Information trapping in quantum communications

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Many quantum systems display Markovian and non-Markovian behaviors with the information flow and backflow between the system and the surrounding environment. In this paper, we introduce a definition of the dynamics of open quantum systems called information trapping, which is a special case of the information flow. We show that under specific conditions, the information flow can exhibit behavior beyond Markovian or non-Markovian system dynamics. The physical reason behind this phenomenon may arise from the entanglement between the system and the environment, such that the rates of entangling and disentangling can equalize over time, influenced by the quantum memory of the system and environmental decoherence effects. This proposal is investigated by considering some witnesses of the behavior of the system dynamics such as fidelity, trace distance, the Holevo quantity, and Hilbert-Schmidt speed in the quantum teleportation and dense coding protocols based on the open quantum system consisting of an XXZ chain Heisenberg affected by intrinsic decoherence. The main achievement of this work is focused on facilitating access to faithful quantum communication.

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I. INTRODUCTION

The interaction of quantum systems with their environment causes the exchange of information, which can lead to dissipating energy and the loss of quantum coherence [1–6]. Nonetheless, the process is not required to be monotonic as the quantum system may provisionally regain some of the lost energy or information because of memory effects throughout the evolution [7–20]. This well-known non-Markovianity dynamical behavior may appear in various quantum information roles, including teleportation related to mixed states [21], enhancement of the capacity for quantum channels [22], optimal entangling protocols [23–25], and work extraction from an Otto cycle [26].

In fact, the dynamics of open quantum systems related to the interaction of the system with its environment is split into two classes: Markovian and non-Markovian dynamics. The continuous flow of information from the system to its environment is known as Markovian dynamics. However, information backflow to the system from the environment during certain time intervals due to quantum memory effects refers to non-Markovian dynamics [8,27,28].

The evaluation of the non-Markovianity of dynamics in quantum systems has been a topic of extensive research [8,9,29,30]. One approach is to identify temporary increases in the entanglement shared between the open quantum system and an isolated ancilla by measuring the deviation from the complete-positivity divisibility of the dynamical map that represents the system's evolution, which was proposed by Rivas *et al.* [31]. The next method depends on determining the distinguishability of two optimal initial states evolving via the same quantum channel or resource, and probing any nonmonotonicity (backflows of information) is characterized by the trace distance (TD) and was first suggested by Breuer *et al.* [27,32]. Other non-Markovianity witnesses have been suggested according to various dynamical figures of merit, including quantum mutual information [33], the flow of Fisher information [34,35], local quantum uncertainty [36], negative time-dependent decoherence rates in the standard shape of the master equation [37], channel capacities [38], coherence [39,40], quantum interferometric power [41–43], the fidelity of the quantum states [44,45], Choi states [46], changes in the volume of the set of accessible states in the evolved system [47], correlation measures [48], spectral analysis [49], quantum evolution speedup [50–52], entropy production rates [53], and the Hilbert-Schmidt speed (HSS) [54]. This array of witnesses and approaches emphasizes the diverse nature of non-Markovian behavior, making it impossible to attribute to a single system-environment interaction feature, hindering its characterization with a single tool for this phenomenon.

The dynamics of open quantum systems is crucial when communication protocols are influenced by system evolution. One of the most popular protocols in quantum communication is quantum teleportation, first proposed by Bennett *et al.* [55], which is a technique for transmitting quantum information from a sender (Alice) in one place to a receiver (Bob) located some distance away, utilizing a classical or nonclassical channel [56–61]. Another favorite protocol in quantum communication, which shares entanglement between Alice and Bob, is quantum dense coding [59,62–64], initially suggested by Bennett and Wiesner [62]. In this method, two bits of classical information are transmitted through a single encoded qubit sharing the initial maximum entanglement of the channel (or resource) state related to the Bell state. Having an initial entangled state shared between the sender(s) and receiver(s) is crucial for enhancing the efficiency of quantum communication protocols.

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In this article, we address the information-flow problem between the system and environment in two protocols of quantum communications such as quantum dense coding and teleportation based on an open quantum system including an XXZ Heisenberg model of spin-$1/2$ particles arranged on a ring under intrinsic decoherence. In this investigation, we examine the witnesses of non-Markovianity based on Holevo capacity, TD, the fidelity of quantum teleportation, and phase estimation via HSS in the output of quantum teleportation to detect memory effects corresponding to system-environment information backflows. We show that under specific conditions, the information flow can exhibit behavior beyond Markovian or non-Markovian system dynamics, which is crucial in quantum communication. The most important reason for investigating this idea is to facilitate faithful experimental quantum teleportation [65,66] that helps to improve securely transmitting quantum information.

This article is organized as follows: In Sec. II, the preliminaries of quantum communication protocols and the formulations governing them are expressed. Moreover, in Sec. III, we describe the theoretical model that can be used as a resource (or channel) for quantum teleportation and quantum dense coding protocols. Last, in Sec. IV we summarize the main findings and provide a discussion.

II. PRELIMINARIES

A. Quantum communication protocols

1. Quantum teleportation

In the normal protocol [67], remote transmission is accomplished using a two-qubit mixed state ρ_{ch} serving as a channel (resource) and is prepared by a generalized depolarized quantum channel $\Lambda(\rho_{\text{ch}})$ concerning a single-qubit input state ρ_{in}. Alice plans to send the encoded qubit state to Bob using this technique. We can consider the unknown input (initial) state of teleportation for any arbitrary pure single-qubit state:

$$|\psi_{\text{in}}\rangle = \cos\left(\frac{\theta}{2}\right)|0\rangle + e^{i\phi}\sin\left(\frac{\theta}{2}\right)|1\rangle, \tag{1}$$

where θ and ϕ denote the amplitude and phase of the initial state of the single-qubit teleportation, respectively. The output state of the teleportation when transmitting an arbitrary single-qubit state (input state $\rho_{\text{in}} = |\psi_{\text{in}}\rangle\langle\psi_{\text{in}}|$) can be given by [67]

