# Phase entanglement negativity for bipartite fermionic systems

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We discuss the behavior of positive linear maps in fermionic systems and then propose the phase partial transpose and the phase entanglement negativity. We show that every fermionic state which mixes local fermion-number parity must have nonvanishing nontrivial phase entanglement negativity, which gives an affirmative answer to a conjecture proposed by Shapourian and Ryu [Phys. Rev. A **99**, 022310 (2019)]. In addition, we prove that the phase entanglement negativity is an entanglement monotone and establish some equalities and inequalities related to the phase entanglement negativity which, particularly, provide some upper bounds and lower bounds of the fermionic entanglement negativity. A more detailed discussion of the $(1+M)$-mode case is also presented, and our results generalize some known findings.

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## I. INTRODUCTION

Quantum entanglement plays a fundamental and important role in quantum information theory [1,2]. Various measures to quantify the degree of entanglement have been proposed [3,4]. Among these, one of the most frequently mentioned quantifications is the entanglement negativity [5,6], which is based on an operation called the partial transpose [7,8]. As a result, the partial transpose and entanglement negativity were introduced into bosonic systems. For bosons, the partial transpose of any bipartite density operator $\rho_{AB} = \sum_{ijkl} \rho_{ijkl}|e_i^A, e_j^B\rangle\langle e_k^A, e_l^B|$ with respect to subsystem $A$ is given by

$$\rho_{AB}^{\mathrm{T_A}} = \sum_{ijkl} \rho_{ijkl}\big|e_k^A, e_j^B\big\rangle\big\langle e_i^A, e_l^B\big|, \tag{1}$$

where $|e_i^A, e_j^B\rangle$ is a chosen product basis. The related entanglement negativity is defined as $\mathcal{N}(\rho_{AB}) = (\|\rho_{AB}^{\mathrm{T_A}}\|_1 - 1)/2$ [3]. It has turned out to be even more useful than in the case of systems of distinguishable particles: Entanglement negativity became an exhaustive bipartite entanglement witness for Gaussian states and particle-number-conserving states of two-mode bosonic systems [9,10]. Subsequently, this result was generalized to any $(1+M)$-mode bosonic Gaussian state [11] and any $(1+M)$-mode bosonic state with a conserved particle number [12]. However, in the attempt to introduce partial transpose and entanglement negativity in fermionic systems, things became exceptionally confusing [13–16]. For example, with the definition of simply swapping the indices of the first subsystem as in Eq. (1), the partial transpose is not Gaussian preserving [13,14], and the entanglement negativity cannot witness any entanglement in the topological phase of the Kitaev Majorana chain [15,16]. Therefore, the authors of [15] introduced a new definition of the partial transpose called the *fermionic partial transpose*. The new definition overcomes the issues mentioned earlier and performs well in many respects [17]. In [18], the authors proved that the entanglement negativity based on the fermionic partial transpose, called *fermionic entanglement negativity*, once again becomes an exhaustive entanglement witness in the $(1+M)$-mode case a parity constraint is considered [19]. As expected, for general bipartite fermionic systems, some entangled states for which the fermionic entanglement negativity is zero exist, and thus, this new negativity fails to capture entanglement in them. However, due to the parity constraint, the authors of [18] believed that the following conjecture is true.

*Conjecture 1.* The states which mix local fermion-number parity (the definition will be presented below) must have a nonvanishing fermionic entanglement negativity [18].

One of the main purposes of the present paper is to give an affirmative answer to the above conjecture, but we do much more. In fact, we introduce the so-called *phase partial transpose* and *phase entanglement negativity* for each real number $\theta$. The fermionic entanglement negativity can be considered a special case of the phase entanglement negativity with $\theta = \pi/2$ (up to a local unitary transformation called partial particle-hole transformation [17]). And the ordinary entanglement negativity corresponds to the case of $\theta = k\pi$, $k \in \mathbb{Z}$. Based on the study of the behavior of positive linear maps in fermionic systems, we find that a universal way to improve the entanglement detection of a given positive linear map between fermion algebras exists. This enables us to show that every state which mixes local fermion-number parity has nonvanishing nontrivial phase entanglement negativity. Furthermore, we establish several equalities and inequalities concerning the phase partial transpose and phase entanglement negativity and prove that the phase entanglement negativity is an

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entanglement monotone. We also find the relation between the ordinary entanglement negativity and the phase entanglement negativity and provide some lower and upper bounds of fermionic entanglement negativity. For the case of a $(1+M)$-mode fermionic system, more precise results are obtained, part of which generalize and strengthen some known results obtained in [18].

This paper is organized as follows. In Sec. II, we mainly give some preliminaries for the second quantization and fermionic systems. In Sec. III, we introduce the phase partial transpose and phase entanglement negativity. Based on the discussion on positive linear maps in fermionic systems, we provide a sufficient condition for a fermionic state to be entangled. Then we apply it to the phase entanglement negativity and answer the conjecture proposed in [18] affirmatively. Section IV is devoted to giving some equalities and inequalities related to the fermionic entanglement negativity and provides lower and upper bounds of the fermionic entanglement negativity, in particular, for the $(1+M)$-mode case. A short discussion and summary are given in Sec. V.

## II. PRELIMINARIES

In this section, we recall some basic notions about and notations for fermionic systems with $N$ modes, where $2 \leqslant N \leqslant \infty$.

### A. Fermionic Fock spaces

The second-quantization description for an $N$-mode fermionic system is associated with fermion creation operators $f_j^\dagger$ and annihilation operators $f_j$ of each mode $j$. These operators act on the fermionic Fock space $\mathcal{H}$ and satisfy the canonical anticommutation relations

$$\{f_j, f_k^\dagger\} = \delta_{jk}, \quad \{f_j, f_k\} = \{f_j^\dagger, f_k^\dagger\} = 0, \quad 1 \leqslant j, k \leqslant N.$$

Here, $\{x, y\}$ stands for the Jordan product $xy + yx$. The Fock vacuum $|0\rangle$ is defined as the vector state that is annihilated by all $f_j$. The subspace $\mathcal{H}^{(1)}$ of one fermion, called the *one-fermion space*, is the closed linear subspace of $\mathcal{H}$ spanned by the basis vectors $f_j^\dagger|0\rangle$, $1 \leqslant j \leqslant N$. Similarly, for each positive integer $k$, $\mathcal{H}^{(k)}$ stands for the $k$-fermion space, which is the subspace spanned by the basis vector $(f_1^\dagger)^{n_1} \cdots (f_N^\dagger)^{n_N}|0\rangle$ with $n_j \in \{0, 1\}$ and $\sum_{j=1}^N n_j = k$. Note that $\mathcal{H}^{(0)}$ is the one-dimensional subspace spanned by the vacuum $|0\rangle$. Thus, the *fermionic Fock space* $\mathcal{H}$ is the Hilbert space $\mathcal{H} = \oplus_{k=0}^N \mathcal{H}^{(k)}$, which has an orthonormal basis called the *Fock basis*,

$$|n_1, \ldots, n_N\rangle = |\{n_j\}\rangle := (f_1^\dagger)^{n_1} \cdots (f_N^\dagger)^{n_N}|0\rangle,$$

where $n_j \in \{0, 1\}$ is the occupation number of the $j$th mode and $\sum_{j=1}^N n_j < \infty$. Clearly, if $N < \infty$, $\mathcal{H}$ is finite-dimensional with $\dim \mathcal{H} = 2^N$. Sometimes, we say that $\mathcal{H}$ is the fermionic Fock space with the one-fermion space $\mathcal{H}^{(1)}$.

Denote by $\mathcal{G}(\mathcal{H})$ *the algebra of all bounded linear operators* on $\mathcal{H}$. For any $\theta \in \mathbb{R}$, the real-number field, the unitary operator on $\mathcal{H}^{(1)}$, defined by

$$|\phi^{(1)}\rangle \mapsto e^{i\theta}|\phi^{(1)}\rangle, \quad |\phi^{(1)}\rangle \in \mathcal{H}^{(1)},$$

together with the graded structure of the Fock space, allows an extension $\Gamma(e^{i\theta})$ to $\mathcal{H}$ by a method called the *second quantization* [20]. Note that $\Gamma(e^{i\theta}) \in \mathcal{G}(\mathcal{H})$ is unitary, which induces a $*$-automorphism of $\mathcal{G}(\mathcal{H})$ by

$$\mathcal{U}_\theta : X \mapsto \Gamma(e^{i\theta})X\Gamma(e^{-i\theta}), \quad X \in \mathcal{G}(\mathcal{H}).$$

Clearly, $\mathcal{U}_\theta$ satisfies

$$\mathcal{U}_\theta(f_j^\dagger) = e^{i\theta} f_j^\dagger, \quad 1 \leqslant j \leqslant N.$$

A very special and important case is $\theta = \pi$. $\Gamma(e^{i\pi})$ is called the *parity operator* and is often written as $(-1)^F$. The parity operator $(-1)^F$ is an involution and determines $\mathbb{Z}_2$ gradings [21] in $\mathcal{H}$ and $\mathcal{G}(\mathcal{H})$ as

$$\mathcal{H}_0 := \{|\phi\rangle \in \mathcal{H} \,|\, (-1)^F|\phi\rangle = |\phi\rangle\},$$
$$\mathcal{H}_1 := \{|\phi\rangle \in \mathcal{H} \,|\, (-1)^F|\phi\rangle = -|\phi\rangle\}$$

and

$$\mathcal{G}_0(\mathcal{H}) := \{X \in \mathcal{G}(\mathcal{H}) \,|\, (-1)^F X (-1)^F = X\},$$
$$\mathcal{G}_1(\mathcal{H}) := \{X \in \mathcal{G}(\mathcal{H}) \,|\, (-1)^F X (-1)^F = -X\},$$

respectively.

### B. Physical maps

The *physical operators* of a fermionic system are those operators in $\mathcal{G}(\mathcal{H})$ which commute with the parity operator, i.e., the operators in $\mathcal{G}_0(\mathcal{H})$. A state of fermions is described by a density operator $\rho$ which is physical, positive, and has a trace of 1. This physical restriction, known as the parity superselection rule, can be regarded as a reasonable requirement from various perspectives [22,23]. We denote the set of states of fermions by

$$\mathcal{S}(\mathcal{H}) := \{\rho \in \mathcal{G}_0(\mathcal{H}) \,|\, \rho \geqslant 0 \quad \text{with } \operatorname{Tr}\rho = 1\},$$

where $\rho \geqslant 0$ means that $\rho$ is positive, i.e., $\rho = \rho^\dagger$, with the spectrum falling in the interval $[0, +\infty)$. $\rho$ is a pure state if $\rho = \rho^2$; otherwise, $\rho$ is a mixed state [1].

