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General hydrodynamic approach for a cold Bose gas

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The aim of this paper is to derive the hydrodynamics for a cold Bose gas from the microscopic platform based on the many-body Schrödinger equation and general assumptions of the hydrodynamic approach (HA) applicable to any dimension. We develop a general HA for a cold spatially inhomogeneous Bose gas assuming two different temporal and spatial scales and obtain the energy as a functional of both fast inner quantum mode and slow macroscopic mode. The equations governing the fast and slow modes are obtained from this functional by their independent variations. The fast mode is the wave function in the stationary state at local density which can be ground, excited with a nonzero atom momenta, or a superposition of more than one state. The energy eigenvalue (or expectation value) of this local wave function universally enters the hydrodynamic equation for the slow mode in the form of the local chemical potential which incorporates the inner local momentum. For zero inner momenta and particular choices of this eigenvalue as a function of gas density, this equation reduces to the known equations based on the local density approximation. If, however, the inner momenta are nonzero, the equation includes the interaction between these momenta and the slow mode velocity. Relation between this general HA and the standard local density approximation is elaborated. Two effects of the local momenta and their density dependence on the soliton solutions are demonstrated.

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I. INTRODUCTION

The clouds of ultracold Bose and Fermi gases in electromagnetic traps naturally invoke the idea of a hydrodynamic-type description in terms of a local order parameter, density, and velocity. To a great extent, the successful development of this area of quantum physics has been based on the hydrodynamic approach (HA) which intrinsically assumes the local density approximation (LDA). However, at present the hydrodynamic-type approach to quantum gases in terms of smoothly varying local quantities not only continues to be used as one of the main theoretical tools, but also is a subject of ongoing developments, modifications, and amendments. This work is also motivated by the idea that not all capabilities of the HA and the related LDA have already been fully explored and a further development is still possible. There are (roughly) four main arguments that motivated this work.

First, there has been an ongoing chain of modifications and amendments to the main mean field approach to quantum gases in the form of the Gross-Pitaevskii equation (GPE). The GPE with a cubic nonlinearity was proposed in the early 1960s [1–3] for description of a Bose-Einstein condensate. The idea underlying the GPE derivation is to associate and replace the Heisenberg field operator with a classical mean field order parameter which is assumed to vary slowly on distances of the order of the interatomic force [3]. The square of the modulus of this order parameter was shown to represent gas density which in this way entered the theory. As a result, via Madelung's ansatz, there have been obtained the equivalent alternative hydrodynamic equations which formed the HA for a superfluid. Although the GPE derivation does not resort to the LDA and the equivalent superfluid HEs essentially express the mass and momentum conservation [3,4], the superfluid HA describes cold atomic gas in terms of a local density and velocity, which has paved way to the natural further development in the spirit of the conventional HA and the underlying LDA. Different HAs have been developed and successfully applied to ultracold gases. But even before GPE appeared, Lee, Huang, and Yang [5] found what is now called a quantum correction to the GPE nonlinearity, which appeared to be the lowest power of the gas density expansion; the GPE with this correction is now known as the extended GPE (eGPE).

In 2000 Kolomeisky *et al.* [6], based on a renormalization group analysis [7], argued that in low dimensions strong atoms' repulsion actually results in a quintic nonlinearity. The square of the modulus of the mean field introduced in [6] is associated with the local density and, as a result, the theory again becomes equivalent to certain HA. It turned out, however [8], that this theory incorrectly predicts highly contrast interference patterns in the one-dimensional split dynamics [9] and shock wave formation [10,11] in a strongly interacting Bose gas in the Tonks-Gerardeau regime. Moreover, a similar situation was found to take place also in a weakly

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interacting Bose gas in the GPE regime [10,11]. The reason is that the interference effects involve short length oscillations which break the applicability of the LDA in general for any interaction strength and, mathematically, the so-called quantum pressure term is responsible for this problem [11,12] (see Conclusion section for more discussion). Nevertheless, the HA explicitly or indirectly exploiting the LDA remained attractive as it correctly describes the situations with smooth density variations that often take place in the experiment.

The intensive studies of a quasi-one-dimensional Bose gas in the early 2000s brought about the idea, suggested by Dunjko, Lorent, and Olshanii [13], that in this geometry actual nonlinearity in the HA depends on the local gas energy density [13–15] and can be taken from the Lieb-Liniger solution [16,17] (alternatively, it is given by the equation of state obtained independently [18]). The advantage of such HA was recently demonstrated both for gas with a short-range repulsion [19,20] and for a dipolar gas [21]. In [19–21], the repulsion-induced nonlinearity in the one-dimensional hydrodynamic equation was determined by the local energy density of the Lieb-Liniger gas. In these papers, the starting point was the classical hydrodynamic equations for local density and velocity. Then the quantum pressure term was formally added to these equations because it is necessary to convert them to a Schrödinger-type equation by means of the Madelung ansatz. The result of such a conversion was dubbed modified nonlinear Schrödinger equation (MNLSE) [19] and Lieb-Liniger GPE (LLGPE) [21]. This conversion procedure was described in [19] as merely a convenient numerical tool for simulating MNLSE instead of the hydrodynamic equations with the remark that the hydrodynamic equations themselves should be carefully examined in the future. Note that the conventional HA does not account for the infinitely many integrals of motion of the Lieb-Liniger gas which is adequately described by the recently developed generalized hydrodynamics [22–24]. Nevertheless, the standard HA based on LDA, which is much simpler than the generalized hydrodynamics, remains valid in many situations of interest and, in particular, for zero temperature and in absence of velocity multivaluedness (see [25]).

We see that the modifications of the GPE depend on the specific system and its dimension. The general HA in contrast is expected to predict a hydrodynamic equation (HE) with certain term whose origin is universal and only its specific form depends on the system and its geometry. In the classical HEs such term is the pressure gradient which connects the macroscopic motion with microscopic nature of the medium via the equation of state. As such, the HA comprises two separate problems: first, finding the pressure as a function of density and temperature for a given liquid with its specific statistics, dimensions, molecular structure, and so on, and, second, solving the HE with this pressure. Such a HA, which is general in the above sense, has not been developed for a generic cold Bose gas of any origin and dimension.

Second, the fundamental relation between the microscopic molecular theory and HA has been well established in the classical statistical physics. The classical hydrodynamic equations can be derived from the many-body distribution function by its reduction to the one-particle distribution and then momentum integration. At the same time, the main idea of the derivation of the HEs proposed for quantum gases in [1–3,13–15,20,21] is based on the essentially one-particle description and its hydrodynamic form in the Madelung representation of the one-particle wave function.

Third, the LDA, applied to a cold gas at zero temperature, assumes that locally gas is in its ground state at the local density. However, a local equilibrium at zero temperature implies only that the gas is in its stationary state which is not necessarily ground state. For instance, a stationary state can be an excited state with a nonzero local momentum or it can be a superposition of different states. Such states can be incorporated only on the level of many-particle descriptions. In particular, because in these cases the total many-particle phase of system's wave function does not have the form of the sum of individual atoms' phases and the one-particle phase cannot be introduced.

The above arguments motivated us to derive the general HA to a cold Bose gas which, on the one hand, would have a general form to justify the different known equations on the common ground and, on the other hand, would provide the connection with the many-body quantum mechanical description which is a counterpart of the microscopic foundation of the classical HA. Our fourth motivation argument is that, in principle, establishing such connection and incorporating local nonground states could reveal new effects and, in particular, novel modifications of the known equations. In this paper this program has been performed. As we have seen above, on the one hand, some large-scale descriptions of a cold quantum gas are explicitly HAs. On the other hand, the large-scale mean field description in terms of Schrödinger-type or GPE-type equations via the Madelung transform also reduce to certain equivalent HA. For this reason and for brevity, we will refer to these approaches as known or standard HAs or known or standard HEs to distinguish them from our HA and HE proposed in this paper.

The standard HA assumes that local small but still macroscopic subvolumes are in the thermodynamic equilibrium with some local temperature and density, and that such subvolumes move as a whole with the local average velocity. All these quantities change very little over distances of order of the interatomic separation and thus are slow variables depending on the position X of the center of mass of the subvolumes. Thus, the LDA applied for a quantum gas at finite temperature assumes the local thermodynamic equilibrium [26]. In the case of a quantum gas at zero temperature, we assume that it is locally in a stationary state (*ground or excited*) corresponding to the local atoms' density n, that it moves with a local velocity $\mathbf{v}$, so that n and $\mathbf{v}$ practically do not change within a small local subvolume and are slow functions of the position of its center of mass and time. To this end we separate a slow amplitude A from the reduced one-body density matrix and find the HE for this A. The nonlinearity in this HE is specified by the density dependence of the local chemical potential which in turn is determined by the eigenvalues of the Schrödinger equation for the locally homogeneous states. It is this chemical potential that universally enters the HE while the nonlinearity depends on the specific system and its dimension via the specific density dependence of the local energy eigenvalues.

In the HA, only small macroscopic subsystems are in the locally stationary states. This approach presupposes two different relaxation processes, a fast microscopic and much slower hydrodynamic. At the first stage, small subsystems equilibrate to their stationary states because their size ΔV is much smaller than the total system size V and their local microscopic inner equilibrium is achieving much faster than the total one. At the second, hydrodynamic stage, the density and velocity difference between stationary subvolumes drives the slow macroscopic dynamics of the whole system. We derive the energy as a functional of both fast mode and slow mode. The fast mode is the wave function ψ_n which is the stationary state of the system Hamiltonian at the local density n of the subvolume. If the local stationary state has momentum excitations, then the local Hamiltonian in addition to the standard short-range part also has the term describing the interaction of the local momenta $\langle \mathbf{p} \rangle$ with the velocity of the slow mode. After integrating out the fast mode, we obtain the general HE only for slow mode A. This HE contains the contribution from the local energy eigenvalue and the term $\langle \mathbf{p} \rangle \cdot \nabla A$ describing interaction between the fast and slow motions. The form of this HE is universal and reduces the HA to the two universal problems: first finding the stationary states ψ for a given cold Bose gas at given density and, second, solving the HE with their energy eigenvalues and finding A. For $\langle \mathbf{p} \rangle = 0$ and particular choice of the energy eigenvalues as functions of gas density, the HE reduces to the known equations. If, however, gas momenta are not due to its motion as a whole and if these momenta depend on the gas density, the derived HA is the only tool applicable in this case. Two examples of the influence of such momenta on gray solitons are presented as an illustration to prompt possible related effects.

