (d). The blue line is obtained by Eq. (7), and the solid red line is obtained by Eqs. (6) and (8). For the two coupling MRRs shown in Fig. 3(a), the antiresonant cavity can be regarded as a waveguide in panel (e). The coupling between the resonant MRR2 and the waveguide is derived by the TMM, and the power enhancements at the points L and R are shown in panel (f). The orange line corresponds to the power enhancement at point L, and the power enhancement at point R is displayed by the green dashed line.

We then compared the outcomes to those obtained from the TMM to generate an error diagram, shown in Fig. 5, where the color bar indicates the absolute value of the difference. The red circle labels the experiment state. The ATCMT is



FIG. 5. Simulation of the relationship between the ATCMT transmission errors and the coupling coefficients. The red circle labels the state of the experiment. The color bar represents the errors.

valid for most situations; however, when the waveguide is strongly coupled to MRR1 and MRR1 is weakly coupled to MRR2, due to the breakdown of field uniformity, the ATCMT will completely fail.

## **IV. CONCLUSION**

In conclusion, we have extended TCMT into the antiresonance regime to analyze the coupling cavity systems that include both antiresonant and resonant elements. Unlike in conventional TCMT where all the information inside the antiresonant elements gets lost, we show in this work that our developed ATCMT can predict the transmission as well as power enhancement response of all elements including antiresonant elements, by comparing to TMM as well as experimental results. In the end, the ATCMT is shown to be valid in a wide coupling strength range. The approach is expected to provide a possible way to extend the application of the TCMT for the design of advanced coupled-cavity structures.

#### ACKNOWLEDGMENTS

This work is supported by the National Natural Science Foundation of China (NSFC) (Grant No. 62275087) and the Knowledge Innovation Program of Wuhan-Shuguang Project (Grant No. 2022010801010082).

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