$$\rho_{\text{out}} = \Lambda(\rho_{\text{ch}})\rho_{\text{in}} = \sum_{i=0}^{3}\text{Tr}[\mathcal{B}_i\rho_{\text{ch}}]\sigma_i\rho_{\text{in}}\sigma_i, \tag{2}$$

where $\Lambda(\rho_{\text{ch}})$ is a generalized depolarized channel and $\mathcal{B}_i$ represents the Bell state corresponding to the Pauli matrix σ_i that is defined as follows:

$$\mathcal{B}_i = \left(\sigma_0 \otimes \sigma_i\right)\mathcal{B}_0\left(\sigma_0 \otimes \sigma_i\right), \quad i = 1, 2, 3, \tag{3}$$

where $\sigma_0 = \mathbb{I}$, $\sigma_1 = \sigma_x$, $\sigma_2 = \sigma_y$, $\sigma_3 = \sigma_z$, and $\mathbb{I}$ denotes the identity matrix. In addition, for any two arbitrary qubits where each is prepared in the standard basis $\{|0\rangle, |1\rangle\}$, we have $\mathcal{B}_0 = \frac{1}{2}(|00\rangle + |11\rangle)(\langle 00| + \langle 11|)$.

The fidelity criterion can be used to evaluate the similarity between the input state and the teleported state. Hence, the quality of the teleported state is characterized by the fidelity $f(\rho_{\text{in}}(t), \rho_{\text{out}}(t))$, which is defined as [68,69]:

$$f(\rho_{\text{in}}(t), \rho_{\text{out}}(t)) = \left(\text{Tr}\left[\sqrt{\sqrt{\rho_{\text{in}}(t)}\rho_{\text{out}}(t)\sqrt{\rho_{\text{in}}(t)}}\right]\right)^2. \tag{4}$$

where the limit for fidelity becomes $0 \leqslant f(\rho_{\text{in}}(t), \rho_{\text{out}}(t)) \leqslant 1$. For $f = 1$, the optimal fidelity required for achieving optimal teleportation can be attained.

2. Quantum dense coding

In quantum dense coding, there is no need to physically transmit the quantum state. The quantum state is transferred to Bob after Alice's local unitary transformation is completed. The main goal of this process is to determine the quantum advantage of the initial resource state ρ shared between Alice and Bob for transmitting classical information. The quantum dense coding protocol can be analytically formulated by the Holevo quantity [70], which can be referred to as the dense coding capacity, as follows [71,72]:

$$\chi(\rho(t)) := S(\bar{\rho}(t)) - S(\rho(t)), \tag{5}$$

where

$$\bar{\rho}(t) = \frac{1}{4}\sum_{i=0}^{3}(\sigma_i \otimes \mathbb{I})\,\rho(t)\,(\mathbb{I} \otimes \sigma_i) \tag{6}$$

is the ensemble average state and the significant quantity in Eq. (5)

$$S(\rho(t)) = -\text{Tr}[\rho(t)\log_2\rho(t)] = -\sum_i \lambda_i \log_2(\lambda_i), \tag{7}$$

where $S(\rho(t))$ represents the von Neumann entropy of the density matrix $\rho(t)$ and λ_i denotes the eigenvalues of the density matrix. Achieving the quantum advantage in the valid quantum dense coding required $\chi(\rho(t)) > 1$, and for the optimal quantum dense coding, it needed $\chi(\rho(t))_{max} = 2$ [73,74]. The maximum classical information that can be sent to Bob for a given initial state is known as the dense coding capacity [75].

B. Non-Markovianity witnesses

1. Trace distance

The distinguishability between two evolved quantum states ρ_{in} and ρ_{out} can be detected by the TD [7,8,27], first proposed by Breuer *et al.* [27,32]. The TD is one of the most well-known metrics for identifying the non-Markovianity of the system dynamics:

$$D(\rho_{\text{in}}, \rho_{\text{out}}) = \tfrac{1}{2}\text{Tr}|\rho_{\text{in}} - \rho_{\text{out}}|, \tag{8}$$

where the modulus of the operator is determined by $|A| = \sqrt{A^\dagger A}$. The bounds of the TD are $0 \leqslant D(\rho_{\text{in}}, \rho_{\text{out}}) \leqslant 1$, where $D(\rho_{\text{in}}, \rho_{\text{out}}) = 0$ if and only if $\rho_{\text{in}} = \rho_{\text{out}}$ and $D(\rho_{\text{in}}, \rho_{\text{out}}) = 1$ if and only if ρ_{in} and ρ_{out} are orthogonal. One key property of the TD is that it provides a clear physical explanation of the distinguishability between two quantum states.

In quantum communication theory, the accuracy of the signal transmission is measured through the statistical distance in the Hilbert space between transmitted and received states

[76]. The statistical distances play a crucial role in characterizing signal degradation, channel noise, and the information obtained from measurements [77]. Key distance measures in quantum information encompass fidelity (Bures metrics) and the TD. The trace distance is related to fidelity through an inequality applicable to two mixed states [78,79],

$$1 - f(\rho_{\rm in}, \rho_{\rm out}) \leqslant D(\rho_{\rm in}, \rho_{\rm out}) \leqslant \sqrt{1 - f(\rho_{\rm in}, \rho_{\rm out})^2}. \tag{9}$$

When both states are pure, the upper bound of the trace distance is saturated. It is important to note that we can utilize two dependent input and output states of teleportation as quantum states to detect non-Markovian dynamics through fidelity and trace distance, as suggested in Ref. [45].

2. Hilbert-Schmidt speed

The HSS is acknowledged as a powerful tool for enhancing quantum parameter estimation in quantum information theory. As mentioned in [54,80], the HSS can be used as a non-Markovianity witness that detects memory effects well.