In Sec. II A, we develop the general HA to a cold inhomogeneous Bose gas, then in Sec. II B derive energy functional of both fast and slow modes and impose local stationary states; in Sec. II C the HE is derived, discussed, and its equivalent standard hydrodynamic form is obtained via the Madelung transform. In Sec. III A, we introduce the effective combined wave function of both slow and fast modes, introduce its common coordinate description, and, in Sec. III B, present two examples of such function; then some important general properties of our HA and its relation to the known GPE-type equations are addressed in Sec. III C. In Sec. IV, it is shown that, if local momenta are nonzero, the periodic soliton train considered in [27] can be made a propagating wave. In Sec. V, we consider the effect of density-dependent local momentum on a soliton in a one-dimensional Lieb-Liniger gas, and Sec. VI. concludes the paper.

II. FROM THE MICROSCOPIC TO HYDRODYNAMIC DESCRIPTION OF A COLD BOSE GAS

A. General consideration

The system of our interest is an *inhomogeneous* gas as its density slowly varies in space. Consider such gas of N cold boson atoms of mass m in the volume V, $\dim V = d$ (d can be 1, 2, or 3). Let x_i be a d -dimensional vector of the position of ith atom, $x_i \in V$, and $x = (x_1, x_2, \dots, x_i, \dots, x_N)$ be the Nd-dimensional vector describing the total system in the Nd-dimensional volume $V \otimes V \otimes V \cdots = V^N$, $x \in V^N$; notation x^{N-k} is used for the set of coordinates of some $N-k$ atoms, $d^N x = dx_1 \dots dx_N$ and $d^{N-k}x = dx_{k+1} \dots dx_N$. The gas wave function $\psi(x,t)$ depends on Nd spatial variables and is normalized on unity, $\langle \psi | \psi \rangle = 1$. The gas Hamiltonian $\widehat{h}$ is the sum

$$\widehat{h} = \widehat{K} + \widehat{U} = -\frac{\hbar^2}{2m} \sum_{i=1}^{N} \Delta_i + \sum_{i>j=1}^{N} U(x_i - x_j), \tag{1}$$

where $\hbar$ is Planck's constant, m is atom's mass, $\widehat{K}$ and $\widehat{U}$ are the operators of kinetic and potential energy, and U is a short-range potential; external and long-range potentials will be addressed later. The local density $n(y)$ is introduced through the reduced one-body density matrix

$$\rho(y, y') = \int_{V^{N-1}} d^{N-1}x \psi^*(y, x^{N-1}) \psi(y', x^{N-1}). \tag{2}$$

The probability density $f(y)$ at y is $f(y) = \rho(y,y)$ and the atom density $n(y) = N f(y)$.

To make a contact with subvolumes with different densities, for each subvolume with the atom density n we also introduce an *auxiliary homogeneous* system V_n of N atoms at density n which, by this definition, has the volume $V_n = N/n$ (Fig. 1). The idea is that, by virtue of a short-range interaction, the energy of a subvolume with density n can be computed as a fraction of the energy of the associated auxiliary system V_n with the same density n. Consider the homogeneous system V_n. Its wave function $\psi_n(x)$, which is normalized on unity, and the energy E_n are functions (functionals) of the density n which is indicated by a subscript n. The system energy is

$$E_n = \int_{V_n^N} d^N x \, \psi_n^*(x) \widehat{h} \psi_n(x). \tag{3}$$

If the system is in a stationary state, then $E_n = N\varepsilon_n$ where E_n is the eigenvalue of the equation

$$\widehat{h}\psi_n = E_n \psi_n \tag{4}$$

and ε_n is that per atom. The energy of a short-range interaction is additive, i.e., the energy of a number of macroscopic subvolumes is the sum of their energies. To see this explicitly in the case of potential energy, the formula $U_n = \langle \psi_n | \widehat{U} | \psi_n \rangle$ can be presented in the following equivalent way possible due to the symmetry with respect to atoms' coordinates:

$$U_n = \frac{N(N-1)}{2} \int_{V_n} dy \int_{V_n} dy' U(y - y') G_2(y, y'), \tag{5}$$

where

$$G_2(y, y') = \int_{V_n^{N-2}} d^{N-2}x \, \psi_n^*(y, y', x^{N-1}) \psi_n(y, y', x^{N-2}) \tag{6}$$

is the pair distribution. If the potential range is much shorter than the size of the macroscopic subvolumes ΔV then

$$U_n = \frac{N(N-1)}{2} \sum_{\Delta V} \int_{\Delta V} dy \int_{\Delta V} dy' U(y - y') G_2(y, y'). \tag{7}$$

In particular, the potential energy ΔU_n of a macroscopic subsystem with the number of atoms ΔN is the fraction of the total U_n:

$$\Delta U_n = \frac{\Delta N}{N} \langle \psi_n | \widehat{U} | \psi_n \rangle. \tag{8}$$

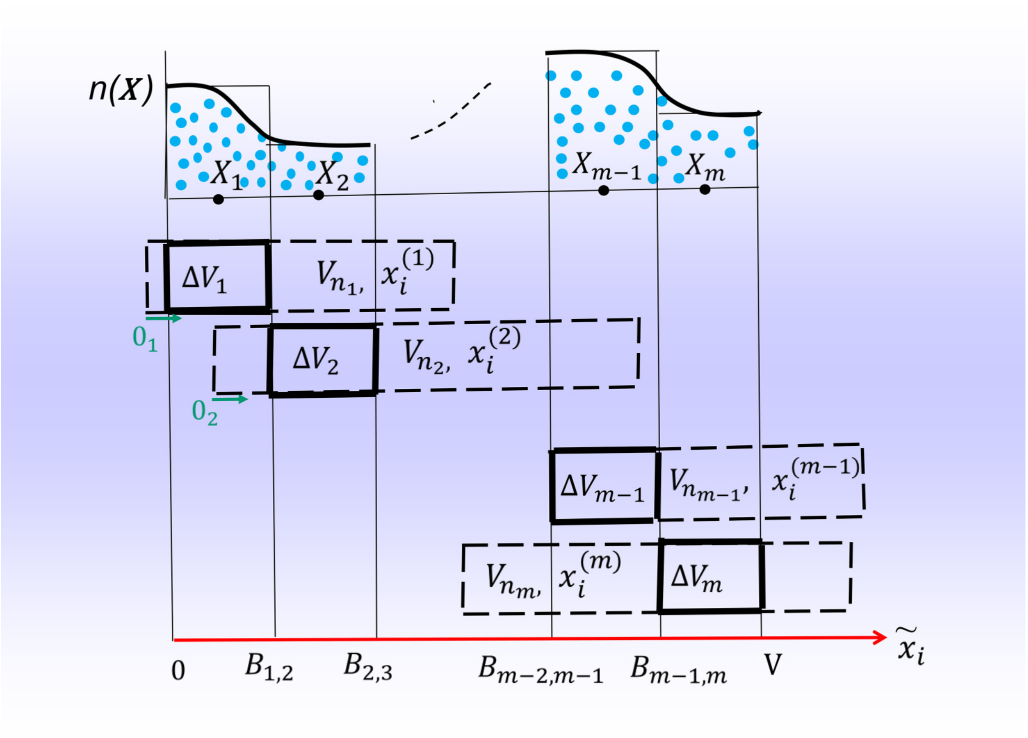

FIG. 1. One-dimensional sketch of the main construction of our approach. An inhomogeneous Bose gas divided into m quasihomogeneous subvolumes ΔV_k with the atom density n_k and which are centered at X_k. At the first stage, each ΔV_k (bold rectangles) is considered a part of an auxiliary homogeneous gas of N atoms (dashed rectangles), which has the density n_k and volume $V_{n_k} = N/n_k$; V_{n_k} is larger for lower density n_k. The d-dimensional vector of the coordinate $x_i^{(k)}$ of i th atom, $i = 1, 2, \dots, N$, is defined in the entire auxiliary volume V_{n_k} relative to some reference frame O_k. The onsets of O_k in different V_{n_k} are not correlated; for instance, they can be connected to the left ends of V_{n_k} as shown by the arrows from O_1 and O_2. Thus, the atoms' coordinates $x_i^{(k)}$ in different V_{n_k} are not mutually adjusted. But, at the second, hydrodynamic stage, the atoms are described only in the actual volume V which requires the vectors $\widetilde{x}_i \in V$, continuously extrapolating local $x_i^{(k)}$ between different ΔV_k. This $\widetilde{x}_i$ can be obtained by synchronizing $x_i^{(k)}$in different ΔV_k by taking the common onset O for all O_k, as shown by the bottom arrow. Then at the boundary $B_{1,2}$ between ΔV_1 and ΔV_2, $x_i^{(1)} = x_i^{(2)}, \dots$, and so on. As a result, defining $\widetilde{x}_i$ to be equal to $x_i^{(k)}$ within ΔV_k gives the desired continuous position vector of ith atom in the actual volume V.

It is instructive to present a similar relation for the kinetic energy in the form that will help us to make a contact with the varying density. To this end we separate the first or first two arguments of the wave function $\psi_n(x)$: $\psi_n(y, x_2, x_3, \dots) = \psi_n(y, x^{N-1})$, $\psi_n(y, y', x_3, \dots) = \psi_n(y, y', x^{N-2})$ where symbols y and y' stand for the d-dimensional vector of position of a single atom, a point in V. Then the kinetic energy $K_n = \langle\psi_n|\widehat{K}|\psi_n\rangle$ can be presented in terms of the reduced density matrix $\rho_n(y, y')$ of the system V_n:

$$K_n = -\frac{\hbar^2 N}{2m}\int_{V_n} dy \int_{V_n} dy'\delta(y-y')\Delta_{y'}\rho_n(y, y'), \tag{9}$$

where

$$\rho_n(y, y') = \int_{V_n^{N-1}} d^{N-1}x\psi_n^*(y, x^{N-1})\psi_n(y', x^{N-1}) \tag{10}$$

is the reduced one-body density matrix in a homogeneous gas with the density n. It follows that the kinetic energy ΔK_n of a macroscopic susbsystem of ΔN atoms is

$$\begin{aligned}\Delta K_n &= -\frac{\hbar^2 N}{2m}\int_{\Delta V} dy \int_{\Delta V} dy'\delta(y-y')\Delta_{y'}\rho_n(y, y')\\ &= \frac{\Delta N}{N}K_n.\end{aligned} \tag{11}$$

The probability density $f_n(y) = \rho_n(y, y)$ and atom density $n = Nf_n$ in a homogeneous system are constant. Equations (9)–(11) will be used to introduce slow density variations and obtain the energy functional in which the slow and fast modes are separated.