The universally existing $\mathbb{Z}_2$ gradings in fermionic systems suggest an additional requirement for maps between fermion operator algebras. In this paper, $\mathcal{H}$ and $\mathcal{K}$ are fermionic Fock spaces with, respectively, the separable complex Hilbert spaces $\mathcal{H}^{(1)}$ and $\mathcal{K}^{(1)}$ as the one-fermion space if no specific assumption is made. A map $\Phi : \mathcal{G}(\mathcal{H}) \to \mathcal{G}(\mathcal{K})$ is said to be *physical* (i.e., $\mathbb{Z}_2$ symmetric [24] or grading equivariant [25]) if

$$\Phi \circ \mathcal{U}_\pi^H = \mathcal{U}_\pi^K \circ \Phi,$$

where $\mathcal{U}_\pi^H$ and $\mathcal{U}_\pi^K$ are implemented, respectively, by the parity operators $(-1)^{F_H}$ and $(-1)^{F_K}$. A linear map $\Phi : \mathcal{G}(\mathcal{H}) \to \mathcal{G}(\mathcal{K})$ is said to be *positive* if $X \in \mathcal{G}(\mathcal{H})$ is positive, implying that $\Phi(X)$ is positive in $\mathcal{G}(\mathcal{K})$. It is clear that a physical positive linear map transforms physical (Hermitian, positive) operators into physical (Hermitian, positive) operators.

### C. Bipartite fermionic systems

To study the quantum correlations of fermions, one needs to consider composite fermionic systems. Here, we adhere to the mode-based approach to fermionic entanglement theory [26–31]. For the particle-based approach, see Refs. [32–36] or the review paper in [37].

Let us start by splitting the set of modes $\{1,\dots,N\}$ into two disjoint subsets $A=\{j_1,\dots,j_{m_A}\}$ and $B=\{j'_1,\dots,j'_{m_B}\}$, with $m_A+m_B=N$. Then the Fock space $\mathcal{H}$ is naturally isomorphic to the tensor product $\mathcal{H}^A\otimes\mathcal{H}^B$ of Fock spaces $\mathcal{H}_A$ and $\mathcal{H}_B$ of subsystems $A$ and $B$ in the following way:

$$|\{n_j\}_A,\{n_j\}_B\rangle\mapsto\{n_j\}_A\otimes\{n_j\}_B, \tag{2}$$

where

$$|\{n_j\}_A,\{n_j\}_B\rangle=\left(f^\dagger_{j_1}\right)^{n_{j_1}}\cdots\left(f^\dagger_{j_{m_A}}\right)^{n_{j_{m_A}}}\left(f^\dagger_{j'_1}\right)^{n_{j'_1}}\cdots\left(f^\dagger_{j'_{m_B}}\right)^{n_{j'_{m_B}}}|0\rangle. \tag{3}$$

In terms of creation operators, this isomorphism can be simply expressed as (see [38], Theorem 7.14)

$$|0\rangle\mapsto|0\rangle_A\otimes|0\rangle_B,\quad f^\dagger_j\mapsto\begin{cases}\tilde{f}^\dagger_j\otimes I_B & \text{for } j\in A,\\ (-1)^{F_A}\otimes\tilde{f}^\dagger_j & \text{for } j\in B,\end{cases}$$

where $(-1)^{F_A}$ is the parity operator of $A$, $I_B$ is the identity operator of $B$, and $\{\tilde{f}^\dagger_j\}_{j\in A/B}$ is the set of creation operators of subsystem $A$ or $B$, which satisfies $\tilde{f}^\dagger_j|0\rangle_{A/B}=f^\dagger_j|0\rangle$. With Eq. (2), the set of fermionic states $\mathcal{S}(\mathcal{H})$ can be identified as $\mathcal{S}(\mathcal{H}^A\otimes\mathcal{H}^B)$ in the following way:

$$|\{n_j\}_A,\{n_j\}_B\rangle\langle\{n_j\}_A,\{n_j\}_B|\mapsto|\{n_j\}\rangle_A\langle\{n_j\}|\otimes|\{n_j\}\rangle_B\langle\{n_j\}|. \tag{4}$$

Hereafter, we will use this identification to denote states of a bipartite fermionic system until the end of Sec. III. Finally, it is worth noting that

$$\mathcal{S}(\mathcal{H}^A)\otimes\mathcal{S}(\mathcal{H}^B)\subseteq\mathcal{S}(\mathcal{H}^A\otimes\mathcal{H}^B).$$

*Separable states*. A fermionic state $\rho\in\mathcal{S}(\mathcal{H}^A\otimes\mathcal{H}^B)$ is said to be *separable* if $\rho$ can be approximated in the trace norm by states of the form

$$\left\{\sum_{k=1}^n p_k\rho_k^A\otimes\rho_k^B : p_k>0\quad\text{with}\ \sum_{k=1}^n p_k=1,\right.$$
$$\left.\rho_k^A\in\mathcal{S}(\mathcal{H}^A),\ \rho_k^B\in\mathcal{S}(\mathcal{H}^B)\right\}. \tag{5}$$

Otherwise, $\rho$ is called *entangled* [30]. Note that Eq. (5) restricts the possible convex decomposition by the condition that $\rho_k^A$ and $\rho_k^B$ must be fermionic states on the subsystems, and so the definition is different from the separability in systems of distinguishable particles [28]. Some detailed explanations can be found in [28,30].

*Fermionic partial transpose*. The fermionic partial transpose was first introduced in [15] and has various representations [15,17]. For our purposes, it is more useful and convenient to present it in the occupation-number basis. The fermionic partial transpose with respect to subsystem $A$ in the occupation-number basis (3) is defined as

$$(|\{n_j\}_A,\{n_j\}_B\rangle\langle\{\bar{n}_j\}_A,\{\bar{n}_j\}_B|)^{\mathrm{T_A}}$$
$$=(-1)^{\phi(\{n_j\},\{\bar{n}_j\})}U_A^\dagger|\{\bar{n}_j\}_A,\{n_j\}_B\rangle\langle\{n_j\}_A,\{\bar{n}_j\}_B|U_A, \tag{6}$$

where the phase factor $(-1)^{\phi(\{n_j\},\{\bar{n}_j\})}$ is determined by

$$\phi(\{n_j\},\{\bar{n}_j\})=\frac{(\tau_A+\bar{\tau}_A)\bmod 2}{2}+(\tau_A+\bar{\tau}_A)(\tau_B+\bar{\tau}_B), \tag{7}$$

$\tau_{A(B)}=\sum_{j\in A(B)}n_j$, $\bar{\tau}_{A(B)}=\sum_{j\in A(B)}\bar{n}_j$, and $U_A=\prod_{j\in A}(f_j^\dagger+f_j)$ is the partial particle-hole transformation, which is a local unitary operator [17]. As far as our research is concerned, $U_A$ can be ignored. In fact, for physical operators, as $\tau_A+\bar{\tau}_A+\tau_B+\bar{\tau}_B$ is always an even number, one sees that the phase factor in Eq. (6) actually has only two values,

$$(-1)^{\phi(\{n_j\},\{\bar{n}_j\})}=\begin{cases}1 & \text{if } \tau_A+\bar{\tau}_A \text{ is even},\\ -\mathrm{i} & \text{if } \tau_A+\bar{\tau}_A \text{ is odd}.\end{cases} \tag{8}$$

*Fermionic entanglement negativity*. The fermionic entanglement negativity of a fermionic state $\rho$ is formally defined as

$$\mathcal{N}(\rho)=\frac{\|\rho^{\mathrm{T_A}}\|_1-1}{2}, \tag{9}$$

where $\rho^{\mathrm{T_A}}$ is the fermionic partial transpose of $\rho$ and $\|\cdot\|_1$ is the trace norm. Similarly, one can consider the logarithmic negativity $\mathcal{E}(\rho)=\log_2\|\rho^{\mathrm{T_A}}\|_1$. Note that the fermionic partial transpose of a Hermitian operator may no longer be Hermitian, and so its eigenvalues may *not be real*, which poses additional difficulties.

*Mixed local fermion-number parity*. A fermionic state $\rho\in\mathcal{S}(\mathcal{H}^A\otimes\mathcal{H}^B)$ is said to have a *mixed local fermion-number parity* if

$$[\rho,\,(-1)^{F_A}]\neq 0, \tag{10}$$

where $[\cdot,\cdot]$ stands for the commutator. Clearly, such fermionic states must be entangled [18]. Furthermore, the authors of [18] conjectured that such entangled states can be witnessed by fermionic entanglement negativity. In this paper, we will give an affirmative answer to this.

For simplicity, we write

$$\rho_c=\tfrac{1}{2}[\rho+(-1)^{F_A}\rho\,(-1)^{F_A}],$$
$$\rho_a=\tfrac{1}{2}[\rho-(-1)^{F_A}\rho\,(-1)^{F_A}]. \tag{11}$$

Notice that $\rho_c\geqslant 0$ with $\mathrm{Tr}(\rho_c)=1$ and $\rho_a$ is Hermitian with $\mathrm{Tr}(\rho_a)=0$. Moreover, Eq. (10) holds if and only if $\rho_a\neq 0$.

*Local operations and classical communication (LOCC)*. In a fermionic system, a *physical* LOCC is a transformation determined by

$$\Lambda(\rho)=\sum_k E_k\rho E_k^\dagger, \tag{12}$$

where the Kraus operators $E_k$ satisfy $\sum_k E_k^\dagger E_k=I$ and $E_k=E_k^A\otimes E_k^B$ for some $E_k^A\in\mathcal{G}_r(\mathcal{H}^A)$ and $E_k^B\in\mathcal{G}_s(\mathcal{H}^B)$, in which $r,s\in\{0,1\}$ [23,39].

## III. PHASE PARTIAL TRANSPOSE AND PHASE ENTANGLEMENT NEGATIVITY

In this section, we discuss the behavior of positive linear maps in fermionic systems. Based on this, we establish a sufficient condition for a bipartite fermionic state with one subsystem of finite modes to be entangled. A characterization of the states which mix local fermion-number parity is provided, and as an application, an affirmative answer to the conjecture proposed in [18] is achieved. In addition, we consider a more generalized entanglement negativity called phase entanglement negativity and obtain some properties.

We start with a one-body fermionic system described by the fermionic Fock space $\mathcal{H}$.