To describe the HSS, we can assume the distance measure $d(p,q)$, given by $[d(p,q)]^2 = \frac{1}{2}\sum_x |p_x - q_x|^2$ [81], in which $p = \{p_x\}_x$ and $q = \{q_x\}_x$, with respect to the unknown parameter ϑ, denote the probability distributions, leads to the determination of the classical statistical speed (CSS) $s[p(\vartheta_0)] = \frac{d}{d\vartheta} d(p(\vartheta_0+\vartheta), p(\vartheta_0))$. We consider a given pair of quantum states ρ and σ to expand to the quantum case and define $p_x = \mathrm{Tr}[\Pi_x \rho]$ and $q_x = \mathrm{Tr}[\Pi_x \sigma]$, which are the measurement probabilities related to the positive operator-valued measure (POVM) $\{\Pi_x \geqslant 0\}$ fulfilling $\sum_x \Pi_x = \mathbb{I}$.

Maximizing the classical statistical distance $d(p,q)$ over all possible measurements of POVMs [82], we determine the corresponding quantum statistical distance known as the Hilbert-Schmidt distance [83] denoted by $D_{\rm HS}(\rho,\sigma) \equiv \max_{\{\Pi_x\}} d(p,q) = \sqrt{\frac{1}{2}\mathrm{Tr}[(\rho-\sigma)^2]}$. Therefore, the related quantum statistical speed (QSS) is defined by maximizing the CSS over all possible measurements of POVMs [81,84]. Considering the quantum state $\rho(\vartheta)$, the HSS is given by [54,80,81]

$$\mathrm{HSS}(\vartheta) = \max_{\{\Pi_x\}} s[p(\vartheta)] = \sqrt{\frac{1}{2}\mathrm{Tr}\left[\frac{d\rho(\vartheta)}{d\vartheta}\right]^2}, \tag{10}$$

which does not require diagonalizing the derivative of $d\rho(\vartheta)/d\vartheta$. It should be noted that the HSS is a type of QSS corresponding to the Hilbert-Schmidt distance [81].

3. Holevo quantity

As is well known, non-Markovian effects can cause faster quantum evolution from an initial state to a subsequent state [50,85–89]. It is thus obvious that the criterion of the Holevo quantity can play the role of a suitable determiner of memory effects occurring during the system dynamics. Here, we focus on exploiting the Holevo capacity [70] as a kind of merit for the non-Markovian feature of quantum evolutions, with resultful practical advantages in the analysis.

As proposed in Ref. [38], concerning the concept that the nonmonotonic speed (positive acceleration) of the quantum dynamics is a signature of memory effects in the system dynamics, one can introduce the non-Markovianity witness based on the Holevo quantity as

$$\mathcal{F}(t) := \frac{d\chi(\rho(t))}{dt} > 0, \tag{11}$$

where $\rho(t)$ represents the evolved state of the system and $\chi(\rho(t))$ is defined in Eq. (5). If the channel of quantum dense coding interacts with an environment, $\chi(\rho(t))$ decreases monotonously, and in this situation the dynamics is called Markovian. So we have $\mathcal{F}(t) < 0$ for some times. On the other hand, every positive value of $\mathcal{F}(t)$ is a witness of non-Markovianity.

Corresponding to this witness, in analogy with what has been accomplished for other measures [27,32,45,54,90], a determiner of the degree of non-Markovianity can be expressed by

$$\mathcal{N} := \max \int_{\mathcal{F}(t)>0} \mathcal{F}(t)dt, \tag{12}$$

where the maximization is taken over all the possible parametrizations of the initial state.

It is important to note that we aim to investigate only non-Markovian effects using the witness based on $\chi(\rho(t))$. In this context, the actual value is not significant, and no optimization of the initial state parameters is performed.

III. THE PHYSICAL MODEL

Consider a paradigmatic open quantum system including an XXZ Heisenberg model of N spin-1/2 particles arranged on a ring. The Hamiltonian of this system can be defined as follows [91,92]:

$$\hat{H}_{(XXZ)} = \epsilon \sum_{i=1}^{N} \hat{\sigma}_i^z - \sum_{i=1}^{N} \left[J\left(\hat{\sigma}_i^x \otimes \hat{\sigma}_{i\oplus1}^x + \hat{\sigma}_i^y \otimes \hat{\sigma}_{i\oplus1}^y\right) + J_z\left(\hat{\sigma}_i^z \otimes \hat{\sigma}_{i\oplus1}^z\right)\right], \tag{13}$$

where the positive constants ϵ, J, and J_z denote the local energy contribution and the coupling terms of the model and $\oplus$ represents the sum particles N (such that $i = 1, 2$). Considering two particles, the Hamiltonian in Eq. (13) reduces to

$$\hat{H}_{(XXZ)} = \begin{pmatrix} -2J_z + 2\epsilon & 0 & 0 & 0 \\ 0 & 2(J_z+\epsilon) & -4J & 0 \\ 0 & -4J & 2(J_z-\epsilon) & 0 \\ 0 & 0 & 0 & -2(J_z+\epsilon) \end{pmatrix}. \tag{14}$$

The eigenvalues and the associated eigenvectors of the aforementioned Hamiltonian $\hat{H}_{(XXZ)}$ are

$$E_{1,2} = \mp 2(\pm J_z + \epsilon), \quad E_{3,4} = 2(J_z \mp \sqrt{4J^2+\epsilon^2}), \tag{15}$$

and

$$|\phi_1\rangle = |11\rangle \quad |\phi_2\rangle = |00\rangle, \quad |\phi_{3,4}\rangle = |10\rangle + \frac{-\epsilon \pm \sqrt{4J^2+\epsilon^2}}{2J}|01\rangle \tag{16}$$

where $|0\rangle = \binom{1}{0}$ and $|1\rangle = \binom{0}{1}$. In addition, $|ij\rangle$ denotes $|i\rangle \otimes |j\rangle$, where $i, j = 0, 1$.