Consider now our inhomogeneous system with slowly modulated $f(y)$ and $n(y)$. We divide the volume V into small macroscopic subvolumes of equal volume $\Delta V \ll V$ with the linear size $\Delta V^{1/d}$ much larger than the average interatomic separation $n^{-1/d}$, so that the number of atoms $n\Delta V$ in ΔV is large. Let $y = X$ be the position of the center of ΔV_X, $\sum_X \Delta V_X = V$ (Fig. 1). We will indicate quantities related to ΔV_X centered at X by a subscript X, i.e., if $y \in \Delta V_X$ then $f(y) = f_X(y)$, $n(y) = n_X(y)$; these functions computed at

$y = X$ will be denoted by the same symbols but without the argument, i.e., $f_X(X) = f_X, n_X(X) = n_X$. Clearly, at different X the functions f_X and n_X can be very different, but their dependence on X is slow and the total energy is the sum of energies of the subvolumes ΔV_X:

$$E = \sum_X \Delta E_X = \sum_X (\Delta K_X + \Delta U_X). \tag{12}$$

As for $y \in \Delta V_X$ one has $f_X(y) \simeq f_X, n_X(y) \simeq n_X$, $\sum_X f_X \Delta V_X = 1$, the energy terms without derivatives can be taken at $y = X$ which accounts for their dependence on the coarse-grained coordinate X since their variations within ΔV_X can be neglected. However, differentiation of $\rho(y, y')$ in ΔK_n (11) can result in the kinetic energy terms due to a slow but finite density variation $n_X(y) - n_X$ even within each ΔV_X. To obtain these terms, in Appendix A we separate from the density $n_X(y)$ a slow amplitude $A_X(y)$ which accounts for the density deviation relative to n_X within ΔV_X. Namely, for each X and atom's coordinates y and y' *within* ΔV_X, in the full local one-body density matrix $\rho_n(y, y')$ in ΔV_X we separate the factor $A_X^*(y)A_X(y')$ due to the variation $n_X(y) - n_X$ and obtain the following relation:

$$\rho_n(y, y') = \frac{A_X(y)^* A_X(y')}{f_X} \rho_X(y, y'), \qquad y, y' \in \Delta V_X \tag{13}$$

where $f_X = A_X{}^*(X) A_X(X) = A_X{}^* A_X$ and the reduced one-body density matrix $\rho_X(y, y')$ is computed at the central density n_X. This $\rho_X(y, y')$ is defined by the formula (10) with $\psi_{n_X} = \psi_X$ which corresponds to the homogeneous system V_{n_X} of N atoms with the density n_X.

Now we compute the y derivatives of ρ_X. Setting $\partial A_X(y)/\partial y = \partial A_X/\partial X$ and $\partial^2 A_X(y)/\partial y^2 = \partial^2 A_X/\partial X^2$, the kinetic energy ΔK_X [Eq. (11)] of the subsystem with ΔN_X atoms and density n_X obtains in the following form:

$$\Delta K_X = -\frac{N\hbar^2}{2m}\left(\frac{\Delta N_X}{N f_X}\right)\int_{V_X} dy\, dy' \delta(y - y')[|A_X|^2 \triangle_{y'} \rho_X(y, y') + (A_X^* \triangle_X A_X)\rho_X(y, y') + 2A_X^* \nabla_X A_X \cdot \nabla_{y'} \rho_X(y, y')]. \tag{14}$$

B. Local equilibrium and the stationary states

Now we can obtain the total energy E [Eq. (12)] as a functional of both fast and slow modes. To this end, in ΔK_X (14) we return to the expression of ρ_X (10) in terms of the function $\psi_{n_X} = \psi_X$. Adding ΔU_X (8) to this equation and setting $\Delta N_X = N f_X \Delta V_X$, one obtains the total energy E in the form of the following sum:

$$E\{\psi_X, A_X\} = \sum_X \Delta V_X \int_{V_n^N} d^N x\, A_X^* \psi_X^* \widehat{H} A_X \psi_X, \tag{15}$$

$$\widehat{H} = \widehat{h} - \frac{i\hbar}{m} \nabla_X \cdot \sum_{i=1}^N \widehat{\mathbf{p}}_i - \frac{N\hbar^2}{2m} \triangle_X, \tag{16}$$

where $\widehat{h}$ is the short-range Hamiltonian (1) and $\widehat{\mathbf{p}}_i = -i\hbar\nabla_i$ is the momentum operator in x_i which both act only on $\psi_X(x)$, whereas the operators ∇_X and $\triangle_X$ act on $A_X = A(X)$. This E is a functional of both fast variables $\psi_X(x)$, $\psi_X^*(x)$, and slow variables $A(X)$, $A^*(X)$, both normalized to unity:

$$\int_{V_n^N} d^N x\, \psi_X^* \psi_X = 1, \qquad \sum_X \Delta V_X A_X^* A_X = 1. \tag{17}$$

We are now ready to impose a local equilibrium which is determined by the local density n_X. As the variations $\delta\psi_X$ and δA_X are very different in their temporal and spatial scales, the variational equations must be applied separately to each of them. As now the total wave function is $A_X\psi_X$, the variational equation for the fast component at constant slow component A is

$$i\hbar A_X \partial\psi_X/\partial t = \frac{\delta E}{A_X^* \delta\psi_X^*}, \tag{18}$$

which gives the following Schrödinger equation:

$$i\hbar\partial\psi_X/\partial t = \left(\widehat{h} - \frac{i\hbar}{m}\nabla_X \ln A_X \cdot \sum_{i=1}^N \widehat{\mathbf{p}}_i\right)\psi_X(x). \tag{19}$$

By our assumption, locally the system is in a stationary state for which the Schrödinger equation (19) reduces to the eigenproblem

$$\left(\widehat{h} - \frac{i\hbar}{m}\nabla_X \ln A_X \cdot \sum_{i=1}^N \widehat{\mathbf{p}}_i\right)\psi_X = N\varepsilon_{X,A}\psi_X. \tag{20}$$

As A_X is slow, the second term can be treated as a perturbation of the Hamiltonian $\widehat{h}$. For this reason, the function ψ_X will be considered as the eigenfunction of the unperturbed operator $\widehat{h}$ with the eigenvalue $N\varepsilon_X$ which corresponds to the homogeneous density n_X as defined in Eq. (4). Then the total perturbed eigenvalue $\varepsilon_{X,A}$ per atom is

$$\varepsilon_{X,A} = \varepsilon_X - \frac{i\hbar}{m}\nabla_X \ln A_X \cdot \langle\mathbf{p}\rangle_X \tag{21}$$

and

$$N\langle\mathbf{p}\rangle_X = \int_{V_n^N} d^N x\, \psi_X^* \sum_{i=1}^N \widehat{\mathbf{p}}_i \psi_X, \tag{22}$$

where $\langle\mathbf{p}\rangle_X$ is the corresponding average momentum per particle (at the density n_X). If gas in a subvolume ΔV_X is in the ground state then of course $\langle\mathbf{p}\rangle_X = 0$, but in general the average momentum per atom in ΔV_X is nonzero and depends on the density $n_X = N|A_X|^2$, i.e., $\langle\mathbf{p}\rangle_X = \langle\mathbf{p}\rangle_X(|A_X|^2)$.

So far we have assumed that the stationary state is a pure eigenstate, but the stationary state can also be a superposition of more than one pure state with an obvious weighted form of the average energy ε_X per atom. For instance, a two-state superposition is

$$\psi_X(x, t) = a_1\psi_{1,X}(x)e^{-iN\varepsilon_1 t/\hbar} + a_2\psi_{2,X}(x)e^{-iN\varepsilon_2 t/\hbar}, \tag{23}$$

where $\psi_{1,X}$ is the eigenstate with the per atom eigenvalue $\varepsilon_X = \varepsilon_1$, $\psi_{2,X}$ is the eigenstate with $\varepsilon_X = \varepsilon_2$, t is the fast time, and $|a_1|^2 + |a_2|^2 = 1$. In particular, $\psi_{1,X}$ can be a ground state with zero momentum and $\psi_{2,X}$ can be a state with nonzero $\langle\mathbf{p}\rangle$. We see that the two states have different exponential time dependence. However, on averaging over fast time, these exponentials play no role in the HA as they disappear in the final hydrodynamic energy functional (25). Instead of wave function of the form (23) the expressions of the HA derived below contain the effective wave function of the form

$$\psi_X = a_1\psi_{1,X}(x) + a_2\psi_{2,X}(x) \tag{24}$$

and the average per atom energy $\varepsilon_X = |a_1|^2\varepsilon_1 + |a_2|^2\varepsilon_2$. The case of a superposition of two states can be directly generalized to the superposition of any number of eigenstates. For simplicity, in what follows we will continue our consideration in terms appropriate for a single pure inner state.

C. Hydrodynamic equation

Now we are ready to establish the equation describing the slow component A_X. Making use of Eqs. (20) and (21) allows one to perform averaging over the fast time t and the x integration in each ΔV_X to eliminate the fast mode ψ_X. Then we change ΔV for dX and obtain the total energy $E_{\rm sr}$ of the short-range interaction as a functional of A^* and A in the form of the following X integral:

$$E_{\rm sr}\{A_X\} = N\int dX\bigg[-\frac{\hbar^2}{2m}A_X^*\Delta_X A_X + |A_X|^2 e_X - \frac{i\hbar}{m}A_X^*\nabla_X A_X\cdot\langle\mathbf{p}\rangle_X\bigg]. \tag{25}$$

If the external potential $U_{\rm ext}$ changes rapidly, i.e., over the scale of an order of the interparticle distance, then it has to be included in the short-range Hamiltonian operator $\widehat{h}$, otherwise, it can be incorporated along with the long-range interaction potential $U_{\rm lr}$ which is assumed here. A long-range potential changes over the large X scale, it interacts with the total number of particles $N|A_X|^2dX$ in ΔV_X, and its source in another subsystem X' is the total number of particles $N|A_{X'}|^2dX'$ in $\Delta V_{X'}$. If both $U_{\rm ext}$ and $U_{\rm lr}$ are long-range interactions, then the total energy due to this interactions have the following form:

$$E_{\rm lr}\{A_X\} = \frac{N(N-1)}{2}\int dX\int dX'|A_X|^2|A_{X'}|^2(1+g_{2,XX'}) \times U_{\rm lr}(X-X') + N\int dX|A_X|^2U_{\rm ext}(X-X'), \tag{26}$$

where $g_{2,XX'}$ is the coarse-grained hydrodynamic pair correlation obtained by averaging G_2 (6) over ΔV_X and $\Delta V_{X'}$ (see Appendix B). The total energy of the inhomogeneous system is the sum $E = E_{\rm sr} + E_{\rm lr}$. The dynamics of the slow modulation A_X can be obtained from the following variational equation:

$$i\hbar\sqrt{N}\partial A_X/\partial t = \frac{\delta E}{\sqrt{N}\delta A_X^*}. \tag{27}$$