The $\mathbb{Z}_2$ grading of $\mathcal{G}(\mathcal{H})$ allows us to write any $X \in \mathcal{G}(\mathcal{H})$ as

$$X = X_0 + X_1, \quad X_0 \in \mathcal{G}_0(\mathcal{H}), \quad X_1 \in \mathcal{G}_1(\mathcal{H}),$$

and define a transformation on $\mathcal{G}(\mathcal{H})$ as

$$\mathcal{R}_\theta(X) = X_0 + \mathrm{e}^{\mathrm{i}\theta} X_1 \ \ \forall \theta \in \mathbb{R}.$$

Obviously, $\mathcal{R}_\theta$ is physical and trace preserving, and the restriction of $\mathcal{R}_\theta$ on the Hilbert space $\mathcal{C}_2(\mathcal{H})$, the Hilbert-Schmidt class of $\mathcal{H}$, is unitary. In fact, we have

$$\mathrm{Tr}[\mathcal{R}_\theta(X)^\dagger \mathcal{R}_\theta(Y)] = \mathrm{Tr}(X^\dagger Y) \ \forall\, X, Y \in \mathcal{C}_2(\mathcal{H}).$$

To see this, one needs to note only that

$$\begin{aligned}\mathrm{Tr}(X_0Y_1) &= \mathrm{Tr}[(-1)^F X_0 (-1)^F Y_1] \\ &= \mathrm{Tr}[X_0(-1)^F Y_1 (-1)^F] = -\mathrm{Tr}(X_0Y_1),\end{aligned}$$

and thus,

$$\mathrm{Tr}(X_0Y_1) = 0$$

for all Hilbert-Schmidt classes $X_0 \in \mathcal{G}_0(\mathcal{H})$ and $Y_1 \in \mathcal{G}_1(\mathcal{H})$. In addition,

$$\mathcal{R}_\pi = \mathcal{U}_\pi, \ \ \mathcal{R}_{\pi/2}^2 = \mathcal{R}_{\pi/2} \circ \mathcal{R}_{\pi/2} = \mathcal{U}_\pi.$$

However, $\mathcal{R}_\theta$ is *not Hermitian preserving* unless $\theta = k\pi,\ k \in \mathbb{Z}$. In the case $\theta = k\pi$, one can see that $\mathcal{R}_{k\pi} = \mathcal{U}_\pi$ or $I$, the identity operation.

Now, let us consider another fermionic system described by the Fock space $\mathcal{K}$. For any positive linear map $\Phi : \mathcal{G}(\mathcal{H}) \to \mathcal{G}(\mathcal{K})$ and any $\theta \in \mathbb{R}$, we define a linear map

$$\Phi_\theta := \Phi \circ \mathcal{R}_\theta. \tag{13}$$

Then it is easily observed that $\Phi_\theta$ is the same as $\Phi$ on physical operators in $\mathcal{G}(\mathcal{H})$ as

$$\Phi_\theta(X) = \Phi(X) \ \ \forall\, X \in \mathcal{G}_0(\mathcal{H}).$$

However, when we consider a composite fermionic system and regard $\Phi_\theta$ as a local operation on one of the subsystems, the situation is completely different.

Assume that $\mathcal{H} = \mathcal{H}^A \otimes \mathcal{H}^B$ is the fermionic Fock space corresponding to a bipartite fermionic system with $\mathcal{H}^B$ being finite-dimensional, and consider the operation

$$\Phi_\theta \otimes I : \mathcal{G}(\mathcal{H}^A \otimes \mathcal{H}^B) \to \mathcal{G}(\mathcal{K} \otimes \mathcal{H}^B),$$

where $I$ is the identity operation on $\mathcal{H}^B$. Let $\rho \in \mathcal{S}(\mathcal{H}^A \otimes \mathcal{H}^B)$ be a fermionic state. As $\rho \in \mathcal{C}_2(\mathcal{H}^A \otimes \mathcal{H}^B)$ and $\dim \mathcal{H}_B < \infty$, there are two finite sequences of Hermitian operators $\{X_k^A\}_{k=1}^n \subset \mathcal{C}_2(\mathcal{H}^A)$ and $\{X_k^B\}_{k=1}^n \subset \mathcal{C}_2(\mathcal{H}^B)$ such that

$$\rho = \sum_{k=1}^n X_k^A \otimes X_k^B = \sum_{k=1}^n \big(X_{0,k}^A \otimes X_k^B + X_{1,k}^A \otimes X_k^B\big),$$

where $X_{0,k}^A \in \mathcal{G}_0(\mathcal{H}^A)$ and $X_{1,k}^A \in \mathcal{G}_1(\mathcal{H}^A)$ are both Hermitian operators [40]. A straightforward computation shows that

$$(\Phi_\theta \otimes I)\rho = \sum_{k=1}^n \big[\Phi\big(X_{0,k}^A\big) \otimes X_k^B + \mathrm{e}^{\mathrm{i}\theta}\Phi\big(X_{1,k}^A\big) \otimes X_k^B\big].$$

It follows that

$$(\Phi_\theta \otimes I)\rho = (\Phi \otimes I)\rho_c + \mathrm{e}^{\mathrm{i}\theta}(\Phi \otimes I)\rho_a, \tag{14}$$

where $\rho_c$ and $\rho_a$ are defined in Eq. (11). As $\Phi$ is positive, one can observe that, for any $\theta \neq k\pi,\ k \in \mathbb{Z}$,

$$(\Phi_\theta \otimes I)\rho \geqslant 0 \iff \begin{cases}\sum_k \Phi\big(X_{1,k}^A\big) \otimes X_k^B = 0, \\ \sum_k \Phi\big(X_{0,k}^A\big) \otimes X_k^B \geqslant 0,\end{cases}$$

which is equivalent to

$$(\Phi_\theta \otimes I)\rho \geqslant 0 \iff \begin{cases}(\Phi \otimes I)\rho_a = 0, \\ (\Phi \otimes I)\rho_c \geqslant 0.\end{cases} \tag{15}$$

Thus, through positivity, any entangled state that can be detected by $\Phi$ must also be detected by $\Phi_\theta$. Hence, we have proved the following sufficient criterion for a bipartite fermionic state to be entangled in terms of positive linear maps.

*Theorem 1.* Let $\mathcal{H} = \mathcal{H}^A \otimes \mathcal{H}^B$ be a fermionic Fock space with $\dim \mathcal{H}^B < \infty$ and $\rho \in \mathcal{S}(\mathcal{H}^A \otimes \mathcal{H}^B)$ be a fermionic state. Then the following statements are equivalent.

(1) A positive linear map $\Phi : \mathcal{G}(\mathcal{H}^A) \to \mathcal{G}(\mathcal{K})$ exists for some fermionic Fock space $\mathcal{K}$ such that either $(\Phi \otimes I)\rho_a \neq 0$ or $(\Phi \otimes I)\rho_c \ngeqslant 0$.

(2) A positive linear map $\Phi : \mathcal{G}(\mathcal{H}^A) \to \mathcal{G}(\mathcal{K})$ exists for some fermionic Fock space $\mathcal{K}$ and some $\theta \in \mathbb{R}$ with $\theta \neq k\pi,\ k \in \mathbb{Z}$, such that $(\Phi_\theta \otimes I)\rho \ngeqslant 0$.

In addition, if one of the above statements holds, then $\rho$ must be entangled.

The next result provides a criterion for states which mix local fermion-number parity.

*Theorem 2.* Let $\mathcal{H} = \mathcal{H}^A \otimes \mathcal{H}^B$ be a fermionic Fock space with $\dim \mathcal{H}^B < \infty$. Then a fermionic state $\rho \in \mathcal{S}(\mathcal{H}^A \otimes \mathcal{H}^B)$ has a mixed local fermion-number parity if and only if, for any trace preserving positive linear injective map $\Phi : \mathcal{G}(\mathcal{H}^A) \to \mathcal{G}(\mathcal{H}^A)$ and for any $\theta \in \mathbb{R}$ with $\theta \neq k\pi,\ k \in \mathbb{Z}$,

$$\|(\Phi_\theta \otimes I)\rho\|_1 > 1.$$

*Proof.* For the "if" part, take $\Phi = I$, the identity map, and $\theta = \pi/2$. Then

$$\|(\Phi_{\pi/2} \otimes I)\rho\|_1 = \|\rho_c + \mathrm{i}\rho_a\|_1 > 1,$$

which implies $\rho_a \neq 0$ as $\|\rho_c\|_1 = \mathrm{Tr}(\rho_c) = 1$.

To check the "only if" part, assume $\rho_a \neq 0$. Then for any trace preserving positive linear injective map $\Phi : \mathcal{G}(\mathcal{H}^A) \to \mathcal{G}(\mathcal{H}^A)$, we have $(\Phi \otimes I)\rho_a \neq 0$. If $\theta \neq k\pi,\ k \in \mathbb{Z}$, we see that $(\Phi_\theta \otimes I)\rho$ is not positive by Theorem 1. Furthermore, since $\Phi$ is trace preserving, we have

$$\mathrm{Tr}[(\Phi \otimes I)\rho_c] = \mathrm{Tr}(\rho_c) = 1$$

and

$$\mathrm{Tr}[(\Phi \otimes I)\rho_a] = \mathrm{Tr}(\rho_a) = 0,$$

which imply $\mathrm{Tr}[(\Phi_\theta \otimes I)\rho] = 1$.

Here, we need a result from mathematics which states that, for any trace-class operator $X$, one always has $\|X\|_1 \geqslant |\operatorname{Tr} X|$, and $\|X\|_1 = \operatorname{Tr} X$ if and only if $X \geqslant 0$ (see [41], Lemma 2.2). With this result, we obtain

$$\|(\Phi_\theta \otimes I)\rho\|_1 > 1$$

as $(\Phi_\theta \otimes I)\rho$ is not positive. ■

Note that, restricted to physical operators, the fermionic partial transpose of the first subsystem up to a local unitary transformation is exactly the operation

$$T_{\pi/2} \otimes I : \mathcal{G}(\mathcal{H}^A \otimes \mathcal{H}^B) \to \mathcal{G}(\mathcal{H}^A \otimes \mathcal{H}^B),$$

where $T : \mathcal{G}(\mathcal{H}^A) \to \mathcal{G}(\mathcal{H}^A)$ is the usual transpose with respect to the Fock basis of $\mathcal{H}^A$. Clearly, $T$ is injective, positive, and trace preserving. The fermionic entanglement negativity of a fermionic state $\rho$ is the same as

$$\mathcal{N}_{\frac{\pi}{2}}(\rho) = \frac{\|\rho^{\mathrm{T}_1}\|_1 - 1}{2}, \tag{16}$$

where $\rho^{\mathrm{T}_1} = (T_{\pi/2} \otimes I)\rho$ is the fermionic partial transpose of $\rho$ [here, we keep the symbol $\mathcal{N}(\rho)$ for the usual entanglement negativity, and $\mathrm{T}_1$ is the fermionic partial transpose with respect to the first subsystem, $A$]. Similarly, one can consider the logarithmic negativity $\mathcal{E}_{\frac{\pi}{2}}(\rho) = \log_2 \|\rho^{\mathrm{T}_1}\|_1$.