Based on the consideration of the effect of intrinsic decoherence, the time evolution of the density matrix of the system is obtained by the Milburn equation [93] as

$$\rho(t) = \sum_{m,n} \exp\left[-\frac{\gamma t}{2}(E_m - E_n)^2 - i(E_m - E_n)t\right]|\phi_m\rangle\langle\phi_m|\rho(0)|\phi_n\rangle\langle\phi_n|, \tag{17}$$

where $E_{m,n}$ and $|\phi_{n,m}\rangle$ are, respectively, the eigenvalues and the eigenstates of the Hamiltonian of the system (7) and γ is the intrinsic decoherence rate. Furthermore, $\rho(0)$ represents the initial density matrix. Then, we consider that the system is initially prepared in the mixed state as follows:

$$\rho(0) = r|\psi_p\rangle\langle\psi_p| + \frac{1-r}{4}\mathbb{I}_4, \tag{18}$$

where $|\psi_p\rangle = \sqrt{p}|01\rangle + \sqrt{1-p}|10\rangle$ is a pure state, with p being the degree of entanglement, and $0 < r \leqslant 1$ is the level of the purity in the initial state.

In the standard basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$, one can obtain $\rho(t)$ with respect to Eqs. (15)–(18) as

$$\rho(t) = \begin{pmatrix} \rho_{11} & 0 & 0 & 0 \\ 0 & \rho_{22} & \rho_{23} & 0 \\ 0 & \rho_{32} & \rho_{33} & 0 \\ 0 & 0 & 0 & \rho_{44} \end{pmatrix}, \tag{19}$$

where the elements of the aforementioned evolved density matrix are given by

$$\begin{aligned}
\rho_{11} = \rho_{44} = -\frac{J^4(r-1)}{\omega}, \quad \rho_{22} &= \frac{1}{\omega}\{\beta\epsilon^4 + J^2\epsilon^2[(12p-1)r+5] + 4J^4 e^{-4t(2\gamma\epsilon^2+8\gamma J^2+i\Omega)}[(r+1)e^{4t(2\gamma\epsilon^2+8\gamma J^2+i\Omega)} \\
&\quad + (2p-1)re^{8it\Omega} + (2p-1)r] + 4\Gamma J^3 r\epsilon(e^{-4t(2\gamma\epsilon^2+8\gamma J^2+i\Omega)} + e^{4t\Omega(i-2\gamma\Omega)} - 6) - 8\Gamma J r\epsilon^3\}, \\
\rho_{33} &= \frac{1}{\omega}(J^2\{\beta\epsilon^2 + 4J^2(r+1) - 4Jre^{-4t\Omega(i+2\gamma\Omega)}[\Gamma\epsilon + J(2p-1)] - 8\Gamma J r\epsilon - 4\Theta e^{4t\Omega(i-2\gamma\Omega)}\}), \\
\rho_{23} &= \frac{1}{4\omega}J(8\Theta(\Omega+\epsilon)e^{-4t(2\gamma\epsilon^2+8\gamma J^2+i\Omega)} + M_- - M_+ + 8\Theta(\epsilon-\Omega)e^{4(i-2\gamma)t\Omega}), \\
\rho_{32} &= \frac{1}{4\omega}J[8\Theta(\epsilon-\Omega)e^{-4t(2\gamma\epsilon^2+8\gamma J^2+i\Omega)} + M_- - M_+ + 8\Theta(\Omega+\epsilon)e^{4t\Omega(i-2\gamma\Omega)}],
\end{aligned} \tag{20}$$

where

$$\begin{aligned}
&\Gamma = \sqrt{-[(p-1)p]}, \quad \Omega = \sqrt{4J^2+\epsilon^2}, \quad \beta = (4p-1)r+1, \quad M_- = (\Omega-\epsilon)[-\beta\epsilon(\Omega-\epsilon)+4J^2(r+1)+8\Gamma Jr(\Omega-\epsilon)], \\
&M_+ = (\Omega+\epsilon)[\beta\epsilon(\Omega+\epsilon) + 4J^2(r+1) - 8\Gamma Jr(\Omega+\epsilon)], \quad \omega = \beta\epsilon^4 + 2J^4(3r+5) - 32\Gamma J^3 r\epsilon \\
&\quad + 2J^2\epsilon^2[(8p-1)r+3] - 8\Gamma J r\epsilon^3, \\
&\Theta = Jr[\Gamma\epsilon + J(2p-1)].
\end{aligned} \tag{21}$$

The important point is that, in this paper, $\hbar = 1$ is considered and all parameters are nondimensionalized to plot the figures, as discussed in Ref. [94]. Note that the parameter values in this paper align with those in practical works [95–97] and theoretical papers [60,98,99].

IV. DISCUSSION AND RESULTS

Substituting Eqs. (1), (3), and (19) in Eq. (2), one can calculate the output state of single-qubit quantum teleportation for Eq. (19) as a resource of quantum teleportation as

$$\rho_{\text{out}}(t) = \begin{pmatrix} \rho_{\text{out},11} & \rho_{\text{out},12} \\ \rho_{\text{out},21} & \rho_{\text{out},22} \end{pmatrix}, \tag{22}$$

where elements of the aforementioned matrix can be calculated as

$$\begin{aligned}
\rho_{\text{out},11} &= \frac{1}{\omega}\left\{\Omega^2\left[\sin^2\left(\frac{\theta}{2}\right)\right][\beta\epsilon^2 + 2J^2(r+1) - 8\Gamma J r\epsilon] - 2J^4(r-1)\left[\cos^2\left(\frac{\theta}{2}\right)\right]\right\},\\
\rho_{\text{out},12} &= \frac{1}{\omega}[J\sin(\theta)e^{(-64\gamma J^2 t - 16\gamma t\epsilon^2 - 4it\Omega + i\phi)}(Ke^{8t(\gamma\epsilon^2+4\gamma J^2+i\Omega)} + Qe^{4t(4\gamma\epsilon^2+16\gamma J^2+i\Omega)} + Ke^{8\gamma \mathrm{t}\Omega^2})],\\
\rho_{\text{out},21} &= \frac{1}{\omega}[J\sin(\theta)e^{\{-64\gamma J^2 t - i[\phi+4t(\Omega-4i\gamma\epsilon^2)]\}}(Ke^{8t(\gamma\epsilon^2+4\gamma J^2+i\Omega)} + Qe^{4t(4\gamma\epsilon^2+16\gamma J^2+i\Omega)} + Ke^{8\gamma t\Omega^2})],\\
\rho_{\text{out},22} &= \frac{1}{\omega}\left\{\Omega^2\left[\cos^2\left(\frac{\theta}{2}\right)\right][\beta\epsilon^2 + 2J^2(r+1) - 8\Gamma J r\epsilon] - 2J^4(r-1)\left[\sin^2\left(\frac{\theta}{2}\right)\right]\right\},
\end{aligned} \tag{23}$$

where

$$\begin{aligned}
Q &= -\beta\epsilon^3 + 16\Gamma J^3 r - 4J^2(2pr\epsilon + \epsilon) + 8\Gamma J r\epsilon^2,\\
K &= 2\epsilon\Theta,
\end{aligned} \tag{24}$$

where $\beta, \omega, \Omega, \Gamma$, and Θ are given by Eq. (21).