Performing this variation we have to remember that the local energy eigenvalue ε_X and average momentum $\langle\mathbf{p}\rangle_X$ depend on the local probability density $|A_X|^2$. The coarse-grained hydrodynamic equation (27) obtains in the form

$$\begin{aligned} i\hbar\frac{\partial A}{\partial t} &= -\frac{\hbar^2}{2m}\Delta A + A\left(\varepsilon + |A|^2\frac{\partial\varepsilon}{\partial|A|^2}\right) \\ &\quad - \frac{i\hbar}{m}\nabla A\cdot\left(\langle\mathbf{p}\rangle + |A|^2\frac{\partial\langle\mathbf{p}\rangle}{\partial|A|^2}\right) \\ &\quad + AU_{\rm ext} + (N-1)A\int_V dX'|A_{X'}|^2(1+g_{2,XX'}) \\ &\quad \times U_{lr}(X-X'), \end{aligned} \tag{28}$$

where the arguments X and t of the functions A and $U_{\rm ext}$ are omitted for brevity. Here ε and $\langle\mathbf{p}\rangle$ are, respectively, the per-atom eigenvalue of the operator $\widehat{h}$ (1) and the average particle momentum (22) in a homogeneous system of density $N|A|^2$, which are certain functions of $A^*A = |A|^2$. The second and third terms in the right-hand side of Eq. (28) are the contributions to the local chemical potential, respectively, from the unperturbed local eigenvalue ε and its perturbation: $\mu_X = \partial(N_X\varepsilon_{X,A})/\partial N_X$ where $N_X = N|A_X|^2dX$ is the number of atoms in ΔV_X and $\varepsilon_{X,A}$ is the total eigenvalue (21). This derivative is equal to

$$\mu_{X,A} = \partial(|A_X|^2\varepsilon_{X,A})/\partial|A_X|^2, \tag{29}$$

which results in the terms in question. It is this μ_X that universally appears in the HE and, in this sense, plays the role similar to the pressure gradient in the classical HEs. The specific form of its A dependence determines the nonlinearity in A and depends on the specific system via the specific density dependence of the local eigenvalues of the stationary Schrödinger equation (20).

Equation (28) shows that the flows of the fast and slow modes interact via the product $\propto\nabla A\cdot\langle\mathbf{p}\rangle$. If e_X is the eigenvalue of an excited state with a nonzero momentum, then this term can play its role. The GPE obtains from (28) for $\langle\mathbf{p}\rangle = 0$ in the weak interaction limit: both in three dimensions (3D) [5] and one dimension (1D) [16,17] the first term in the energy expansion is proportional to the density $n \propto |A|^2$ so that $\varepsilon|A|^2 \propto |A|^4$, which results in the GPE with the qubic nonlinearity. If ε is the ground-state eigenvalue for the Lieb-Lineger gas, then (28) gives the equation considered in [13–15,20,21]. In particular, this approach reduces to the eGPE for a low density and to the fifth-power nonlinearity proposed in [6] in the strong-coupling regime. However, Eq. (28) is not restricted to a specific model and dimension and is a general HE for a cold Bose gas.

Equation (28) can be presented in the classical hydrodynamic form. Setting $A = a\exp(i\theta/\hbar)$ with the real amplitude a and phase θ, we multiply (28) by $a\exp(-i\theta/\hbar)$ and separate the imaginary and real parts. The imaginary part can be reduced to the form

$$\frac{\partial a^2}{\partial t} + \nabla[a^2(\mathbf{v} + \langle\mathbf{p}\rangle/m)] = 0, \tag{30}$$

where $\mathbf{v} = \nabla\theta/m$. This equation is interpreted as that of continuity and $\mathbf{v}$ as velocity of the slow mode. The real part can be interpreted as the Hamilton-Jacobi equation with the quantum corrections [28]. Applying operator ∇ to this equation, one obtains the analog of the standard hydrodynamic

Navier-Stokes equation for an inviscid fluid:

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\nabla)\mathbf{v} = -\frac{1}{m}\nabla\left[\mu_{\rm tot} + \mathbf{v}\cdot\left(\langle\mathbf{p}\rangle + a^2\frac{\partial\langle\mathbf{p}\rangle}{\partial a^2}\right)\right], \tag{31}$$

where

$$\mu_{\rm tot} = \varepsilon + |A|^2\frac{\partial\varepsilon}{\partial|A|^2} - \frac{\hbar^2\Delta a}{2ma} + V_{\rm ext} + (N-1)\int_V dX'|A_{X'}|^2 U_{lr}(X-X')(1+g_{2,XX'}) \tag{32}$$

and $\langle\mathbf{p}\rangle$ is the average momentum per atom. We see that the right-hand side has the form (pressure gradient$/N$) $=\nabla(\mu_{X,A}+\mu_q+\mu_{lr})/m$ where $\mu_q \sim \hbar^2/m$ is the quantum pressure term and μ_{lr} is the contribution to the local chemical potential due to the long-range interaction. Note that the relation between Schrödinger's equation and hydrodynamics has been revealed by Madelung long ago [29] and since then has been extensively used as the motivation for the HA and LDA. The hydrodynamic Madelung representation is derived from Schrödinger's equation for the one-particle wave function [28,29]. In contrast, we arrived at the above HEs (30) and (31) in the reverse procedure, i.e., first deriving the HE (28) from the many-particle Schrödinger's equation (20) and only then applying Madelung's ansatz to the slow mode A which is indeed a one-particle quantity.

III. TOTAL EFFECTIVE WAVE FUNCTION

A. Introduction of the common coordinate

The general energy functional (15) can be presented in the following suggesting form:

$$E_{sr}\{\Psi\} = \int_V dX \int_{V_X^N} d^N x\, \Psi_X^*(x)\widehat{H}\Psi_X(x), \tag{33}$$

$$\Psi_X(x) = A_X\psi_X(x). \tag{34}$$

Here $\Psi_X(x)$ is the total effective inhomogeneous wave function and $E_{sr}\{\Psi\}$ can be interpreted as the expectation value of the Hamiltonian $\widehat{H}$ (16) in the state $\Psi_X(x)$. The function ψ_X is of the type (24) without fast t exponentials. The effective wave function depends on the Nd-dimensional vector $x = (x_1, x_2, \ldots, x_N)$ and the coarse-grained coordinate X. The d-dimensional components x_i of the argument x in $\Psi_X(x)$ are defined in the auxiliary volume V_{nx} (Fig. 1), which we denote simply as V_X, and X is a d-dimensional vector defined in the actual system's volume V. The dependence of the effective wave function $\Psi_X(x) = A_X\psi_X(x)$ on the coarse-grained coordinate X through A_X is explicit, and is in general functional one via dependence of $\psi_X(x)$ on the local density n_X. Thus, each local subvolume ΔV_X is represented by a homogeneous system V_X of N atoms at density n_X, the number of atoms in ΔV_X is set by the factors A_X^* and A_X, so that the operator $\widehat{H}$ acts on the vector $X \oplus x \in \Delta V_X \otimes V_X^N$. The total space in which our inhomogeneous gas is described is $\oplus_X(\Delta V_X \otimes V_X^N)$. It is appropriate to associate X and ΔV_X with the macrospace and x and V_X with the inner space: in the inner space the system is in the stationary quantum state $\psi_X(x)$ and in the macrospace space the system is in the state A_X.

The coordinates of the inner vectors x in different subsystems ΔV_X are independent as they are defined in different domains V_X and so far there was no need in their "synchronization." However, after the inner space has been used and the HE (28) already obtained, there is no need to resort to the auxiliary domains V_X, but instead there is a need of description in terms of the atoms' coordinates in the actual volume V. Then it is convenient and, for some reason even necessary, to use the common coordinate system in which vector $\widetilde{x} = (\widetilde{x}_1, \ldots, \widetilde{x}_N)$ has components that are smoothly interpolating those of the local inner vectors $x^{(X)}$ in different subsystems ΔV_X through the entire system. For instance, the phase of the total wave function depends on the atoms' positions in the actual space V and thus is a function of such $\widetilde{x}$ with the components $\widetilde{x}_i$ from V. To introduce $\widetilde{x}$, the inner coordinates x in different subsystems ΔV_X can be "synchronized" by choosing the common coordinate onset in all V_X as sketched in Fig. 1. Let $x^{(X)} = (x_1^{(X)}, x_2^{(X)}, \ldots, x_N^{(X)})$ be the vector of a point related to ΔV_X and defined in its V_X. If the onset is common, then at the boundary B_{12} between two adjacent ΔV_{X_1} and ΔV_{X_2}, the components of $x^{(2)} = (x_1^{(2)}, x_2^{(2)}, \ldots, x_N^{(2)})$ with $x_i^{(2)} \in \Delta V_{X_2}$ are equal to the components of $x^{(1)} = (x_1^{(1)}, x_2^{(1)}, \ldots, x_N^{(1)})$ with $x_i^{(1)} \in \Delta V_{X_1}$; similar matching takes place at all boundaries between neighboring ΔV's. Thanks to this pairwise adjustment, combining coordinates in different ΔV_X in the common reference frame one obtains the vector $\widetilde{x} = (\widetilde{x}_1, \widetilde{x}_2, \ldots, \widetilde{x}_N)$ that continuously interpolates between the local coordinates through the entire system. The vector $\widetilde{x}$ does not need the superscript X and is such that (i) its components $\widetilde{x}_i$ are continuous and defined in the total system's volume V, (ii) its components $\widetilde{x}_i$ in ΔV_X coincide with the local components $x_i^{(X)}$ of this point, i.e., $\widetilde{x}_i = x_i^{(X)}$ if $\widetilde{x}_i \in \Delta V_X$. As the densities at both sides of the boundaries $B_{XX'}$ are equal, $\psi_X(\widetilde{x})$ is a continuous function of the variable $\widetilde{x}$. In particular, the phase of $\psi_X(\widetilde{x})$, $\arg(\psi_X)$, is a function of the coordinates of all atoms in the actual volume V. In what follows we omit tilde and use notation x for the common coordinate $\widetilde{x}$.

B. Two forms of states with a nonzero momentum

As usual, in the HA the system is specified in its inner space, and here we present two important examples of inner states (34) which we are going to implement below. For clarity, the d-dimensional vectors such as components x_i of x, as well as y and X, will be presented in boldface.