More generally, for any real number $\theta$, we can define the *phase partial transpose with respect to* $\theta$ of $\rho$ by $\rho^{\mathrm{T}_1}(\theta) = (T_\theta \otimes I)\rho$ and define the *phase entanglement negativity*, denoted by $\mathcal{N}_\theta(\rho)$, as

$$\mathcal{N}_\theta(\rho) = \frac{\|\rho^{\mathrm{T}_1}(\theta)\|_1 - 1}{2}.$$

From the proof of Theorem 2, it is clear that $\mathcal{N}_\theta(\rho) \geqslant 0$ for all $\rho$.

Note that the ordinary partial transpose of $\rho$ is exactly $\rho^{\mathrm{T}_1}(0)$, which differs from $\rho^{\mathrm{T}_1}(k\pi), k \in \mathbb{Z}$, by (at most) a local unitary transformation. So we have

$$\mathcal{N}_{k\pi}(\rho) = \mathcal{N}_0(\rho) = \mathcal{N}(\rho).$$

We say that a phase entanglement negativity $\mathcal{N}_\theta$ is *nontrivial* if $\theta \neq k\pi$ for some $k \in \mathbb{Z}$.

The phase partial transpose of the second subsystem $\rho^{\mathrm{T}_2}(\theta)$ is defined similarly by $\rho^{\mathrm{T}_2}(\theta) = (I \otimes T_\theta)\rho$. Clearly, $[\rho^{\mathrm{T}_1}(\theta)]^{\mathrm{T}_2}(-\theta) = \rho^{\mathrm{T}}$.

A direct application of Theorem 2 gives

$$\|\rho^{\mathrm{T}_1}(\theta)\|_1 = \|(T_\theta \otimes I)\rho\|_1 > 1$$

for all $\rho$ with $\rho_a \neq 0$ and $\theta \in \mathbb{R}$ with $\theta \neq k\pi$, $k \in \mathbb{Z}$. Therefore, we have proved the following result.

*Theorem 3.* Let $\mathcal{H} = \mathcal{H}^A \otimes \mathcal{H}^B$ be a fermionic Fock space with $\dim \mathcal{H}^B < \infty$. For any $\theta \in \mathbb{R}$ with $\theta \neq k\pi$, $k \in \mathbb{Z}$, the phase entanglement negativity $\mathcal{N}_\theta(\rho)$ of any fermionic state $\rho \in \mathcal{S}(\mathcal{H}^A \otimes \mathcal{H}^B)$ with mixed local fermion-number parity is nonvanishing. Particularly, the conjecture of [18] has an affirmative answer.

In addition, our result says that the conjecture is true even for the case $\dim \mathcal{H}^A = \infty$. Also, note that the phase partial transpose is usually not Hermitian preserving, but the positivity of the phase partial transpose is equivalent to the zero negativity based on it.

Now, we show further that the phase entanglement negativity is an entanglement monotone.

*Theorem 4.* Let $\mathcal{H} = \mathcal{H}^A \otimes \mathcal{H}^B$ be a fermionic Fock space with $\dim \mathcal{H}^B < \infty$ and $\theta \in \mathbb{R}$. Then, for any fermionic state $\rho \in \mathcal{S}(\mathcal{H}^A \otimes \mathcal{H}^B)$ and any physical LOCC $\Lambda(\cdot) = \sum_k E_k(\cdot)E_k^\dagger$, we have

$$\mathcal{N}_\theta(\rho) \geqslant \sum_k \operatorname{Tr}(E_k\rho E_k^\dagger)\mathcal{N}_\theta\left(\frac{E_k\rho E_k^\dagger}{\operatorname{Tr}(E_k\rho E_k^\dagger)}\right) \geqslant \mathcal{N}_\theta(\Lambda(\rho)).$$

The proof is given in Appendix.

Therefore, the nontrivial phase entanglement negativity is a genuine fermionic entanglement measure for any $(1+M)$-mode fermionic system with $1 \leqslant M < \infty$ (this can be easily seen from Proposition 3 in Sec. IV and Theorem 3 in [18]).

At the end of this section, we provide an example to illustrate how the phase entanglement negativity varies with the angle $\theta$ for a given fermionic state.

In $(1+1)$-mode fermionic systems, a generic fermionic state $\rho$ can be written in the occupation-number basis $\{|00\rangle, |10\rangle, |01\rangle, |11\rangle\}$ as

$$\rho = \rho(p, \mu, \eta, \upsilon, \varphi) = \begin{bmatrix} p\cos^2\upsilon & 0 & 0 & p\mu\cos\upsilon\sin\upsilon \\ 0 & (1-p)\cos^2\varphi & (1-p)\eta\cos\varphi\sin\varphi & 0 \\ 0 & (1-p)\bar{\eta}\cos\varphi\sin\varphi & (1-p)\sin^2\varphi & 0 \\ p\bar{\mu}\cos\upsilon\sin\upsilon & 0 & 0 & p\sin^2\upsilon \end{bmatrix}, \tag{17}$$

where $\upsilon, \varphi \in [0, \pi/2]$; $\mu, \eta \in \mathbb{C}$ with $0 \leqslant |\mu|, |\eta| \leqslant 1$; and $0 \leqslant p \leqslant 1$. The above $\rho$ reduces to a pure fermionic state if $p = |\mu| = 1$ or $1 - p = |\eta| = 1$. For any $\theta \in \mathbb{R}$, the phase partial transpose of $\rho$ associated with $\theta$ is

$$\rho^{\mathrm{T}_1}(\theta) = \begin{bmatrix} p\cos^2\upsilon & 0 & 0 & \mathrm{e}^{\mathrm{i}\theta}(1-p)\eta\cos\varphi\sin\varphi \\ 0 & (1-p)\cos^2\varphi & \mathrm{e}^{\mathrm{i}\theta}p\mu\cos\upsilon\sin\upsilon & 0 \\ 0 & \mathrm{e}^{\mathrm{i}\theta}p\bar{\mu}\cos\upsilon\sin\upsilon & (1-p)\sin^2\varphi & 0 \\ \mathrm{e}^{\mathrm{i}\theta}(1-p)\bar{\eta}\cos\varphi\sin\varphi & 0 & 0 & p\sin^2\upsilon \end{bmatrix}.$$

So

$$\|\rho^{\mathrm{T}_1}(\theta)\|_1 = \sqrt{p^2 + \tfrac{1}{2}[|x^2 - \mathrm{e}^{2\mathrm{i}\theta}|\eta|^2y^2| - (x^2 - |\eta|^2y^2)]} + \sqrt{(1-p)^2 + \tfrac{1}{2}[|y^2 - \mathrm{e}^{2\mathrm{i}\theta}|\mu|^2x^2| - (y^2 - |\mu|^2x^2)]}, \tag{18}$$

where $x = p\sin 2\upsilon$ and $y = (1-p)\sin 2\varphi$. In particular, for $\theta = k\pi$, $k \in \mathbb{Z}$, we see that

$$\|\rho^{\mathrm{T}_1}(k\pi)\|_1 = \sqrt{p^2 + \max\{0, |\eta|^2y^2 - x^2\}} + \sqrt{(1-p)^2 + \max\{0, |\mu|^2x^2 - y^2\}},$$

and for $\theta = \pi/2 + k\pi$, $k \in \mathbb{Z}$, we get

$$\|\rho^{\mathrm{T}_1}(\pi/2 + k\pi)\|_1 = \sqrt{p^2 + |\eta|^2y^2} + \sqrt{(1-p)^2 + |\mu|^2x^2}.$$

So it is clearly seen that the ordinary entanglement negativity $\mathcal{N}(\rho)$ vanishes if and only if $|\eta|y \leqslant x$ and $|\mu|x \leqslant y$, but the fermionic entanglement negativity $\mathcal{N}_{\frac{\pi}{2}}(\rho)$ vanishes if and only if $|\eta|y = |\mu|x = 0$; i.e., $\rho$ is in a diagonal form [18,28]. Therefore, the ordinary partial transpose cannot capture entanglement as well as before [15]. For example,

$$\rho = \begin{bmatrix} \frac{3}{8} & 0 & 0 & \frac{\sqrt{3}}{8} \\ 0 & \frac{3}{8} & \frac{\sqrt{3}}{8} & 0 \\ 0 & \frac{\sqrt{3}}{8} & \frac{1}{8} & 0 \\ \frac{\sqrt{3}}{8} & 0 & 0 & \frac{1}{8} \end{bmatrix}$$

is entangled, which cannot be captured by the ordinary partial transpose as $\mathcal{N}(\rho) = 0$. However, we notice that all the nontrivial phase entanglement negativity (i.e., $\theta \neq k\pi$, $k \in \mathbb{Z}$) can effectively distinguish entangled fermionic states in this simple situation, and $\theta = \pi/2 + k\pi$, $k \in \mathbb{Z}$, is the largest one for the value of $\|\rho^{\mathrm{T}_1}\|_1$ (the same as the phase entanglement negativity introduced by it) among them.

Thus, we achieve the following result.

*Proposition 1.* For any $(1+1)$-mode fermionic state $\rho$ and any $\theta \in \mathbb{R}$, we have

$$\mathcal{N}(\rho) \leqslant \mathcal{N}_\theta(\rho) \leqslant \mathcal{N}_{\frac{\pi}{2}}(\rho).$$

Proposition 1 reveals that the fermionic entanglement negativity $\mathcal{N}_{\frac{\pi}{2}}$ is the best one to detect entanglement in the fermionic states. However, this does not mean that the introduction of the phase entanglement negativity is nonessential. In the process of quantum computation, the input quantum state $\rho$ is transformed into $\sigma = U\rho U^\dagger$ through the quantum gate $U$. During the evolution of a closed quantum system, the system evolves from the initial state $\rho$ to $\sigma = \rho_t = U_t\rho U_t^\dagger$ at time $t$. Generally, we need to know the change in $\sigma$, that is, to distinguish $\sigma$ from $\rho$. However, since $\sigma$ is unitarily equivalent to the state $\rho$, it has the same eigenvalues as $\rho$, and only the eigenvectors can undergo changes. Therefore, the task of distinguishing quantum states with the same eigenvalues is fundamental. We provide some specific examples to illustrate that $\mathcal{N}_\theta$ can distinguish fermionic states that $\mathcal{N}$ and $\mathcal{N}_{\frac{\pi}{2}}$ cannot.