Here, we explore the nature of the system dynamics in processes of quantum dense coding and quantum teleportation based on resource equation (14). This proposal is investigated by considering some witnesses of behavior of the system dynamics based on the fidelity f, TD, Holevo quantity χ, and Hilbert-Schmidt speed with respect to initial phase HSS_ϕ.

The straightforward expressions for fidelity and HSS are given in Appendix. However, due to the cumbersome form of the straightforward expressions of the Holevo quantity and TD, we omit reporting them.

In Figs. 1(a)–1(c), the temporal variations of the fidelity f, TD, Holevo quantity χ, and Hilbert-Schmidt speed with respect to initial phase HSS_ϕ for different values of the system parameters are displayed. The fidelity and trace distance between the input [Eq. (1) such that $\rho_{\text{in}} = |\psi_{\text{in}}\rangle\langle\psi_{\text{in}}|$] and output [Eq. (22)] states in single-qubit quantum teleportation using Eq. (4) are computed. The Holevo quantity utilizing Eqs. (5) and (19), which is channel capacity in quantum dense coding, is calculated. Moreover, the phase estimation is applied via the Hilbert-Schmidt speed [Eq. (10)] corresponding to the output state of the single-qubit quantum teleportation [Eq. (22)]. These quantities enable us to thoroughly explore the dynamic nature of the system in quantum communication. For example, fidelity and trace distance allow us to check the dynamics of the system in quantum teleportation between the input and output states. In addition, the Holevo quantity allows us to understand the dynamic nature of the system without sending the quantum state and only using the channel in quantum dense coding. In addition, by using the HSS, we can estimate the initial phase of quantum teleportation in the output state so that we can get the dynamic nature of the system. The first result that can be obtained from Figs. 1(a)–1(c) is that the qualitative behaviors of f, the TD, χ, and HSS_ϕ are exactly the same because the minimum and maximum points of oscillations coincide. This result indicates that in the time intervals when the quality of the quantum teleportation is good, i.e., the fidelity value is equal to unity, we have good channel capacity to send qubits in quantum dense coding, and the HSS value is maximum, which means that we have a good phase estimate, which leads to optimal information extraction of the initial phase in the present quantum teleportation protocol. The next result that can be inferred from Fig. 1 is that when the intrinsic decoherence rate is negligible, i.e., $\gamma = 0.001$ in Fig. 1(a), both the fidelity value and the Holevo capacity value are at their maximum, which indicates that we have a good quality of quantum teleportation and quantum dense coding channel capacity, respectively. Moreover, with increasing intrinsic decoherence rates in Figs. 1(b) and 1(c), the oscillations in the qualitative behaviors of f, the TD, χ, and HSS_ϕ decrease, and over time, they show a uniform behavior without oscillations [Fig. 1(c)]. A very significant question that arises here is why the qualitative behaviors of f, the TD, χ, and HSS_ϕ become uniform over time in Fig. 1(c) and the oscillations are suppressed over time. The more important question is, What behavior does the system dynamics show?

To answer these key questions, we must return to the discussion of information flow between the environment and the system. Consider the flow of information $\mathcal{F}(t)$ as defined by the Holevo quantity $\frac{d\chi}{dt}$ proposed in Ref. [38]. When $\mathcal{F}(t) > 0$, this indicates the presence of non-Markovian effects. Reference [45] suggested that fidelity can act as a witness for these non-Markovian effects in quantum teleportation between input and output states. Additionally, in