First, we want to set gas in motion as a whole. Let the local ground-state wave function of the initial state be $\Psi_{X,0} = A_0(\mathbf{X},t)\psi_{X,0}(x)$. In this state, the energy density is $N|A_X|^2\varepsilon_X$ where ε_X is given in (4), momentum density is 0, and A_0 is governed by Eq. (28) with $\langle\mathbf{p}\rangle = 0$. If this gas is set in motion as a whole so that all atoms have the same momentum $\mathbf{p}$, then its new inner state $\psi_{X,p}$ and hence the total $\Psi_{X,p}$ acquires the factor $\exp\sum_i^N i(\mathbf{p}\mathbf{x}_i)/\hbar$ where $(\mathbf{p}\mathbf{x}_i)$ is the scalar product of two d-dimensional vectors. Then the total wave function of the gas moving as a whole is

$$\Psi_{X,p} = A_X\psi_{X,p}(x) = A_X\psi_{X,0}(x)\exp\sum_{i=1}^N i(\mathbf{p}\mathbf{x}_i)/\hbar. \tag{35}$$

This $\Psi_{X,p}$ is the local state with the energy density $N|A_X|^2(\varepsilon_{X,0}+\ p^2/2m)-N\frac{i\hbar}{m}A_X^*(\mathbf{p}\cdot\nabla_X)A_X$ where $\varepsilon_{X,0}$ is the eigenvalue for the eigenfunction $\psi_{X,0}$, it has the inner momentum $N\mathbf{p}$ with density $N\mathbf{p}|A_X|^2$; the macrostate A is now the solution of Eq. (28) with $\langle\mathbf{p}\rangle=\mathbf{p}$. Thus, in the state (35), the momentum density is changing along with the atom density $\propto|A_X|^2$, but the individual momenta $\mathbf{p}=\text{const}$. As the ground-state function is real and its phase is zero, the fast phase at some point $\mathbf{y}$ in the gas volume V is equal to $\theta_{\text{fm}}(\mathbf{x}_k=\mathbf{y})=\arg[\exp iN(\mathbf{py})/\hbar]=N(\mathbf{py})/\hbar$ and the total phase $\theta_{\text{tot}}(\mathbf{y})=\theta_{\text{sm}}+\theta_{\text{fm}}$ where $\theta_{\text{sm}}(\mathbf{y})=\arg[A(\mathbf{y})]$. It can be noticed that in this state the fast phase is additive so that it is possible and natural to assign an individual fast phase $(\mathbf{px}_i)/\hbar$ to each atom. Then the total phase of a single atom is $\theta_1(\mathbf{y})=\theta_{\text{sm}}(\mathbf{y})+(\mathbf{py})/\hbar$. The related problem is that in a ring geometry stretched, say, along y, the periodic boundary condition implies that the total phase change $\theta_{\text{tot}}(L)$ along the system length L must be a multiple of 2π. However, if $\Delta\theta_{\text{sm}}=\theta_{\text{sm}}(L)\neq 0$, which is the case of gray solitons, this phase $\theta_{\text{tot}}(L)=\ \Delta\theta_{\text{sm}}+NpL/\hbar$ is different and the excess phase $\text{mod}(\theta_{\text{tot}},2\pi)$ must be compensated. Our result shows that the compensating source can be a very slow flow of the gas as a whole with the velocity v' chosen as to cancel the phase, i.e., to make $\text{mod}[\theta_{\text{tot}}(L)+\ Nmv'L/\hbar,2\pi]=\ 0$. Due to additivity of the fast phase, this periodic boundary condition reduces to the condition $\text{mod}[\theta_1(L)+\ v'mL/\hbar,2\pi]=\ 0$ on the phase $\theta_1(y)$ of single atom, which is equivalent to

$$\Delta\theta_{\text{sm}}+\text{mod}[(p+mv')L/\hbar,2\pi]=0. \tag{36}$$

While the additional kinetic energy $Nmv'^2/2\sim 1/N$ is negligible, the total slow momentum per atom p_{sm}, up to unimportant constant, changes to

$$p'_{\text{sm}}=p_{\text{sm}}-\Delta\theta_{\text{sm}}\hbar/L. \tag{37}$$

This expression for the new momentum p'_{sm} is in line with the so-called momentum renormalization which has always been assumed in the soliton theory [3,30,31]. The advantage of dealing only with the single-atom phase θ_1 makes it possible the description in terms of a one-particle wave function. It is shown below that, both in our HA and in the standard HA, a gas moving as a whole can indeed be described by the same equation for a one-particle wave function.

Second, we want to describe a state with average nonzero momenta of some αN out of N atoms. In this state, the total wave function is similar to (24), i.e., it is a superposition of the local ground state $A_X\psi_{X,0}(x)$ and state with certain momentum:

$$\begin{aligned}\Psi_{X,p}&=A_X\psi_{X,p}(x)\\&=A_X\psi_{X,0}(x)\left[a_0+a_p\exp\sum_{i=1}^{N}i(\mathbf{px}_i)/\hbar\right],\end{aligned} \tag{38}$$

where $(a_p/a_0)^2=\alpha$. If the ground state as a whole is set in motion, then $a_0=0$, $a_p=\alpha=1$, and the wave function (38) reduces to the wave function (35). The state (38) has the energy density $N|A_X|^2(\varepsilon_{X,0}+\ \alpha p^2/2m)-N\frac{i\hbar}{m}A_X^*(\alpha\mathbf{p}\cdot\nabla_X)A_X$ where $\varepsilon_{X,0}$ is the eigenvalue for the eigenfunction $\psi_{X,0}$, the inner momentum density is $N\alpha\mathbf{p}|A_X|^2$. In each ΔV with its $|A|^2$, the average ratio (number of atoms with momentum $\mathbf{p}$)/(number of atoms with zero momentum) $=\ \alpha$ and remains fixed. Now the fast phase at a point $\mathbf{y}$ in V is $\theta_{\text{fm}}(\mathbf{y})=\arg[\psi_{X,p}(\mathbf{x}_i=\mathbf{y})]$:

$$\theta_{\text{fm}}(\mathbf{y})=\arctan\left[\frac{a_p\sin[N(\mathbf{py})/\hbar]}{a_0+a_p\cos[N(\mathbf{py})/\hbar]}\right]. \tag{39}$$

This phase cannot be separated into individual phases of single atoms so that, in contrast to the above case of gas motion as a whole, the one-particle description is impossible. The counterpart of Eq. (36) for the velocity v' of the phase compensating flow in a one-dimensional ring geometry has the form

$$\text{mod}[\Delta\theta_{\text{sm}}+\theta_{\text{fm}}(L)+Nmv'L/\hbar,2\pi]=0, \tag{40}$$

where θ_{fm} is given in (39). In spite of the difference with the case of the wave function (35), the renormalized slow mode momentum is the same [Eq. (37)]. The states with the wave function (38) will be addressed in Sec. IV.

The two states (35) and (38) are useful for our presentation as their many-particle phases are known. At the same time, the phase of a single excited eigenstate of the operator $\widehat{h}$ (1) with nonzero momentum is usually not known. Such a state is employed in Sec. VI where the momentum and energy eigenvalues will suffice for our purpose. The peculiarity of such states is that the number of excited atoms remains constant in all ΔV, but their momenta can depend on the density via $|A|^2$.

C. General properties of the general HA and its relation to the standard HA

As a general HA, the presented description of a cold Bose gas comprises two modes. As a result, not only the energy (25) but also the total momentum $\mathbf{P}$ and total phase θ consist of both slow and fast contributions:

$$\mathbf{P}=\mathbf{P}_{\text{sm}}+N\langle\mathbf{p}\rangle,\quad \theta_{\text{tot}}=\theta_{\text{sm}}+\theta_{\text{fm}}, \tag{41}$$

where $\mathbf{P}_{\text{sm}}=-i\hbar mN\int dV(A^*\nabla A-A\nabla A^*)$ is the slow mode momentum and $N\int dV\langle\mathbf{p}\rangle_X/V$ is the fast mode momentum, the phases are defined at points $\mathbf{y}$ in the actual volume V, $\theta_{\text{sm}}(\mathbf{y})=\arg A(\mathbf{y})$ and $\theta_{\text{fm}}(\mathbf{y})=\arg[\psi_X(\mathbf{x}_i=\mathbf{y})]$. We see that our HA which explicitly describes an inhomogeneous system in terms of both microscopic inner many-particle fast mode $\psi(x)$ and the slow macroscopic hydrodynamic mode $A(X)$ is substantially different from the standard HA which describes the system solely in terms of the slow one-particle mode. We will now show that, nevertheless, at least in a one-dimensional geometry, both descriptions are consistent with one another as long as the gas moves as a whole and $\langle p\rangle$ is a constant momentum p of every atom, which is independent of the density.

To address this and our next problems we introduce the renormalized time τ, coordinate y, momentum p, and new function $\phi(\tau,y)$: $t=\tau m/(\pi^2n_0^2\hbar)$, $X=y/(\pi n_0)$, $x_k=y_k/(\pi n_0)$, where y is the renormalized one-dimensional coordinate from V, y_i is the renormalized coordinate of ith atom, $p=\langle p\rangle/(\pi n_0\hbar)$, $\phi=A\sqrt{N/n_0}$, where $n_0=N/L$ is the unperturbed density and L is system's length. In these variables, the energy is in units $\hbar^2\pi^2n_0^2/m$ and $L=\pi N$; the atom momentum, related velocity, which is now in units of the sound velocity $v_S = \pi n_0 \hbar/m$, and wave number $\langle p\rangle/\hbar$ in units πn_0 are equal numbers (which one should keep in mind). For a constant p, the general form of Eq. (28) in one dimension is

$$i\partial_\tau \phi = -\tfrac{1}{2}\phi'' + \mu(|\phi|^2)\phi - ip\phi', \tag{42}$$

where prime of ϕ stands for the y derivative and $\mu(|\phi|^2) = \varepsilon + |\phi|^2\partial\varepsilon/\partial|\phi|^2$ is the unperturbed local chemical potential. As we showed above, if ψ_g is the initial local inner ground state of the gas, but the gas is set in motion as a whole and all atoms have the same momenta p, then the ψ_g acquires the factor $\exp(\sum_k^N ipy_k)$, the inhomogeneous system is described by the total function of the form (35), i.e., $\Psi = \phi(y,\tau)\psi_g \exp(\sum_i^N ipy_i)$, all atoms have similar fast phase py, and the slow mode ϕ is the solution of Eq. (42). At the same time, the corresponding equation of the standard HA is similar to (42), but without p term:

$$i\partial_\tau \phi = -\tfrac{1}{2}\phi'' + \mu(|\phi|^2)\phi. \tag{43}$$

However, now both slow and fast modes appear as the solution of the standard equation (SE) (43) which describes the one-atom state and therefore must have the form $\phi_{\rm SE} = \phi_{\rm sm}(y,t)\exp i(py - p^2t/2)$ where $\phi_{\rm sm}(y,t)$ is the slow mode and p is an arbitrary momentum. Substituting this $\phi_{\rm SE}$ in Eq. (43) we find that the $\phi_{\rm sm}$ satisfies exactly Eq. (42) of our HA. Thus, the slow component of $\phi_{\rm SE}$ and ϕ satisfy the same equation (42), the total spatial phase of the function $\phi_{\rm SE}$ and that of an individual atom in ψ are similar and equal to $\arg(\phi) + py$. The difference is that both momentum p and coordinate y in the standard HA and our HA have different origin. In the standard HA, py is the common linear phase of all atoms whereas in our HA, py_i is the fast phase of ith atom; the momentum p in the standard HA is the parameter of the solution $\phi_{\rm SE}$ which is interpreted as the wave number of the background oscillation [31], whereas in our HA, p is the local momentum eigenvalue per atom. Thus, the SE (43) for the supposedly macroscopic variable $\phi_{\rm SE}$ describes both fast oscillations and actual slow variable $\phi_{\rm sm}$, the last alone being governed by our Eq. (42). Thanks to this fact both the LDA and our HA are equivalent in the case when the gas is moving as a whole. However, in the state with the wave function (38) when certain gas fraction is in the ground state and the total momentum is that of the local excitations, and/or when this momentum depends on the local density, then such a gas can be described only by our HA as the standard HA is not applicable in this case. Below we illustrate possible effects in such systems by two examples.