In the following, we use $\sigma(\rho)$ to denote the eigenvalues of $\rho$.

*Example 1.* Consider $(1+1)$-mode fermionic states

$$\rho_1 = \frac{1}{4}\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}, \quad \rho_2 = \frac{1}{3}\begin{bmatrix} 1 & 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{2} & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{2} \end{bmatrix}.$$

It is clear that $\rho_2 = U\rho_1U^\dagger$ as

$$\sigma(\rho_1) = \left\{\frac{1}{2}, \frac{1}{2}, 0, 0\right\} = \sigma(\rho_2).$$

For any $\theta \in \mathbb{R}$, we have

$$\mathcal{N}_\theta(\rho_1) = \tfrac{1}{2}(\sqrt{1 + |\sin\theta|} - 1)$$

and

$$\mathcal{N}_\theta(\rho_2) = \frac{1}{2}\left(\sqrt{1 + \frac{8}{9}|\sin\theta|} - 1\right).$$

Hence, $\mathcal{N}(\rho_1) = \mathcal{N}_{k\pi}(\rho_1) = 0 = \mathcal{N}_{k\pi}(\rho_2) = \mathcal{N}(\rho_2)$, and $\mathcal{N}$ cannot distinguish $\rho_1$ from $\rho_2$. But $\mathcal{N}_\theta(\rho_1) > \mathcal{N}_\theta(\rho_2)$ for $\theta \neq k\pi$.

*Example 2.* Consider $(1+1)$-mode fermionic states

$$\rho_1 = \begin{bmatrix} \frac{1}{5} & 0 & 0 & \frac{1}{10} \\ 0 & \frac{1}{3} & \frac{\sqrt{5}}{6} & 0 \\ 0 & \frac{\sqrt{5}}{6} & \frac{5}{12} & 0 \\ \frac{1}{10} & 0 & 0 & \frac{1}{20} \end{bmatrix},$$

$$\rho_2 = \frac{1}{2}\begin{bmatrix} \frac{1}{6} & 0 & 0 & \frac{\sqrt{2}}{6} \\ 0 & \frac{9}{10} & \frac{3\sqrt{6}}{10} & 0 \\ 0 & \frac{3\sqrt{6}}{10} & \frac{3}{5} & 0 \\ \frac{\sqrt{2}}{6} & 0 & 0 & \frac{1}{3} \end{bmatrix}.$$

In this case, we have

$$\sigma(\rho_1) = \left\{\tfrac{3}{4}, \tfrac{1}{4}, 0, 0\right\} = \sigma(\rho_2),$$

and $\rho_1$ is unitarily similar to $\rho_2$. It is easily checked that

$$\mathcal{N}_\theta(\rho_1) = \frac{1}{2}\left(\sqrt{\frac{1097}{3600} + f(\theta)} + \sqrt{\frac{1153}{3600} + f(\theta)} - 1\right),$$

with

$$f(\theta)=\frac{1}{3}\sqrt{\frac{7853}{11250}-\frac{1}{10}\cos 2\theta},$$

and

$$\mathcal{N}_\theta(\rho_2)=\frac{1}{2}\left(\sqrt{\frac{1097}{3600}+g(\theta)}+\sqrt{\frac{1153}{3600}+g(\theta)}-1\right),$$

with

$$g(\theta)=\frac{1}{10}\sqrt{\frac{29837}{4050}-\frac{3}{2}\cos 2\theta}.$$

It follows that $f(\theta)=g(\theta)=67/225$ if $\theta=\pi/2+k\pi$, and consequently, $\mathcal{N}_{\pi/2}(\rho_1)=\mathcal{N}_{\pi/2}(\rho_2)$. However, $\mathcal{N}_\theta(\rho_1)\neq\mathcal{N}_\theta(\rho_2)$ for $\theta\neq\pi/2+k\pi$. This reveals that the phase entanglement negativity can distinguish $\rho_1$ from $\rho_2$ but the fermionic entanglement negativity cannot.

*Example 3*. Consider $(1+1)$-mode fermionic states

$$\rho_1=\begin{bmatrix}\frac{2+\sqrt{3}}{8} & 0 & 0 & \frac{1}{8}\\ 0 & \frac{2+\sqrt{3}}{8} & \frac{1}{8} & 0\\ 0 & \frac{1}{8} & \frac{2-\sqrt{3}}{8} & 0\\ \frac{1}{8} & 0 & 0 & \frac{2-\sqrt{3}}{8}\end{bmatrix}$$

and

$$\rho_2=\frac{1}{8}\begin{bmatrix}1+\frac{1}{\sqrt{5}} & 0 & 0 & \frac{2}{\sqrt{5}}\\ 0 & 3+\frac{\sqrt{41}}{\sqrt{5}} & \frac{2}{\sqrt{5}} & 0\\ 0 & \frac{2}{\sqrt{5}} & 3-\frac{\sqrt{41}}{\sqrt{5}} & 0\\ \frac{2}{\sqrt{5}} & 0 & 0 & 1-\frac{1}{\sqrt{5}}\end{bmatrix}.$$

Note that

$$\sigma(\rho_1)=\left\{\tfrac{1}{2},\tfrac{1}{2},0,0\right\},\quad \sigma(\rho_2)=\left\{\tfrac{3}{4},\tfrac{1}{4},0,0\right\},$$

which implies that $\rho_1$ is majorized by $\rho_2$. For this case, we have

$$\mathcal{N}_\theta(\rho_1)=\frac{1}{2}\left(\sqrt{1+\frac{1}{4}|\sin\theta|}-1\right)$$

and

$$\mathcal{N}_\theta(\rho_2)=\frac{1}{2}\left(\sqrt{\frac{1}{16}+\frac{1}{20}|\sin\theta|}+\sqrt{\frac{9}{16}+\frac{1}{20}|\sin\theta|}-1\right).$$

Thus, $\mathcal{N}(\rho_1)=\mathcal{N}(\rho_2)=0$, and $\mathcal{N}_{\frac{\pi}{2}}(\rho_1)=\mathcal{N}_{\frac{\pi}{2}}(\rho_2)=(\sqrt{5}-2)/4$, which means that both $\mathcal{N}$ and $\mathcal{N}_{\frac{\pi}{2}}$ cannot distinguish $\rho_1$ from $\rho_2$. However, for $\theta\neq k\pi/2$, $\mathcal{N}_\theta$ can distinguish $\rho_1$ from $\rho_2$ as $\mathcal{N}_\theta(\rho_1)\neq\mathcal{N}_\theta(\rho_2)$.

## IV. BOUNDS OF FERMIONIC ENTANGLEMENT NEGATIVITY

In this section, we discuss further the fermionic entanglement negativity. We first give an expression of the fermionic partial transpose in terms of creation and annihilation operators. Keep in mind that the fermionic partial transpose is defined only on the subalgebra of physical operators which includes all states of fermions.

Let $N$ be a positive integer and consider the $N$-mode fermionic system $\mathcal{H}$ with annihilation operators $\{f_j\}_{j=1}^N$. Consider a bipartition $(J_1,J_2)$ of $N$ modes, that is, divide the set of modes $\{1,\ldots,N\}$ into subsets $J_1=\{j_1,\ldots,j_{N_1}\}$ and $J_2=\{j_1',\ldots,j_{N_2}'\}$, with $N_1+N_2=N$. Obviously, from Eq. (6), the fermionic partial transpose of any operator $X\in\mathcal{G}_0(\mathcal{H})$ can be expressed in terms of the creation and annihilation operators. In fact, by writing $X=X_c+X_a$, we have

$$\begin{aligned}X_c&=\frac{1}{2}[X+(-1)^{F_1}X(-1)^{F_1}]\\ &=\sum_{L,L'\subseteq J_1:|L|+|L'|=\text{even}}P_L\bar{P}_{L^c}XP_{L'}\bar{P}_{L'^c}\end{aligned}$$

and

$$\begin{aligned}X_a&=\frac{1}{2}[X-(-1)^{F_1}X(-1)^{F_1}]\\ &=\sum_{L,L'\subseteq J_1:|L|+|L'|=\text{odd}}P_L\bar{P}_{L^c}XP_{L'}\bar{P}_{L'^c},\end{aligned}$$

where $(-1)^{F_1}=\prod_{j\in J_1}(-1)^{f_j^\dagger f_j}$, $P_L=\prod_{j\in L}f_j^\dagger f_j$, $\bar{P}_L=\prod_{j\in L}f_jf_j^\dagger$, $P_\emptyset=\bar{P}_\emptyset=I$, $L^c=J_1\backslash L$ is the relative complement of $L$ in $J_1$, $|L|$ denotes the cardinality of the subset $L$, and the summation is taken over all possible subsets $L$ and $L'$ of $J_1$.

As the range of $P_L\bar{P}_{L^c}$ is spanned by basis vectors

$$|\{n_j\}_{j\in J_1},\{n_j\}_{j\in J_2}\rangle,\quad n_j=\begin{cases}1, & j\in L,\\ 0, & j\in L^c,\\ 0\text{ or }1, & j\in J_2,\end{cases}$$

it follows from Eq. (6) that

$$\begin{aligned}&(P_L\bar{P}_{L^c}XP_{L'}\bar{P}_{L'^c})^{\mathrm{T}_1}\\ &=\begin{cases}C_{L'}^\dagger C_LP_L\bar{P}_{L^c}XP_{L'}\bar{P}_{L'^c}C_{L'}^\dagger C_L & \text{if }|L|+|L'|\text{ is even},\\ -iC_{L'}^\dagger C_LP_L\bar{P}_{L^c}XP_{L'}\bar{P}_{L'^c}C_{L'}^\dagger C_L & \text{if }|L|+|L'|\text{ is odd},\end{cases}\end{aligned}$$

where $C_L^\dagger$ denotes the product of $f_j^\dagger$ for $j\in L$ on the same order as Eq. (3) and $C_\emptyset^\dagger=I$.