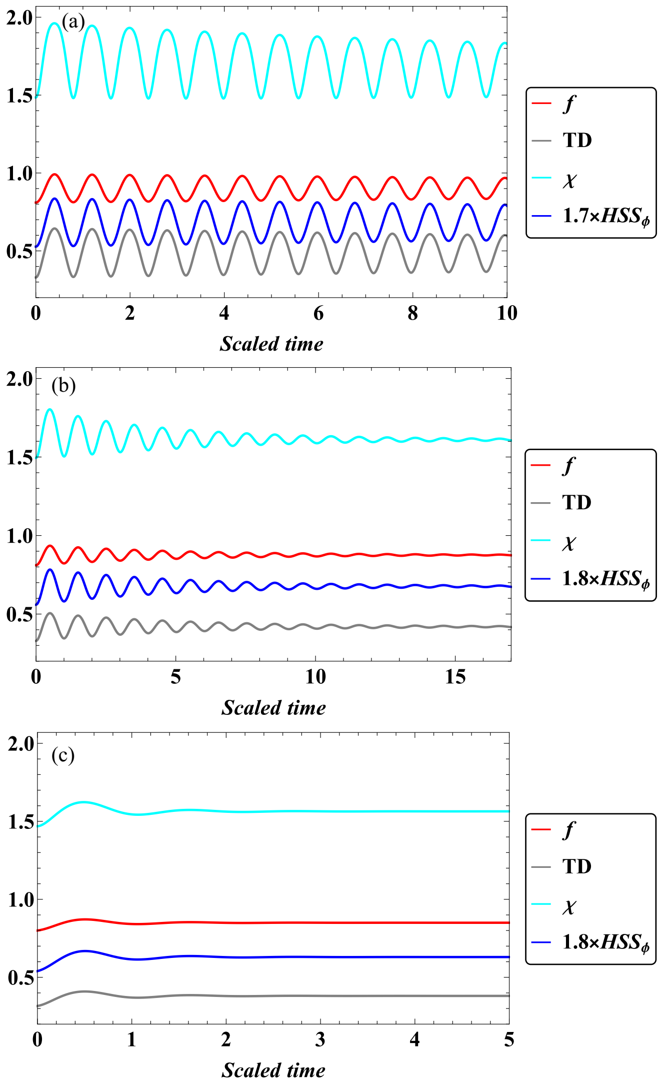

FIG. 1. Comparison between the qualitative behaviors of fidelity f, trace distance (TD), Holevo capacity χ, and Hilbert-Schmidt speed with respect to HSS_ϕ when $r = p = \epsilon = 1$ and $\theta = \phi = \pi/2$ and (a) $J = 0.85$, $\gamma = 0.001$, (b) $J = 0.6$, $J_z = 1.8$, $\gamma = 0.01$, and (c) $J = 0.5$, $\gamma = 0.1$.

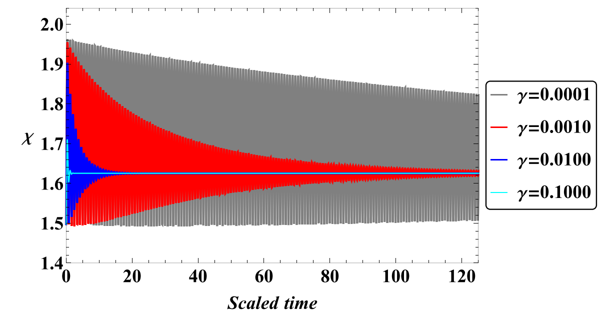

FIG. 2. The dynamics of Holevo capacity χ in the quantum dense coding protocol with increasing of the intrinsic decoherence rate γ when $r = p = \epsilon = 1$ and $J = 0.8$.

Ref. [100], HSS quantity was utilized as a non-Markovian witness in the context of quantum teleportation output. Given the qualitative similarities in the behaviors of fidelity, the Holevo quantity, TD, and HSS, which exhibit coinciding maximum and minimum points, we can also employ other metrics such as the fidelity, TD, and HSS as witnesses for monitoring non-Markovian effects in this scenario. Suppose the flow of information $\mathcal{F}(t)$ is based on fidelity $\frac{df}{dt}$ [45], trace distance $\frac{d\text{TD}}{dt}$ [27], and Hilbert-Schmidt speed $\frac{d\text{HSS}_\phi}{dt}$ [54] as witnesses for probing the non-Markovian effects. Recall that, when the information flow with respect to the quantities is positive, $\mathcal{F}(t) > 0$, we face non-Markovian behavior, while, when the information flow is negative, $\mathcal{F}(t) < 0$, we have Markovian behavior. Hence, in Figs. 1(a)–1(c), we see that the oscillations occur due to this fact. For some time intervals, the behavior of the system dynamics is Markovian, while in some other time intervals it is non-Markovian. In fact, when the value of the mentioned witnesses is negative, the Markovian dynamics can be detected such that the information flows from the system to the environment. However, when the value of the mentioned witnesses is positive, the non-Markovian dynamics can be found such that the information back flows from the environment to the system. However, in Fig. 1(c), for $t > 2$ we see that the behavior has no oscillations and it is uniform. To clarify this concept, see Fig. 2.

In Fig. 2, the time evolution of the Holevo capacity χ in quantum dense coding for various intrinsic decoherence rates γ is depicted. It is obvious that with increasing decoherence rate γ, oscillations in qualitative behavior are suppressed and become uniform. It seems that in this situation, with, for example, $\gamma = 0.1$ or 0.01, information trapping occurs in the quantum communication protocol. For a deeper understanding of this important physical concept, the qualitative behavior of the flow of information corresponding to the Holevo capacity witness $\mathcal{F}(t)$ is illustrated in Figs. 3(a) and 3(b) using the same conditions as in Fig. 2. In Fig. 3, which we plotted for different intervals, we see that the oscillations are around zero. Figure 3(b) focuses on oscillations in shorter intervals, and we clearly see that the average value of the oscillations for more time (around 123.5 to 124) is zero. Furthermore, the oscillations of the information flow are trapped between the upper and lower bounds of the oscillations indicated by the red dashed lines. As illustrated, the oscillations are very small but never become zero, although the average value of the oscillations becomes zero. Hence, the term "information trapping" is correct because the information flow is trapped between Markovian and non-Markovian behaviors. This point plays a key role in quantum communications.

So the responses to the two questions above have been expressed. But another key question arises: What is the physical reason for information trapping in reality? We have always faced two types of behavior, Markovian and non-Markovian. But here, we provide a case that could result in a faithful protocol. In response to the question above, we express that the system interacts with the environment over time. At initial times the quality of the quantum communication protocol is good. But over time the quality can be influenced by decoherence effects and becomes weak. However, when the rates of the flow and backflow of the information between the system and the environment are very small, then information trapping can occur. This happens due to the presence of the effects

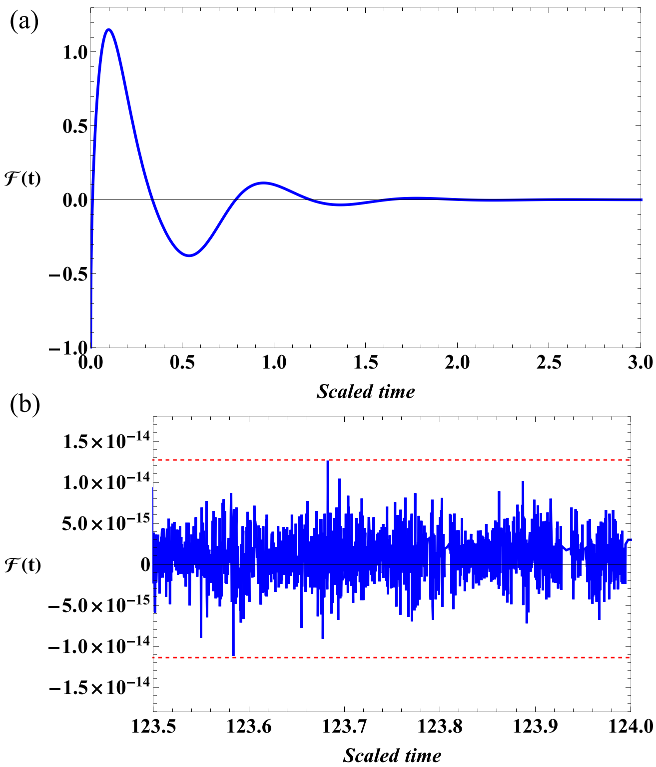

FIG. 3. The temporal variations of information flow based on Holevo capacity $\mathcal{F}(t)$ when $r = p = \epsilon = 1, J = 0.8$, and $\gamma = 0.1$ for different intervals of scaled time: (a) 0–3 and (b) 123.5–124.