IV. CAN A PERIODIC SOLITON TRAIN PROPAGATE?

The GPE has both single-soliton solution and the so-called periodic soliton train solution [27,32]. Here we show that in contrast to the former solution whose phase at the periphery can be of the form v_0y with arbitrary v_0 independent of the soliton velocity v, the latter is always of the form of a standing wave

$$\phi_{\rm train} = \phi(y - vt)\exp i(vy - v^2t/2). \tag{44}$$

Here $\phi(y - vt)$ is the slow mode which describes the periodic train moving with velocity v, and the exponential $\exp(ivy)$ describes the total gas set in motion as a whole with the same velocity v. The reason why we call this a standing wave is that the train's velocity with respect to the medium (the gas) is zero, i.e., the train does not propagate with respect to the medium. We show that the above $\phi_{\rm train}$ is a general solution of the time-dependent GPE.

We are interested in the solution of Eq. (43) in the form $\phi_{\rm train} = \phi(y,t)\exp i(v_0y - v_0^2t/2)$ in which ϕ is a stationary solution $\phi = \phi(y - vt) = R\exp i\theta_{\rm sm}$ where R is the real amplitude and $\theta_{\rm sm}$ is real phase. As shown above, if the gas is moving as a whole with velocity v_0 then the slow mode ϕ is the solution of Eq. (42) with $p = v_0$. The imaginary part of (42) is the phase equation which does not depend on the potential μ and has the form

$$(\theta'_{\rm sm} + v_0 - v)R^2 = C, \tag{45}$$

where C is an integration constant. If the solution is solitary, then, at the periphery, R takes a constant value R_∞, $\theta'_{\rm sm} = 0$, and one has $\theta_{\rm sm}(y) = (v_0 - v)\int_{-\infty}^{y} dy'[R_\infty^2/R^2(y') - 1]$. Adding the fast phase v_0y we obtain $\theta_{\rm tot} = \theta_{\rm sm}(y) + v_0y$, which describes the medium moving with the velocity v_0 and the soliton superimposed upon it. As v in $R = R(y - vt)$ is arbitrary, the soliton is propagating in the medium with the relative velocity $v - v_0$ restricted only by the speed of sound.

In the case of a periodic solution, the phase acquires a term linear in y. For periodic R Eq. (45) is solved by

$$\theta_{\rm sm}(y) = \int_0^y dy' C/R^2(y') + (v - v_0)y. \tag{46}$$

It has been shown that the first integral is expressed in terms of elliptic functions and does not have terms linear in y [27,32]. The total phase is then $\theta_{\rm tot} = \theta_{\rm sm} + v_0y$ and has the linear term vy which indicates that the resulting velocity of the medium is v in accord with (44). Thus, the periodic train is moving with the velocity v exactly equal to that of the medium and therefore is a standing wave. This means that the periodic soliton train can be set in motion with velocity v_0 only along with the whole medium moving with the same velocity and momentum Nmv_0. The wave function of the state considered above is of the form (35), it allows for the one-particle description, and can be considered both in the standard and our HA.

To make the train a propagating wave, its velocity must be made different from the velocity of the ground-state atoms. This is possible if the medium has $\alpha N < N$ local excitations v_0 with small α so that the total momentum $N\langle\mathbf{p}\rangle = \alpha N v_0 < Nv_0$. In this case, the wave function is of the form (38), the slow mode is governed by Eq. (42) with $p = \alpha v_0$, and the slow phase has the form (46) in which v_0 is replaced with αv_0. But now in average $N(1-\alpha)$ atoms remain in the ground-state fraction with zero momenta and only small fraction of αN atoms have average momenta. The total wave function of the fraction of $N(1-\alpha)$ atoms with zero momentum has the form $a_0R(y - vt)\exp(i\theta_{\rm sm})\psi_g$ where $\theta_{\rm sm}$ has the linear term $(v - \alpha v_0)y$. We see that the train velocity v differs from that of the medium: its major fraction which in average consists of $N(1-\alpha)$ atoms is moving with the common velocity $v - \alpha v_0$, hence, the train drifts with respect to the medium with the velocity αv_0. If $\alpha = 1$ then $\theta_{\rm fm} = vy$, the velocities of the train and medium are both v, and the train is a standing wave. Thus, the local excitation momenta can push the train with respect to the bulk of atoms at rest, thereby making the soliton train a propagating wave. The state considered cannot be described in terms of the one-particle HA.

V. EFFECT OF THE DENSITY-DEPENDENT LOCAL MOMENTA ON A SOLITON

A stationary soliton with velocity v in a gas with a constant momentum δ per atom (both due to all atoms in the same motion and only to some atoms with nonzero momenta) has the form $\phi(y - vt|v, \delta)$ which depends on the parameters v and δ. This solution can be obtained from that in a gas at rest, $\phi(y - vt|v)$, by a simple shift of the parameter v by δ, i.e., $\phi(y - vt|v, \delta) = \phi(y - vt|v - \delta)$. This is a general rule that follows from this observation: both in the imaginary part [Eq. (46)] and in the real part of Eq. (42), the parameter v appears in the combination $v - \delta$. But we now want to see how a soliton solution can be modified in a gas with local momenta that depend on the local gas density. To this end, as a simple application of the obtained equation (28), we consider a cold one-dimensional Lieb-Liniger Bose gas in the strong repulsion Tonks-Girardeau regime [16,17]. The energy structure of a Lieb-Liniger gas in this regime is known to be Fermilike with the ground state in which all the energy levels up to certain maximal value are occupied. We assume that a small fraction of αN upper levels are excited and bear momenta p. In principle, this p can depend on the local density in itself. To support this possibility, we suggest the following reasoning. The total momentum of a Lieb-Liniger gas is a multiple of $2\pi/L$ [16]. As the inner state ψ_X of each subvolume ΔV_X is related to a homogeneous system of N atoms and length $L_X = N/n_X$, the possible momenta in a subvolume ΔV_X are multiples of $2\pi n_X/N$ which is proportional to the density n_X. If atoms' momenta p are adiabatically following the momentum levels that are slowly changing in space along with density, then $p(|A|^2) \propto |A|^2$. This suggests a model of an inhomogeneous system, in which both energy and momentum of individual atoms are changing with the density.

Consider the eigenstate with αN exited atomic levels. If p is the excitation momentum at the nonperturbed density n_0, then the excitation momentum at density n is $(n/n_0)p$. The ground-state energy per atom is $e_0 = (n/n_0)^2/6$ [16,17,33]. As the density is $n = N|A|^2 = n_0|\phi|^2$, one has $e_0 = |\phi|^4/6$ and $e_0 + |\phi|^2\partial e_0/\partial|\phi|^2 = |\phi|^4/2$, $\langle p\rangle(n) = \delta|\phi|^2$ and $\langle p\rangle + |\phi|^2\partial\langle p\rangle/\partial|\phi|^2 = 2\delta|\phi|^2$ where $\delta = \alpha p$. Making use of these functions in the energy (25), one gets

$$E = \frac{1}{\pi}\int_0^L dy\left(-\frac{1}{2}\phi^*\phi'' + \frac{1}{6}|\phi|^6 - i\delta\phi^*\phi'|\phi|^2\right). \qquad (47)$$

Substituting $\phi(y - v\tau)\exp(-i\lambda\tau)$ in the HA (42) gives the equation for ϕ:

$$iv\phi' - \tfrac{1}{2}\phi'' + \tfrac{1}{2}|\phi|^4\phi - 2i\delta\phi'|\phi|^2 - \lambda\phi = 0. \qquad (48)$$

The function ϕ has the asymptotics $\phi(y \to \pm\infty) = 1[A(x \to \pm\infty) = 1/\sqrt{L}]$ so that $\int dy|\phi|^2 = \pi N$. We are looking for the stationary solution of Eq. (48) in the form $\phi(\tau, y) = R(y - v\tau)e^{i\theta(y-v\tau)}$ where R and θ are real functions, and v is the wave velocity in units $v_s = \pi n_0\hbar/m$ of sound velocity for $\delta = 0$. The soliton profile $|\phi|^2$ is found to be

$$|\phi|^2 = 1 - \frac{3B}{2 + 3v\delta + D\cosh[2\sqrt{B}(y - v\tau)]}, \qquad (49)$$

where $B = 1 - (v - \delta)^2 + 2v\delta$, $D = \sqrt{(1 + 3v^2)(1 + 3\delta^2)}$. The phase change $\Delta\theta_{\mathrm{sm}}$ along the soliton due to the slow mode is

$$\Delta\theta_{\mathrm{sm}} = -\operatorname{sign}(v - \delta)\cos^{-1}\left(\frac{3(v - \delta)^2 - 1 - 3v\delta}{D}\right) + \sqrt{3}\delta\ln\left(\frac{\sqrt{3B} + 2 + 3v\delta}{D}\right). \qquad (50)$$