Note that $C_LP_L=\bar{P}_LC_L=C_L$. Thus, we have

$$\begin{aligned}P_L\bar{P}_{L^c}&=C_L^\dagger\bar{P}_{L^c}C_L=C_L^\dagger\bar{P}_{L^c}\bar{P}_LC_L=C_L^\dagger\bar{P}_{J_1}C_L,\\ C_{L'}^\dagger C_LP_L\bar{P}_{L^c}&=C_{L'}^\dagger C_LC_L^\dagger\bar{P}_{J_1}C_L\\ &=C_{L'}^\dagger\bar{P}_L\bar{P}_{J_1}C_L=C_{L'}^\dagger\bar{P}_{J_1}C_L,\end{aligned}$$

and

$$\begin{aligned}P_{L'}\bar{P}_{L'^c}C_{L'}^\dagger C_L&=C_{L'}^\dagger\bar{P}_{J_1}C_{L'}C_{L'}^\dagger C_L\\ &=C_{L'}^\dagger\bar{P}_{J_1}\bar{P}_{L'}C_L=C_{L'}^\dagger\bar{P}_{J_1}C_L.\end{aligned}$$

Then the fermionic partial transpose of $P_L\bar{P}_{L^c}XP_{L'}\bar{P}_{L'^c}$ can be reduced to

$$\begin{aligned}&(P_L\bar{P}_{L^c}XP_{L'}\bar{P}_{L'^c})^{\mathrm{T}_1}\\ &=(C_L^\dagger\bar{P}_{J_1}C_LXC_{L'}^\dagger\bar{P}_{J_1}C_{L'})^{\mathrm{T}_1}\\ &=\begin{cases}C_{L'}^\dagger\bar{P}_{J_1}C_LXC_{L'}^\dagger\bar{P}_{J_1}C_L & \text{if }|L|+|L'|\text{ is even},\\ -iC_{L'}^\dagger\bar{P}_{J_1}C_LXC_{L'}^\dagger\bar{P}_{J_1}C_L & \text{if }|L|+|L'|\text{ is odd}.\end{cases}\end{aligned}\tag{19}$$

Indeed, Eq. (19) provides a precise expression for the fermionic partial transpose of any physical operator $X$ with respect to the first subsystem with

$$X^{\mathrm{T}_1} = (X_c)^{\mathrm{T}_1} + (X_a)^{\mathrm{T}_1} = \sum_{L,L'\subseteq J_1:|L|+|L'|=\text{even}} C_{L'}^{\dagger}\bar{P}_{J_1}C_L X C_{L'}^{\dagger}\bar{P}_{J_1}C_L - i\sum_{L,L'\subseteq J_1:|L|+|L'|=\text{odd}} C_{L'}^{\dagger}\bar{P}_{J_1}C_L X C_{L'}^{\dagger}\bar{P}_{J_1}C_L. \tag{20}$$

Since

$$[(-1)^{F_1}X(-1)^{F_1}]^{\mathrm{T}_1} = (-1)^{F_1}X^{\mathrm{T}_1}(-1)^{F_1} \tag{21}$$

and

$$(X^{\mathrm{T}_1})^{\dagger} = (-1)^{F_1}(X^{\dagger})^{\mathrm{T}_1}(-1)^{F_1},$$

the fermionic partial transpose of a Hermitian operator is pseudo-Hermitian with respect to $(-1)^{F_1}$ [42,43] (an operator $D$ is said to be pseudo-Hermitian if some invertible Hermitian operator $\eta$ exists such that $D^{\dagger} = \eta D\eta^{-1}$).

The relations $(X_c)^{\mathrm{T}_1} = (X^{\mathrm{T}_1})_c$ and $(X_a)^{\mathrm{T}_1} = (X^{\mathrm{T}_1})_a$ allow us to simply write them as $X_c^{\mathrm{T}_1}$ and $X_a^{\mathrm{T}_1}$, respectively. If $X$ is Hermitian, then Eq. (20) implies that

$$X_c^{\mathrm{T}_1} = \frac{1}{2}[X^{\mathrm{T}_1} + (X^{\mathrm{T}_1})^{\dagger}] = \sum_{L,L'\subseteq J_1:|L|+|L'|=\text{even}} C_{L'}^{\dagger}\bar{P}_{J_1}C_L X C_{L'}^{\dagger}\bar{P}_{J_1}C_L$$

is Hermitian and

$$X_a^{\mathrm{T}_1} = \frac{1}{2}[X^{\mathrm{T}_1} - (X^{\mathrm{T}_1})^{\dagger}] = -i\sum_{L,L'\subseteq J_1:|L|+|L'|=\text{odd}} C_{L'}^{\dagger}\bar{P}_{J_1}C_L X C_{L'}^{\dagger}\bar{P}_{J_1}C_L$$

is skew Hermitian.

As mentioned above, each physical operator $X \in \mathcal{G}_0(\mathcal{H})$ can be written in the form $X = X_c + X_a$. In addition, each $X$ also admits a direct sum decomposition

$$X = X\Pi_+ + X\Pi_-,$$

where $\Pi_{\pm} = \frac{1}{2}[1 \pm (-1)^F]$. $\{\Pi_+, \Pi_-\}$ forms a complete set of orthogonal projections, and both commute with $X$.

*Lemma 1.* The equality

$$\|X_c^{\mathrm{T}_1}\|_1 + \|X_a^{\mathrm{T}_1}\|_1 = \|(X\Pi_+)^{\mathrm{T}_1}\|_1 + \|(X\Pi_-)^{\mathrm{T}_1}\|_1 \tag{22}$$

holds for all $X \in \mathcal{G}_0(\mathcal{H})$.

*Proof.* Equation (20) implies that

$$\|(X\Pi_+)^{\mathrm{T}_1}\|_1 = \|X_c^{\mathrm{T}_1}\Pi_+ + X_a^{\mathrm{T}_1}\Pi_-\|_1 = \|X_c^{\mathrm{T}_1}\Pi_+\|_1 + \|X_a^{\mathrm{T}_1}\Pi_-\|_1$$

and

$$\|(X\Pi_-)^{\mathrm{T}_1}\|_1 = \|X_c^{\mathrm{T}_1}\Pi_- + X_a^{\mathrm{T}_1}\Pi_+\|_1 = \|X_c^{\mathrm{T}_1}\Pi_-\|_1 + \|X_a^{\mathrm{T}_1}\Pi_+\|_1.$$

Hence,

$$\begin{aligned}\|(X\Pi_+)^{\mathrm{T}_1}\|_1 + \|(X\Pi_-)^{\mathrm{T}_1}\|_1 &= \|X_c^{\mathrm{T}_1}\Pi_+\|_1 + \|X_a^{\mathrm{T}_1}\Pi_-\|_1 + \|X_c^{\mathrm{T}_1}\Pi_-\|_1 + \|X_a^{\mathrm{T}_1}\Pi_+\|_1\\ &= (\|X_c^{\mathrm{T}_1}\Pi_+\|_1 + \|X_c^{\mathrm{T}_1}\Pi_-\|_1) + (\|X_a^{\mathrm{T}_1}\Pi_-\|_1 + \|X_a^{\mathrm{T}_1}\Pi_+\|_1)\\ &= \|X_c^{\mathrm{T}_1}\Pi_+ + X_c^{\mathrm{T}_1}\Pi_-\|_1 + \|X_a^{\mathrm{T}_1}\Pi_- + X_a^{\mathrm{T}_1}\Pi_+\|_1\\ &= \|X_c^{\mathrm{T}_1}\|_1 + \|X_a^{\mathrm{T}_1}\|_1,\end{aligned}$$

where we used the property that $X_c^{\mathrm{T}_1}$ and $X_a^{\mathrm{T}_1}$ commute with $\Pi_+$ and $\Pi_-$. ■

From the proof of Lemma 1, one can see that Eq. (22) also holds for any phase partial transpose.

On the other hand, for a given fermionic state $\rho \in \mathcal{S}(\mathcal{H})$, one may write

$$\rho = \rho\,\Pi_+ + \rho\,\Pi_- = p_+\rho_+ + p_-\rho_-,$$

where $\rho_{\pm}$ are the normalized states of $\rho\Pi_{\pm}$ with probabilities $p_{\pm} = \mathrm{Tr}(\rho\Pi_{\pm})$. Now, we define

$$\mathcal{N}^s(\rho) := p_+\mathcal{N}_{\theta}(\rho_+) + p_-\mathcal{N}_{\theta}(\rho_-).$$

It is clear that $\mathcal{N}_{\theta}(\rho) = \mathcal{N}(\rho)$ for $\rho$ with $\rho_a = 0$; i.e., $\mathcal{N}_{\theta}(\rho)$ does not change with $\theta$ for such states.

Note that Eq. (21) holds for any phase partial transpose, which implies

$$\rho_c^{\mathrm{T}_1} = \rho_c^{\mathrm{T}_1}(\theta) = \frac{\rho^{\mathrm{T}_1}(\theta) + (-1)^{F_1}\rho^{\mathrm{T}_1}(\theta)(-1)^{F_1}}{2}$$

and

$$e^{i\theta}\rho_a^{\mathrm{T}_1} = \rho_a^{\mathrm{T}_1}(\theta) = \frac{\rho^{\mathrm{T}_1}(\theta) - (-1)^{F_1}\rho^{\mathrm{T}_1}(\theta)(-1)^{F_1}}{2},$$

where the first partial transpose is the ordinary one. By the convexity and unitary invariance of the trace norm, we see

$$\max\{\|\rho_c^{\mathrm{T}_1}\|_1, \|\rho_a^{\mathrm{T}_1}\|_1\} \leqslant \|\rho^{\mathrm{T}_1}(\theta)\|_1,$$

which provides an upper bound and a lower bound of the phase negativity $\mathcal{N}_{\theta}$ by

$$\mathcal{N}(\rho_c) \leqslant \mathcal{N}_{\theta}(\rho) \leqslant \mathcal{N}^s(\rho). \tag{23}$$

It is surprising that $\mathcal{N}^s$ is independent of $\theta$.

*Proposition 2.* The equality

$$\mathcal{N}^s(\rho) = \mathcal{N}(\rho_c) + \tfrac{1}{2}\|\rho_a^{\mathrm{T}_1}\|_1 \tag{24}$$

holds for all $\rho \in \mathcal{S}(\mathcal{H})$, where $\mathcal{N}(\rho_c)$ is the ordinary entanglement negativity of $\rho_c$ and

$$\|\rho_a^{\mathrm{T}_1}\|_1 = \left\|\sum_{L,L'\subseteq J_1:|L|+|L'|=\text{odd}} C_{L'}^{\dagger}\bar{P}_{J_1}C_L\rho\, C_{L'}^{\dagger}\bar{P}_{J_1}C_L\right\|_1.$$

*Proof.* Equation (22) implies that

$$\frac{\|(\rho\Pi_+)^{\mathrm{T}_1}\|_1 - p_+ + \|(\rho\Pi_-)^{\mathrm{T}_1}\|_1 - p_-}{2} = \frac{\|\rho_c^{\mathrm{T}_1}\|_1 - 1 + \|\rho_a^{\mathrm{T}_1}\|_1 - 0}{2}.$$

By the definition of $\mathcal{N}^s$, Eq. (24) holds. ■

The following result reveals that $\mathcal{N}^s$ is also a quantification of the fermionic positive partial transpose (PPT) property.