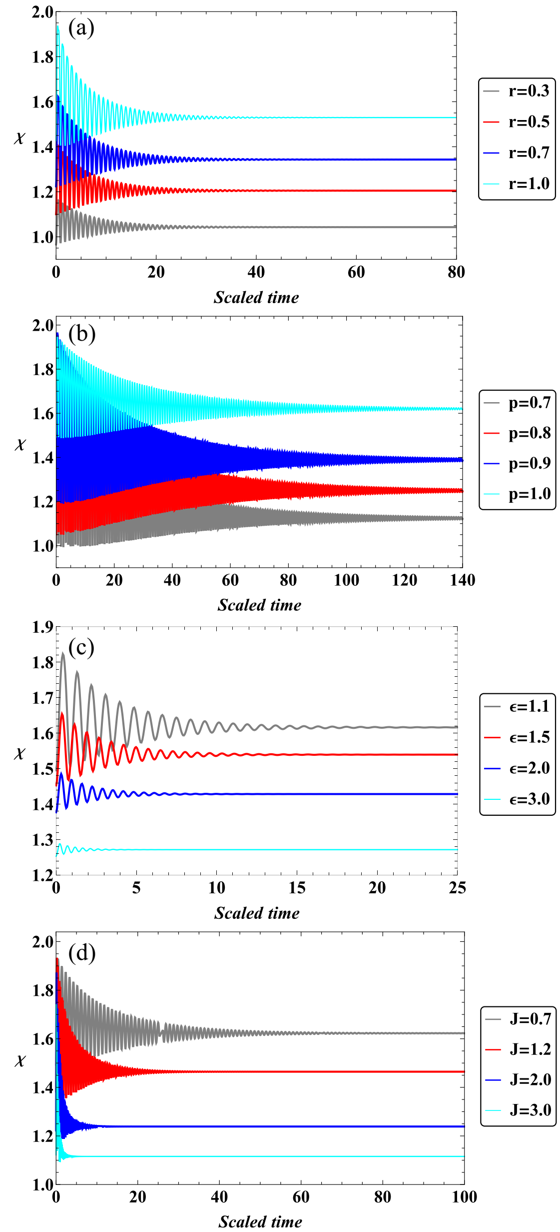

FIG. 4. Dynamics of the Holevo capacity with an increasing (a) r for $p = 1, \epsilon = 0.7, J = 0.8$, and $\gamma = 0.005$, (b) p for $r = \epsilon = 1, J = 0.8$, and $\gamma = 0.001$, (c) local energy contribution ϵ for $r = p = 1, J = 0.7$, and $\gamma = 0.01$, and (d) coupling term J for $r = p = 1, \epsilon = 0.9$, and $\gamma = 0.003$ when $\theta = \phi = \pi/2$.

of decoherence and quantum memory, where a decrease in one causes an increase in the other and creates Markovian or non-Markovian effects; however, when the amount of both effects is very small, information trapping arises.

In Fig. 4, the dynamics of the Holevo quantity χ for increasing r, p, local energy contribution ϵ, and coupling term J are illustrated. Figure 4 shows that the information trapping versus changing the parameters r, p, ϵ, and J is robust. However, we can see that by changing these parameters, it is possible for information trapping to occur at earlier times or at the minimum or the maximum value of the Holevo quantity.

Interpretation of information trapping

Now, we aim to offer an explanation of information trapping using the concept of entanglement, which can aid deeper comprehension of this phenomenon. Von Neumann's entropy is a widely used measure of entanglement between the parts of a bipartite system [101] when the total system (total system = system + environment) is in a pure state [102]. Bennett *et al.* [103] demonstrated that the von Neumann entropy of the reduced density matrix for each part of a bipartite system in its pure state can be utilized as a determination of quantum entanglement and correlations. The same discussion applies to the scenario in which a system interacts with its environment. The total state can be assumed to be a pure state according to the principle of purification [68], with entropy acting as a measure of entanglement that can be computed for the reduced density matrix ρ. This means that, if we suppose the state of the total state, the entanglement between the system and the environment can be determined by the von Neumann entropy as $S(\rho) = -\mathrm{Tr}[\rho \log_2(\rho)] = -\sum_i \lambda_i \log_2(\lambda_i)$, where λ_i is the eigenvalues of the density matrix.