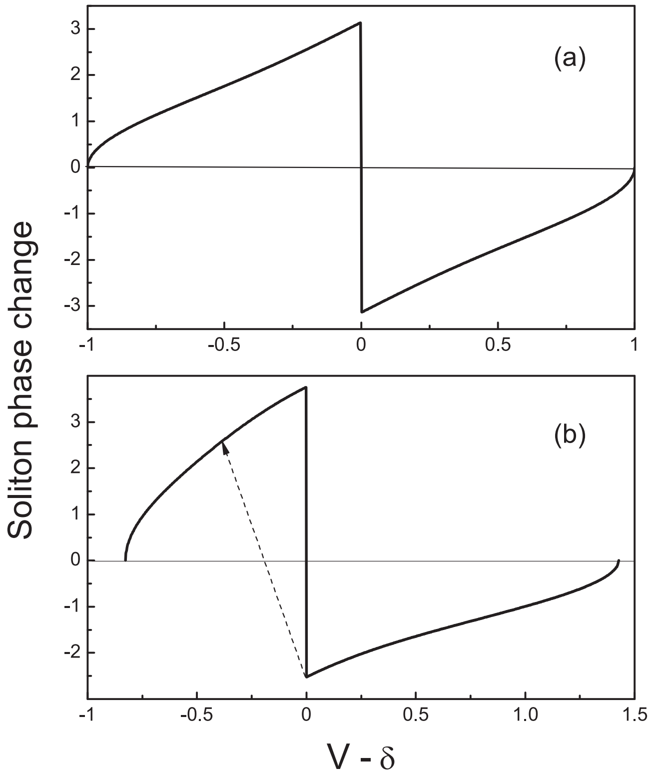

FIG. 2. Total phase change along the soliton $\Delta\theta_{\mathrm{sm}}(v - \delta)$. (a) $\delta = 0$ and $p = \text{const}$ independent of density, every soliton has antisoliton with the opposite phase. (b) $p \propto |\phi|^2$, $\delta = 0.3$, solitons with the phase above the point indicated by the arrow do not have antisolitons.

The condition $B \geqslant 0$ yields the following anisotropic restriction on the soliton velocity which is different in the directions along and opposite to δ: $2\delta - \sqrt{1 + 3\delta^2} < v < 2\delta + \sqrt{1 + 3\delta^2}$. It is important that δ is not the actual atoms' velocity p but the effective one αp. If the excited fraction α is small, then δ is considerably smaller than p which in itself can be larger than unity (i.e., than the speed of sound). Both for $\delta = 0$ and p independent of density, the soliton with any allowed velocity v and phase change $\Delta\theta_{\mathrm{sm}}$ has an antisoliton with the phase change $-\Delta\theta_{\mathrm{sm}}$, and the two velocities are symmetric with respect to $v = \delta$ [Fig. 2(a)]. In contrast, Eq. (50) shows that a density dependent p changes the symmetry: not for all v solitons have antisolitons and the velocities of a soliton and its antisoliton are asymmetric [Fig. 2(b)].

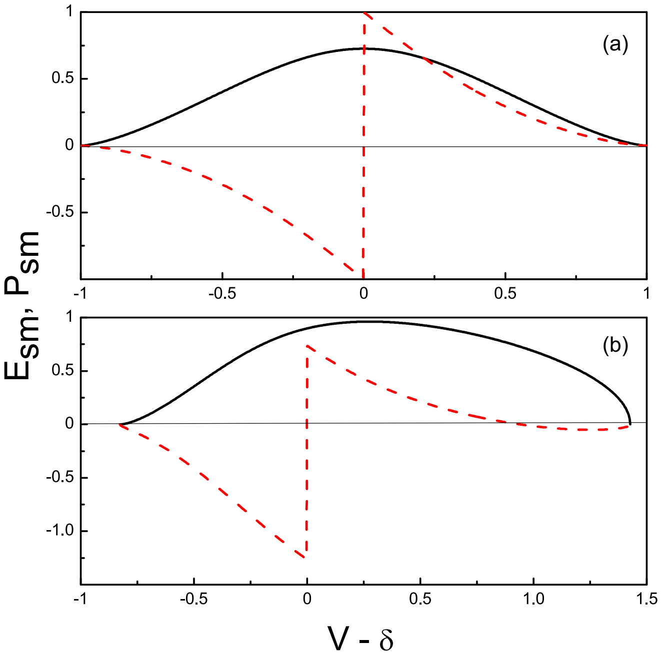

FIG. 3. Soliton energy E_{sm} (solid lines) and momentum P_{sm} (dashed lines) vs $(v-\delta)$. (a) $\delta=0$ and $p=\mathrm{const}$ independent of density, (b) $p \propto |\phi|^2$, $\delta=0.3$.

The energy E_{sm} of the slow mode is equal to the total E [Eq. (47)] minus the contribution from the homogeneous gas with the excitations (see Appendix C) (the excitation energy has to be excluded too, but it was set to zero [33]). This E_{sm} and the soliton momentum P_{sm} (37) are obtained in the form

$$E_{\mathrm{sm}} = \frac{\sqrt{3}}{\pi}\left[1-(v-\delta)^2+G\right]\ln\left[\frac{\sqrt{3B}+2+3v\delta}{D}\right] + \frac{\sqrt{3}v\delta}{\pi}(9\delta^2-1)B, \tag{51}$$

$$P_{\mathrm{sm}} = -\frac{\sqrt{3}}{\pi}v(1+3\delta^2)\ln\left[\frac{\sqrt{3B}+2+3v\delta}{D}\right] + \frac{\sqrt{3}\delta}{\pi}B - \frac{\Delta\theta_{\mathrm{sm}}}{\pi}, \tag{52}$$

where $G = 4v\delta(1+3v\delta) - 2\delta(v-\delta)(1+3v\delta) + 3\delta^2\{2/3+v^2-(1+3v\delta)^2-(v-\delta)^2\}$ and the energy is in units $\hbar^2\pi^2 n_0^2/2m$. For $\delta=0$, for which the above results (49)–(52) reproduce those found in [6], as well as for constant p, both E_{sm} and P_{sm} are symmetric with respect to the value $v=\delta$ [Fig. 3(a)]. However, for $p \propto |\phi|^2$, both $E_{\mathrm{sm}}(v)$ and $P_{\mathrm{sm}}(v)$ are asymmetric which is shown for $\delta=0.3$ in Fig. 3(b).

In usual situation, when a soliton is excited in the gas ground state, its momentum includes only that of a slow mode and the compensating momentum. Our situation is different in that the total excitation comprises the local excitations of the momenta p and the slow soliton mode. However, we are interested in the soliton dispersion relation E_{sm} vs $P_{\mathrm{sm}} = P_{\mathrm{tot}} - \mathrm{mod}(\theta_{\mathrm{tot}}, 2\pi)/\pi$ in which θ_{tot} contains the contribution of the momenta p [Eq. (40)]. Thus, the dispersion relation $E_{\mathrm{sm}}(P_{\mathrm{sm}})$ accounts for the soliton excitation in the gas with local momentum excitations. Figure 4 presents the dependence $E_{\mathrm{sm}}(P_{\mathrm{sm}})$ for $\delta=0$ and a density independent p, and for $\delta=0.3$ in the case of $p \propto |\phi|^2$. For $\delta=0$ and constant p this curve is symmetric with respect to the line $P_{\mathrm{sm}}=0$ [6] [Fig. 4(a)]. This is in line with the above rule that if the momentum p is density independent, then the soliton velocity v is just replaced by $v-\delta$ resulting in symmetric $E_{\mathrm{sm}}(P_{\mathrm{sm}})$. In our case this momentum is proportional to the density and the dependence $E_{\mathrm{sm}}(P_{\mathrm{sm}})$ is deformed and asymmetric with respect to the line $P_{\mathrm{sm}}=0$ [Fig. 4(b)]. Thus, the reason for the asymmetry of the curves in Figs. 2(b), 3(b), and 4(b) as well as for the absence of antisolitons in some velocity range [Fig. 2(b)] is not a nonzero excitation momentum p itself, but its density dependence.

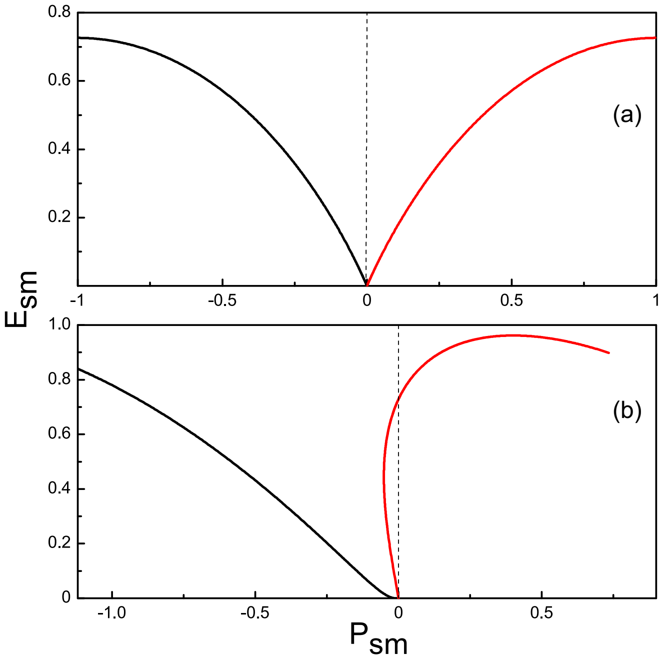

FIG. 4. Soliton dispersion relation $E_{\mathrm{sm}}(P_{\mathrm{sm}})$. (a) $\delta=0$ and $p=\mathrm{const}$ independent of density. (b) $p \propto |\phi|^2$, $\delta=0.3$.

VI. CONCLUSION

The development of the physics of ultracold gases has demonstrated a high efficiency of the large-scale hydrodynamic-type description in terms of local smoothly varying or coarse-grained quantities. A number of equations have been presented, e.g., the mean field GPE [1–3], eGPE with the beyond mean field Lee-Huang-Yang corrections [5], the modified GPE with the quintic nonlinearity [6], as well as based on the hydrodynamic analogy MNLSE [19] and LLGPE [20]. All of them have been successfully applied to certain domain of Bose gas parameters such as dimension, interaction strength, density, and all of them have somewhat or substantially different derivation and mathematical form which reflect such parameters. All these derivations have in common that they have been based on the one-particle quantum description. This motivated us to derive the general HA to a cold Bose gas which would have a general form and provide the connection with the many-body quantum mechanical description.

Starting from the many-body quantum mechanical approach we developed the HA. As any standard HA, it presupposes two different temporal and spatial scales and, respectively, two different modes, the fast and slow. Our HA is grounded on the energy functional $E\{\psi_X, A_X\}$ [Eqs. (15) and (16)] of both modes. The fast mode and the local equilibrium are identified, respectively, with the many-body wave function ψ_n and its stationary state at local density n. The integration over the fast mode (over the short scale) naturally resulted in the HE for the slow mode alone which is the counterpart of the momentum integration in the local distribution function in the classical HA. The HE contains two universal terms, the local chemical potential expressed via the energy eigenvalues of the local ψ_n and the interaction between the local momenta and slow velocity. For different particular forms of the former term the HE (28) reduces to the known equations. The second term is new and its full exploration is a novel problem. In this paper we sketched only two possible effects related to the local excitations and their density dependence.