*Proposition 3.* For any $\theta \in \mathbb{R}$, with $\theta \neq k\pi$, $k \in \mathbb{Z}$, $\mathcal{N}_\theta(\rho) = 0$ if and only if $\mathcal{N}^s(\rho) = 0$.

*Proof.* The "if" part is clear from Eq. (23).

For the "only if" part, assume $\mathcal{N}_\theta(\rho) = 0$. Then, by Theorem 3, $\rho$ cannot have a mixed local fermion-number parity, that is, $\rho_a = 0$. Thus, from Eq. (24), we see that $\mathcal{N}^s(\rho) = \mathcal{N}(\rho) = \mathcal{N}_\theta(\rho) = 0$. This completes the proof. ■

The above proposition leads to a result which is parallel to the known fact that $\rho$ is separable if and only if both $\rho_+$ and $\rho_-$ are separable.

*Proposition 4.* For any nontrivial phase partial transpose, a fermionic state $\rho$ is phase PPT if and only if both $\rho_+$ and $\rho_-$ are phase PPT.

In addition, we can provide upper and lower bounds for fermionic entanglement negativity of a special class consisting of those states $\rho$ with $\rho_c$ PPT, which widely occurs in the case of $(1 + M)$-mode partition.

*Proposition 5.* For any finite-mode bipartite fermionic state $\rho$ with $\rho_c$ PPT, the following inequalities hold:

$$\frac{\sqrt{1 + \|\rho_a^{\mathrm{T}_1}\|_1^2} - 1}{2} \leqslant \mathcal{N}_{\frac{\pi}{2}}(\rho) \leqslant \frac{1}{2}\left\|\rho_a^{\mathrm{T}_1}\right\|_1. \tag{25}$$

*Proof.* The right side of the inequality is clear from Eq. (24). For the left, we use the mathematical result: If $X = X_1 + \mathrm{i}X_2$ is an operator in the trace class with $X_1$ being positive and $X_2$ being Hermitian, then $\|X\|_1^2 \geqslant \|X_1\|_1^2 + \|X_2\|_1^2$ (see Ref. [44], Theorem 1.2). ■

To illustrate how to use Eqs. (20) and (25) to estimate $\mathcal{N}_{\frac{\pi}{2}}$, we give an example here.

Consider the special case in which $J_1 = \{1\}$. The key terms in Eq. (20) that need to be calculated are

$$C_{L'}^\dagger \bar{P}_{J_1} C_L = \begin{cases} f_1 f_1^\dagger, & L = L' = \emptyset, \\ f_1^\dagger f_1, & L = L' = J_1, \\ f_1^\dagger, & L = \emptyset, L' = J_1, \\ f_1, & L = J_1, L' = \emptyset, \end{cases}$$

and thus,

$$\rho_c^{\mathrm{T}_1} = f_1 f_1^\dagger \rho f_1 f_1^\dagger + f_1^\dagger f_1 \rho f_1^\dagger f_1,$$
$$\rho_a^{\mathrm{T}_1} = -i(f_1^\dagger \rho f_1^\dagger + f_1 \rho f_1).$$

Hence, $\rho_c^{\mathrm{T}_1}$ is positive, and

$$\begin{aligned}\left\|\rho_a^{\mathrm{T}_1}\right\|_1 &= \|f_1^\dagger \rho f_1^\dagger + f_1 \rho f_1\|_1 \\ &= \|(f_1^\dagger + f_1)(f_1^\dagger \rho f_1^\dagger + f_1 \rho f_1)(f_1^\dagger + f_1)\|_1 \\ &= \|f_1 f_1^\dagger \rho f_1^\dagger f_1 + f_1^\dagger f_1 \rho f_1 f_1^\dagger\|_1 \\ &= \|f_1 f_1^\dagger \rho f_1^\dagger f_1\|_1 + \|f_1^\dagger f_1 \rho f_1 f_1^\dagger\|_1,\end{aligned}$$

where we used the property that $\{f_1^\dagger f_1, f_1 f_1^\dagger\}$ forms a complete set of orthogonal projections and $f_1^\dagger + f_1$ is a unitary operator. Meanwhile, because

$$\begin{aligned}\|f_1 f_1^\dagger \rho f_1^\dagger f_1\|_1 &= \|f_1^\dagger f_1 \rho f_1 f_1^\dagger\|_1 \\ &= \|(f_1^\dagger + f_1) f_1^\dagger f_1 \rho f_1 f_1^\dagger (f_1^\dagger + f_1)\|_1 \\ &= \|f_1 \rho f_1\|_1,\end{aligned} \tag{26}$$

one gets $\|\rho_a^{\mathrm{T}_1}\|_1 = 2\|f_1 \rho f_1\|_1$. It follows from Eq. (24) that

$$\mathcal{N}^s(\rho) = \|f_1 \rho f_1\|_1.$$

Thus, applying Eq. (25) gives

$$\frac{\sqrt{1 + 4\|f_1 \rho f_1\|_1^2} - 1}{2} \leqslant \mathcal{N}_{\frac{\pi}{2}}(\rho) \leqslant \|f_1 \rho f_1\|_1. \tag{27}$$

Moreover, from the Hölder inequality, we see that

$$\|f_1 \rho f_1\|_1 = \left\|f_1 \rho^{\frac{1}{2}} \rho^{\frac{1}{2}} f_1\right\|_1 \leqslant \left\|f_1 \rho^{\frac{1}{2}}\right\|_2 \left\|\rho^{\frac{1}{2}} f_1\right\|_2,$$

where $\|A\|_2 = [\mathrm{Tr}(A^\dagger A)]^{\frac{1}{2}}$ is the Hilbert-Schmidt norm, and with the mean inequality, we have

$$\|f_1 \rho f_1\|_1 \leqslant \frac{\mathrm{Tr}(\rho f_1^\dagger f_1) + \mathrm{Tr}(\rho f_1 f_1^\dagger)}{2} = \frac{1}{2}.$$

Equation (27) provides a relatively accurate estimate for $\mathcal{N}_{\frac{\pi}{2}}$ in the case when $J_1$ contains one mode.

If we denote the difference between the upper bound and lower bound of Eq. (27) by

$$\Delta(\rho) = \|f_1 \rho f_1\|_1 - \frac{\sqrt{1 + 4\|f_1 \rho f_1\|_1^2} - 1}{2},$$

then the monotonic increasing property of the function

$$f(t) = 1 + t - \sqrt{1 + t^2}$$

implies that

$$\Delta(\rho) = \frac{f(2\|f_1 \rho f_1\|_1)}{2} \leqslant \frac{f(1)}{2} = \frac{2 - \sqrt{2}}{2}.$$

Equation (27) also implies that the following conditions are equivalent: (1) $\mathcal{N}_{\frac{\pi}{2}}(\rho) = 0$, (2) $f_1 \rho f_1 = 0$, (3) $[\rho, (-1)^{f_1^\dagger f_1}] = 0$, and (4) $\rho$ is separable [see Eq. (26)]. Thus, we have proved the following results.

*Proposition 6.* Consider the $N$-mode fermionic system with annihilation operators $\{f_j\}_{j=1}^N$ and a bipartition of modes $J_1 = \{j_0\}$ and $J_2 = \{j : j \neq j_0\}$. Then the inequalities

$$\frac{\sqrt{1 + 4\left\|f_{j_0} \rho f_{j_0}\right\|_1^2} - 1}{2} \leqslant \mathcal{N}_{\frac{\pi}{2}}(\rho) \leqslant \left\|f_{j_0} \rho f_{j_0}\right\|_1$$

hold for all $N$-mode fermionic states $\rho$. In addition, the following statements are equivalent:

(1) $\rho$ is separable.
(2) $\mathcal{N}_{\frac{\pi}{2}}(\rho) = 0$.
(3) $[\rho, (-1)^{f_{j_0}^\dagger f_{j_0}}] = 0$.
(4) $f_{j_0} \rho f_{j_0} = 0$.

We point out that the fact (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) was also obtained in [18]. But our approaches here are different and seem simpler.

At the end of this section, we remark that $\mathcal{N}^s$ has many properties similar to the fermionic negativity $\mathcal{N}_{\frac{\pi}{2}}$, such as Proposition 3 and monotonicity. $\mathcal{N}^s$ is independent of the choice of the partial transpose (fermionic or not) and is a sum of the symmetric and asymmetric parts relative to the local parity operator. In many situations, $\mathcal{N}^s(\rho)$ is easier to calculate [see the $(1+M)$-mode case]. So it may be more convenient in some scenarios to utilize the symmetry-resolved negativity $\mathcal{N}^s$ in place of the fermionic entanglement negativity $\mathcal{N}_{\frac{\pi}{2}}$.

## V. CONCLUSION

Finally, we give a brief conclusion and discussion.

Since the ordinary partial transpose and the entanglement negativity do not work well for fermionic systems, the concept of fermionic partial transpose and the corresponding entanglement negativity were proposed in [15]. Under these notions, some results similar to the ordinary entanglement negativity were obtained in [18] for $(1+M)$-mode fermionic systems with $M<\infty$, and a conjecture for general $(N+M)$-mode systems was raised which states that the states which mix local fermion-number parity must have a nonvanishing fermionic entanglement negativity.

To answer the conjecture, we generalized the notion of the fermionic partial transpose to the phase partial transpose $A^{\mathrm{T}_1}(\theta)$ of physical operators $A$ with any real number $\theta$, and the fermionic partial transpose was the same as $A^{\mathrm{T}_1}(\pi/2)$. Accordingly, we introduced the phase entanglement negativity $\mathcal{N}_\theta(\rho)$. We discussed the behavior of positive linear maps and established a sufficient criterion for a bipartite fermionic state to be entangled. This enabled us to show that every fermionic state which mixes local fermion-number parity must have nonvanishing nontrivial phase entanglement negativity and, in particular, gives an affirmative answer to the conjecture mentioned above. In addition, we proved that the phase entanglement negativity is an entanglement monotone and established some equalities and inequalities related to the phase entanglement negativity which, in particular, provide some upper bounds and lower bounds of the fermionic entanglement negativity. Furthermore, we introduced a symmetry-resolved entanglement negativity $\mathcal{N}^s$, which is an upper bound of $\mathcal{N}_\theta$, has properties similar to $\mathcal{N}_\theta$, and is independent of the choice of $\theta$. We also provided some interesting equalities and inequalities concerning negativities, in particular, for the $(1+M)$-mode case. These relationships can easily reflect the results obtained in [18].