In order to clarify the interpretation of information trapping, we compare the qualitative behaviors of entropy (which represents the entanglement between the system and environment) with respect to the output state ρ_{out} and HSS_ϕ (which is a powerful witness of dynamics effects) in Fig. 5. In the preceding figures (Figs. 1–5), we have demonstrated that increasing values of our witnesses, such as HSS, indicate non-Markovian effects in the system's dynamics, whereas decreasing values indicate Markovian effects. In Fig. 5, we observe

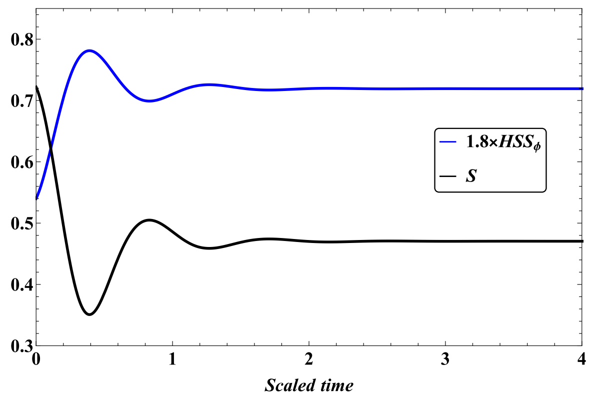

FIG. 5. Comparison between the Hilbert-Schmidt speed with respect to HSS_ϕ and entropy S when $r = p = 1$, $\epsilon = 0.8$, $\theta = \phi = \pi/2$, $J = 0.8$, and $\gamma = 0.1$.

that wherever the entanglement is maximum (minimum), the value of the HSS_ϕ is minimum (maximum). This means that system-environment entangling leads to the flow of information from the system to its environment and Markovian dynamics, while disentangling leads to information backflow to the system from its environment and non-Markovianity [104]. Here, it can also be observed that entanglement is trapped in a stable state. Thus, it can be concluded that the reason for information being trapped in a quantum protocol can be the level of system-environment entanglement. That is, when the amount of information flow between the system and environment and the backflow of information between the environment and system are at their lowest, the system-environment entanglement falls into a trap in a stable state and leads to information trapping. The physical reason for this phenomenon may arise from the entanglement between the system and the environment. Over time, the rates of entangling and disentangling can equalize, influenced by the quantum memory of the system and environmental decoherence effects. This phenomenon is similar to the steady state of entanglement that arises in an open quantum system composed of two identical two-level subsystems in a common stationary environment undergoing Markovian dissipation [105,106] or two trapped ions coupled to the dissipative environment [107], which employs the master-equation method of evolution of dynamics. These results refer to the asymptotic dissipative preparation of an entangled state between the system and the environment.

Briefly, information trapping can be described as follows: In information trapping, there is a boundary between the information flow between the system and environment and the backflow of information between the environment and system, which leads to Markovian and non-Markovian effects. The physical reason for this phenomenon can be the entanglement between the system and the environment, and over time, the rates of entangling and disentangling may become the same, which is due to the existence of the quantum memory of the system and the effects of the environment.

V. CONCLUSION

Quantum communication technologies depend on the faithful transmission of information encoded in quantum states across quantum channels. In this paper, we established a definition of the dynamics of open quantum systems called information trapping, which is a special case of the information flow and backflow between the system and the environment. The idea behind this definition is that information trapping can occur when the rates of information flow and backflow between the system and the surrounding environment are very low. This causes the information flow to become trapped between the upper and lower bands of the maximum and minimum values of the oscillations in non-Markovian and Markovian dynamics, and the average oscillations of the information flow become zero. The physical explanation for this phenomenon could arise from the entanglement between the system and the environment, such that the rates of entangling and disentangling could equalize over time, affected by the quantum memory of the system and environmental decoherence effects.

Introducing some witnesses of the behavior of the system dynamics based on the fidelity, trace distance, Holevo quantity, and Hilbert-Schmidt speed, one can detect the information trapping via powerful tools. The fidelity and TD were used as witnesses of the behavior of the system dynamics in the transmission between two quantum states in single-qubit quantum teleportation. The Holevo quantity was utilized as a witness through the quantum capacity channel in the quantum dense coding protocol. Moreover, the HSS with respect to the initial phase was employed as a witness based on the output of the quantum teleportation.

The model considered is an open quantum system including an XXZ Heisenberg of two spin-$1/2$ particles arranged on a ring affected by intrinsic decoherence. The effect of intrinsic decoherence is applied to the system using the Milburn method.

Our study supplies a useful concept to detect the nature of the system dynamics based on the concept of the flow of information in quantum communication protocols such as quantum teleportation and dense coding. The main reason for exploring this concept is to enable faithful experimental quantum teleportation [65] to improve the secure transmission of quantum information. It thus motivates further analyses of the role of memory effects in open quantum systems, especially for experimental quantum communication protocols and quantum remote sensing.

All data generated or analyzed during this study are included in this paper.

Practical research was conducted by S.M.H. and M.N. Interpretations and comparison of results and writing of the article were done by S.M.H. and M.N. with the help of J.S.-Y. The article was reviewed and edited by J.S.-Y.

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APPENDIX: STRAIGHTFORWARD EXPRESSIONS

By inserting Eqs. (1) and (22) in Eq. (4) and using $\rho_{\rm in} = |\psi_{\rm in}\rangle\langle\psi_{\rm in}|$, one can calculate the straightforward expression for the fidelity of single-qubit quantum teleportation as follows:

$$f = \frac{1}{2\omega}\{\sin^2(\theta)\exp[-2(16\gamma J^2 t + 4\gamma t\epsilon^2 + 2it\Omega + i\phi)][JQ(1+e^{4i\phi})e^{4t(2\gamma\epsilon^2+8\gamma J^2+i\Omega)} + \omega e^{32\gamma J^2 t+8\gamma t\epsilon^2+4it\Omega+2i\phi} + 2\Theta J\epsilon(1+e^{4i\phi})(1+e^{8it\Omega})]\}. \tag{A1}$$

Furthermore, by using Eqs. (10) and (22), one can compute the straightforward expression of quantum estimation with respect to phase ϕ by employing the HSS as

$$\mathrm{HSS}_\phi = \sin(\theta)\left(\frac{1}{\omega^2}\{J^2 e^{-8t(4\gamma\epsilon^2+16\gamma J^2+i\Omega)}[Qe^{4t(4\gamma\epsilon^2+16\gamma J^2+i\Omega)} + 2\Theta\epsilon(e^{8\gamma t\Omega^2} + e^{8t(\gamma\epsilon^2+4\gamma J^2+i\Omega)})]^2\}\right)^{1/2}, \tag{A2}$$

where β, ω, Ω, Γ, Θ, and Q are given by Eqs. (21) and (24).

It should be noted that due to the cumbersome forms of the expressions for the Holevo quantity, TD, and entropy, we refrain from reporting them here.

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