To conclude, we can speculate about possible relevance and applications of the presence of the momentum-related term. We may point to the following experimental situations and problems. Excitations of local momenta in the Lieb-Liniger gas have been induced experimentally by a laser beam in [34]. States with the peaks at nonzero momenta have been observed in experiments on colliding clouds of cold Bose atoms. The occupation inversion obtained in this experiment was attributed to a negative temperature [10,35]. However, the thermodynamic approach to such systems is an approximation and, in principle, in terms of the HA this situation can be thought of as an excited state with nonzero momentum.

In the Introduction, we mentioned the problem related to the quantum pressure term, which violates the consistency of the slow mode equation and results in incorrect interference patterns: on the one hand, it is present in the exact Schrödinger equation and cannot be just discarded, but, on the other hand, it "spoils" the slow mode equation. The problem is that the second-order spatial density derivative should be better attributed to the fast mode equation, but the known HAs consist of the equation only for a slow mode. Our HA, which consists of both slow mode equation (20) and fast mode equation (28), suggests a possible remedy. The quantum pressure term, which is a part of the term $\triangle A$ in Eq. (28), can in principle be relocated from this equation to the fast mode equation (20). In this way, the slow mode equation would be more consistent while the local energy eigenvalue be renormalized. This can hopefully extend the applicability of the HA to the interference-type effects.

As we saw above, the constant phase of a stationary soliton in a ring geometry can be compensated by the global flow of the whole gas. However, in course of a gas dynamics the local phase can rapidly vary and the global flow is hardly a remedy. Can the local phase be compensated by a local momenta excitation? Interestingly, such a possibility was mentioned by Kivshar and coworkers in [31,36]. These papers address gray solitons of a nonlinear Schrödinger equation (similar to the GPE) describing their propagation in a fast oscillating background. Surprisingly, though this equation allows only for a constant spatially independent background wave number, in [31,36] a spatial dependence of the wave number in the soliton vicinity was mentioned as a possible source of the phase compensation. Our result seems to be the right tool to approach this problem.

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APPENDIX A: SEPARATION OF THE EFFECT OF A SLOW DENSITY VARIATION IN THE REDUCED ONE-BODY DENSITY MATRIX

Consider a single subsystem ΔV_X with the center at X. The density $n(y)$ is slowly varying within this volume and the psi function is a functional of n, $\psi(y, x^{N-1}|n) = \psi_n(y)$, where for brevity the dependence on x is omitted. We want to separate the contribution due to this density variation from the one-body density matrix taken at the central density $n_X = n(X)$. A small change in density $n = n_X + \delta n(y)$ results in the change $\psi_n(y, x^{N-1}) = \psi_X(y, x^{N-1}) + \delta\psi(y, x^{N-1})$ of the wave function. Then one has

$$\begin{aligned}\rho_n(y, y') &= \int dx^{N-1}[\psi_X^*(y) + \delta\psi^*(y)] \times [\psi_X(y') + \delta\psi(y')]\\ &\simeq \rho_X(y, y') + \int dx^{N-1}[\psi_X^*(y)\delta\psi(y')\\ &\quad + \psi_X(y')\delta\psi^*(y)]. \qquad \text{(A1)}\end{aligned}$$

Due to the presence of $\delta(y - y')$ in the integrand of (14), we can consider only the case $\Delta y = y' - y \to 0$. Then, neglecting the terms on the order $\delta\psi\,\Delta y$, the above one-body density matrix can be reduced to the form

$$\rho_n(y, y') = \rho_X(y, y') + \delta\rho(y') + \delta\rho^*(y), \qquad \text{(A2)}$$

where $\delta\rho(y') = \int dx^{N-1}\psi_X^*(y')\delta\psi(y')$ and $\delta\rho^*(y) = \int dx^{N-1}\psi_X(y)\delta\psi^*(y)$. Since $\rho_X(y, y) = f_X(y) \neq 0$, there exists a finite vicinity of $y' - y = 0$ in which $\rho_X(y, y') \neq 0$, hence, one can divide by this function. Then one has

$$\rho_n(y, y') = \rho_X(y, y')[1 + \delta\rho(y')/f_X + \delta\rho^*(y)/f_X], \qquad \text{(A3)}$$

where, in the denominators, $\rho_X(y, y')$ is replaced by f_X in within the accuracy up to $O(\Delta y^2)$. Neglecting terms quadratic in $\delta\rho$, this can be cast in the form of Eq. (13), i.e.,

$$\rho_n(y, y') = \frac{A_X(y)A_X^*(y')}{f_X}\rho_X(y, y'), \qquad \text{(A4)}$$

where we introduced the amplitude $A(y)$ and its conjugate according to the following definition:

$$\begin{aligned}A_X(y) &= \sqrt{f_X} + \delta\rho(y)/\sqrt{f_X},\\ A_X^*(y') &= \sqrt{f_X} + \delta\rho^*(y')/\sqrt{f_X}. \qquad \text{(A5)}\end{aligned}$$

For constant density n_X, Eq. (A4) recovers $\rho_X(y, y')$ and for $y = y'$ it reduces to $f_X(y)$. In formulas (A4) and (A5), the contribution of the density variation within ΔV_X is separated from the one-body density matrix at n_X in the form of the product $A_X(y)A_X^*(y')$ which is employed in the calculation of the kinetic energy (14).

APPENDIX B: COARSE-GRAINED CORRELATION FUNCTION AND THE LONG-RANGE INTERACTION ENERGY

In a liquid state, at large separation $y - y' \simeq X - X'$, the pair distribution $G_2(y, y')$ usually tends to the product $f(y)f(y')$ of the probability densities, so that in general $G_2(y, y') = f(y)f(y')[1 + g_2(y, y')]$ where g_2 is the pair correlation. The coarse-grained hydrodynamic pair distribution $\rho_{2,XX'}$ and coarse-grained correlation $g_{2,X,X'}$ are defined by the following equation:

$$\begin{aligned}\rho_{2,XX'}|A_X|^2|A_{X'}|^2 &= \frac{1}{(\Delta V)^2}\int_{\Delta V_X} dy \int_{\Delta V_{X'}} dy' G_2(y, y')\\ &= |A_X|^2|A_{X'}|^2(1 + g_{2,X,X'}),\end{aligned} \tag{B1}$$

where we made use of the relations (A4), $\rho_X(y, y) = f_X(y) \simeq f_X$. Then the energy of the long-range interaction is

$$\begin{aligned}E_{\text{lr}}\{A_X\} &= \frac{N(N-1)}{2}\int_V dy \int_V dy' G_2(y, y')U_{\text{lr}}(y - y')\\ &\quad + N\int_V dy U_{\text{ext}}(y)\rho(y, y)\\ &= \frac{N(N-1)}{2}\sum_{\Delta V_X, \Delta V_{X'}}\int_{\Delta V_X} dy \int_{\Delta V_{X'}} dy' G_2(y, y')\\ &\quad \times U_{\text{lr}}(y - y') + N\sum_{\Delta V_X}\int_{\Delta V_X} dy U_{\text{ext}}(y)\rho(y, y)\\ &\simeq \frac{N(N-1)}{2}\int dX \int dX' |A_X|^2|A_{X'}|^2(1 + g_{2,XX'})\\ &\quad \times U_{\text{lr}}(X - X') + N\int dX' |A_X|^2 U_{\text{ext}}(X - X'),\end{aligned} \tag{B2}$$

which is Eq. (26).

APPENDIX C: THE SOLITON ENERGY

The soliton energy is the difference between the energy E of the system with soliton and $|\phi|^2 = f \to 1$ at the periphery, and the energy E_0 of the homogeneous system with the same number of atoms N. As the density n in the area of gray soliton is lower than $n_0 = N/L$, the actual density far from soliton $\widetilde{f}_\infty = |\widetilde{\phi}_\infty|^2$ is slightly higher than $f_\infty = 1$, so that $\widetilde{f} = f + \delta f$ where δf is nonzero at the periphery and very small in a large system. Then

$$\int dy\, \delta f + \int dy(f - 1) = 0. \tag{C1}$$

Here we find the soliton energy taking into account this correction at the periphery. The total energy with ϕ satisfying the equation (48) can be found in the form of the virial theorem. We multiply (48) with ϕ^*/π and integrate over the system length which, regarding the expression for the energy (47), gives

$$\begin{aligned}0 &= \frac{1}{\pi}\int_0^L dy\Big(iv\phi^*\phi' - \frac{1}{2}\phi^*\phi'' + \frac{1}{2}f^3 - 2ip\phi^*\phi'|\phi|^2 - \lambda|\phi|^2\Big)\\ &= E + \frac{1}{\pi}\int_0^L dy\Big(iv\phi^*\phi' - \lambda f + \frac{1}{3}f^3 - ip\phi^*\phi' f\Big),\end{aligned} \tag{C2}$$

whence

$$E = \frac{1}{\pi}\int_0^L dy\Big(-iv\phi^*\phi' + \lambda f - \frac{1}{3}f^3 + ip\phi^*\phi' f\Big). \tag{C3}$$

Next, from Eq. (48) at the periphery one finds $\lambda = f_\infty^2/2 = \frac{1}{2}$. Now we change f to the actual $f + \delta f$ and λ to $\lambda' = 1/2(1 + 2\delta f_\infty)$ in the expression (C3), taking into account that $\int dy\, f^k\delta f = \int dy\, \delta f + O(\text{soliton width}/L)$ for any k and that the first and last terms in (C3) vanish at the periphery. Retaining only terms linear in δf and making use of (C1), one obtains

$$E = \frac{1}{\pi}\int_0^L dy\Big(-iv\phi^*\phi' + \frac{1}{2} - \frac{1}{3}f^3 + ip\phi^*\phi' f\Big). \tag{C4}$$

The soliton energy E_{sm} is E minus the contribution from the constant background: $E_{\text{sm}} = E - \int dy(-1^3/3 + 1/2)$. Finally, the renormalized soliton (slow mode) energy is

$$E_{\text{sm}} = \frac{1}{\pi}\int_0^L dy\Big[\frac{1}{3}(1 - |\phi|^6) - iv\phi^*\phi' + ip\phi^*\phi'|\phi|^2\Big], \tag{C5}$$

which was computed to give the value (51).

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