Some interesting questions remain. The positive map criterion of entanglement for distinguishable systems is not valid for fermionic systems any longer. Theorem 1 in the present paper suggests a possible version of the positive map criterion for fermions and is worth pursuing. Also, what does a reasonable fermionic entanglement witness criterion looks like?

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## APPENDIX: A PROOF OF THEOREM 4

In this Appendix, we give a proof of Theorem 4, i.e., the monotonicity for the phase entanglement negativity. We need two lemmas.

*Lemma 2.* Let $\mathcal{H}=\mathcal{H}^A\otimes\mathcal{H}^B$ be a fermionic Fock space with $\dim\mathcal{H}^B<\infty$ and $\rho\in\mathcal{S}(\mathcal{H}^A\otimes\mathcal{H}^B)$ be a fermionic state. Then for any operators $X^A,Y^A\in\mathcal{G}_r(\mathcal{H}^A)$ and $X^B,Y^B\in\mathcal{G}_s(\mathcal{H}^B)$, with $r,s\in\{0,1\}$, and every $\theta\in\mathbb{R}$,

$$\begin{aligned}&[(X^A\otimes X^B)\rho(Y^A\otimes Y^B)]^{\mathrm{T}_1}(\theta)\\&\quad=[(Y^A)^{\mathrm{T}}\otimes X^B]\rho^{\mathrm{T}_1}(\theta)[(X^A)^{\mathrm{T}}\otimes Y^B],\end{aligned}$$

where $\rho^{\mathrm{T}_1}(\theta)$ is the phase partial transpose of $\rho$ with respect to the real number $\theta$.

*Proof.* Assume that $\rho\in\mathcal{S}(\mathcal{H}^A\otimes\mathcal{H}^B)$ is a fermionic state. First, one may write

$$\rho=\textstyle\sum_k Z_{0,k}^A\otimes Z_k^B+\sum_k Z_{1,k}^A\otimes Z_k^B,$$

where $Z_{0,k}^A\in\mathcal{G}_0(\mathcal{H}^A)$ and $Z_{1,k}^A\in\mathcal{G}_1(\mathcal{H}^A)$. Then

$$\begin{aligned}&\big[(X^A\otimes X^B)\big(Z_{0,k}^A\otimes Z_k^B\big)(Y^A\otimes Y^B)\big]^{\mathrm{T}_1}(\theta)\\&\quad=[(Y^A)^{\mathrm{T}}\otimes X^B]\big[\big(Z_{0,k}^A\big)^{\mathrm{T}}\otimes Z_k^B\big][(X^A)^{\mathrm{T}}\otimes Y^B]\end{aligned}$$

and

$$\begin{aligned}&\big[(X^A\otimes X^B)\big(Z_{1,k}^A\otimes Z_k^B\big)(Y^A\otimes Y^B)\big]^{\mathrm{T}_1}(\theta)\\&\quad=[(Y^A)^{\mathrm{T}}\otimes X^B]\big[\mathrm{e}^{\mathrm{i}\theta}\big(Z_{1,k}^A\big)^{\mathrm{T}}\otimes Z_k^B\big][(X^A)^{\mathrm{T}}\otimes Y^B].\end{aligned}$$

As

$$\begin{aligned}&\big(Z_{0,k}^A\otimes Z_k^B+Z_{1,k}^A\otimes Z_k^B\big)^{\mathrm{T}_1}(\theta)\\&\quad=\big(Z_{0,k}^A\big)^{\mathrm{T}}\otimes Z_k^B+\mathrm{e}^{\mathrm{i}\theta}\big(Z_{1,k}^A\big)^{\mathrm{T}}\otimes Z_k^B,\end{aligned}$$

by the linearity of the phase partial transpose, this completes the proof. ■

*Lemma 3.* Let $\mathcal{H}=\mathcal{H}^A\otimes\mathcal{H}^B$ be a fermionic Fock space with $\dim\mathcal{H}^B<\infty$ and $\rho\in\mathcal{S}(\mathcal{H}^A\otimes\mathcal{H}^B)$ be a fermionic state. Then for any sequences of operators $\{X_k^A\}_{k=1}^\infty\subset\mathcal{G}_r(\mathcal{H}^A)$ and $\{X_k^B\}_{k=1}^\infty\subset\mathcal{G}_s(\mathcal{H}^B)$, with $\|\sum_{k=1}^\infty(X_k^A\otimes X_k^B)^\dagger(X_k^A\otimes X_k^B)\|<\infty$, $r,s\in\{0,1\}$, and any $\theta\in\mathbb{R}$, we have

$$\begin{aligned}&\left\|\sum_{k=1}^\infty\big[\big(X_k^A\otimes X_k^B\big)\rho\big(X_k^A\otimes X_k^B\big)^\dagger\big]^{\mathrm{T}_1}(\theta)\right\|_1\\&\quad\leqslant\sum_{k=1}^\infty\big\|\big[\big(X_k^A\otimes X_k^B\big)\rho\big(X_k^A\otimes X_k^B\big)^\dagger\big]^{\mathrm{T}_1}(\theta)\big\|_1\\&\quad\leqslant\left\|\left[\sum_{k=1}^\infty\big(X_k^A\otimes X_k^B\big)^\dagger\big(X_k^A\otimes X_k^B\big)\right]^{\mathrm{T}_1}(\theta)\right\|\|\rho^{\mathrm{T}_1}(\theta)\|_1,\end{aligned}$$

where $\|\cdot\|$ is the operator norm and $\|\cdot\|_1$ is the trace norm.

*Proof.* By the convexity of the trace norm, the first inequality is clear. For the second, by Lemma 2, we see

$$\begin{aligned}
&\sum_{k=1}^{\infty}\left\|\left[(X_k^A\otimes X_k^B)\rho(X_k^A\otimes X_k^B)^{\dagger}\right]^{\mathrm{T}_1}(\theta)\right\|_1\\
&=\sum_{k=1}^{\infty}\left\|\{[(X_k^A)^{\dagger}]^{\mathrm{T}}\otimes X_k^B\}\rho^{\mathrm{T}_1}(\theta)[(X_k^A)^{\mathrm{T}}\otimes(X_k^B)^{\dagger}]\right\|_1\\
&=\left\|\begin{bmatrix}\{[(X_1^A)^{\dagger}]^{\mathrm{T}}\otimes X_1^B\}\rho^{\mathrm{T}_1}(\theta)[(X_1^A)^{\mathrm{T}}\otimes(X_1^B)^{\dagger}] & \cdots & 0 & \cdots\\ \vdots & \ddots & \vdots & \vdots\\ 0 & \cdots & \{[(X_k^A)^{\dagger}]^{\mathrm{T}}\otimes X_k^B\}\rho^{\mathrm{T}_1}(\theta)[(X_k^A)^{\mathrm{T}}\otimes(X_k^B)^{\dagger}] & \cdots\\ \vdots & & \vdots & \ddots\end{bmatrix}\right\|_1\\
&\leqslant\left\|\begin{bmatrix}\{[(X_1^A)^{\dagger}]^{\mathrm{T}}\otimes X_1^B\}\rho^{\mathrm{T}_1}(\theta)[(X_1^A)^{\mathrm{T}}\otimes(X_1^A)^{\dagger}] & \cdots & \{[(X_1^A)^{\dagger}]^{\mathrm{T}}\otimes X_1^B\}\rho^{\mathrm{T}_1}(\theta)[(X_k^A)^{\mathrm{T}}\otimes(X_k^B)^{\dagger}] & \cdots\\ \vdots & \ddots & \vdots & \\ \{[(X_k^A)^{\dagger}]^{\mathrm{T}}\otimes X_k^B\}\rho^{\mathrm{T}_1}(\theta)[(X_1^A)^{\mathrm{T}}\otimes(X_1^B)^{\dagger}] & \cdots & \{[(X_k^A)^{\dagger}]^{\mathrm{T}}\otimes X_k^B\}\rho^{\mathrm{T}_1}(\theta)[(X_k^A)^{\mathrm{T}}\otimes(X_k^B)^{\dagger}] & \cdots\\ \vdots & & \vdots & \ddots\end{bmatrix}\right\|_1\\
&=\left\|\begin{bmatrix}[(X_1^A)^{\dagger}]^{\mathrm{T}}\otimes X_1^B\\ \vdots\\ [(X_k^A)^{\dagger}]^{\mathrm{T}}\otimes X_k^B\\ \vdots\end{bmatrix}\rho^{\mathrm{T}_1}(\theta)\left[(X_1^A)^{\mathrm{T}}\otimes(X_1^B)^{\dagger}\ \cdots\ (X_k^A)^{\mathrm{T}}\otimes(X_k^B)^{\dagger}\ \cdots\right]\right\|_1\\
&\leqslant\left\|\sum_{k=1}^{\infty}[(X_k^A)^{\dagger}X_k^A]^{\mathrm{T}}\otimes(X_k^B)^{\dagger}X_k^B\right\|\|\rho^{\mathrm{T}_1}(\theta)\|_1=\left\|\left[\sum_{k=1}^{\infty}(X_k^A\otimes X_k^B)^{\dagger}(X_k^A\otimes X_k^B)\right]^{\mathrm{T}_1}(\theta)\right\|\|\rho^{\mathrm{T}_1}(\theta)\|_1,
\end{aligned}$$

where we used the properties of the trace norm $\|XYX^{\dagger}\|_1\leqslant\|X\|\|Y\|_1\|X^{\dagger}\|=\|X^{\dagger}X\|\|Y\|_1$ and $\|\sum_k P_kXP_k\|_1\leqslant\|X\|_1$ for any set of mutually orthogonal projections $\{P_k\}$ (see [45], Theorem 5.1). ■

*Proof of Theorem 4.* Now, consider physical LOCC transformations:

$$\rho\mapsto\sum_k(X_k^A\otimes X_k^B)\rho(X_k^A\otimes X_k^B)^{\dagger},$$

where the Kraus operators $X_k^A\otimes X_k^B$ satisfy $\sum_k(X_k^A\otimes X_k^B)^{\dagger}(X_k^A\otimes X_k^B)=I$ and $X_k^A\in\mathcal{G}_r(\mathcal{H}^A)$ and $X_k^B\in\mathcal{G}_s(\mathcal{H}^B)$, with $r,s\in\{0,1\}$. By Lemma 3 and the fact that $I^{\mathrm{T}_1}(\theta)=I$, we can conclude that the phase entanglement negativity is nonincreasing under such transformations; i.e., the phase entanglement negativity is an entanglement monotone. ■


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