Quantum resource theory of coding for error correction

Dong-Sheng Wang [1,2,*] Yuan-Dong Liu,[1,2,†] Yun-Jiang Wang,[3,4,5,‡] and Shunlong Luo[6,§]

[1] *CAS Key Laboratory of Theoretical Physics, Institute of Theoretical Physics, Chinese Academy of Sciences, Beijing 100190, China*
[2] *School of Physical Sciences, University of Chinese Academy of Sciences, Beijing 100049, China*
[3] *School of Telecommunication Engineering, Xidian University, Xi'an, Shann Xi 710071, China*
[4] *Guangzhou Institute of Technology, Xidian University, Guangzhou 510555, China*
[5] *Hangzhou Institute of Technology, Xidian University, Hangzhou, Zhejiang 311231, China*
[6] *Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, China*

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Error-correction codes are central for fault-tolerant information processing. Here we develop a rigorous framework to describe various coding models based on quantum resource theory of superchannels. We find, by treating codings as superchannels, a hierarchy of coding models can be established, including the entanglement assisted or unassisted settings, and their local versions. We show that these coding models can be used to classify error-correction codes and accommodate different computation and communication settings depending on the data type, side channels, and pre-/postprocessing. We believe the coding hierarchy could also inspire new coding models and error-correction methods.

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I. INTRODUCTION

To protect information against noises, error-correction codes are needed [1]. This is important for both reliable computation and communication. A fundamental quantity of a noise channel is its capacity, which is the maximal rate of information transmission. Shannon's classical channel coding theorem shows that the capacity is given by the maximal mutual information $I(A:B)$ between the sender Alice, A, and receiver Bob, B, over all possible inputs [2]. Quantum noises are mathematically modeled as quantum channels, and a variety of capacities can be defined [3–6].

A notable difference between the classical and quantum cases are the nonadditivity of some channel capacities [7,8], leading to many interesting phenomena [9,10]. The primary setting of quantum communication is the direct transmission of quantum data, with the asymptotic coherent information as the measure of its quantum capacity [4]. Many efforts have been devoted to understand this. For instance, it has been shown that weakly noisy channels are limited in their nonadditivity [11], and many nonadditive examples have been constructed [12]. It was found that adding forward classical communication does not make a difference, but the back communication can increase its quantum capacity [13]. Meanwhile, other quantities have also been explored, such as the reverse coherent information [14] and Rains information [15].

In recent years, the quantum resource theory (QRT) has become a unifying framework to characterize quantum features [16]. A notable example is bipartite entanglement, for which the bipartite separable states are free, i.e., of zero resource, and stochastic local operations with classical communication are free operations that cannot increase entanglement [17]. For quantum computation, we recently developed the QRT of modeling and logical gates [18–20], which can be used to classify universal quantum computing models and predict new ones.

There have been a few resource-theoretic approaches for quantum communication, e.g., Refs. [21–24], and they are mostly QRTs of channels [16]. In these approaches, some channels are identified as useless for communication tasks, such as the set of replacement channels [22] and unitary channels, with maximal quantum capacity for a given dimension, will be the most resourceful. However, this cannot identify the codings that achieve a quantum capacity of a channel as resources. This motivates us to consider the QRT of codings, which are actually superchannels [25].

In this paper, we develop a QRT of codings which are relevant to quantum communication and error correction, and especially can be applied to various situations. A coding, including a pair of encoding and decoding operations, is a superchannel which converts a channel into another. A free set is the codings that do not work well for a given noise channel, and resources are those that indeed work. The one-side computational ability of Alice and Bob also matters, and different codes can be chosen for different purposes [1]. We find this can be characterized by a hierarchical family of coding models.

We develop a hierarchy of coding models that includes the standard settings for quantum communication and the entanglement-assisted (EA) one [7], and also our two recent refined models [26]. The hierarchy is defined according to two notions of locality, a one-side computational locality, relevant to Alice's and Bob's computational ability, and a communication locality, relevant to the entanglement assistance. The

*Contact author: wds@itp.ac.cn
†Contact author: liuyuandong@itp.ac.cn
‡Contact author: yunjiangw@xidian.edu.cn
§Contact author: luosl@amt.ac.cn

TABLE I. A table of the channel capacities studied and mentioned in this paper.

	Model I	Model II	Model III	Model IV
Quantum	$Q_{\text{I}} = I(\Phi)$	$Q_{\text{II}} = [\log_2 d + I(\Phi)]/2$	$Q_{\text{III}} = \lim_{n\to\infty} \frac{I_c(\Phi^{\otimes n})}{n}$	$Q_{\text{IV}} = J(\Phi)/2$
Classical	$C_{\text{I}} = \chi_{\text{ORT}}(\Phi)$	$C_{\text{II}} = \log_2 d + I(\Phi)$	$C_{\text{III}} = \lim_{n\to\infty} \frac{\chi(\Phi^{\otimes n})}{n}$	$C_{\text{IV}} = J(\Phi)$
Private	$P_{\text{I}} = \Delta C_{\text{I}}$	$P_{\text{II}} = [\log_2 d + I(\Phi)]/2$	$P_{\text{III}} = \Delta C_{\text{III}}$	$P_{\text{IV}} = J(\Phi)/2$

hierarchy has a nontrivial structure. This can be seen by their capacities. A capacity can be understood as a measure of the maximal resource of codings in a coding model. The quantum capacities of a channel Φ are $I(\Phi)$, $(\log_2 d + I(\Phi))/2$, $I_c(\Phi)$, and $J(\Phi)/2$, respectively, for models from I–IV, and for $I(\Phi)$ as the coherent information with the maximally mixed state as input, $I_c(\Phi)$ denoting the asymptotic coherent information, and $J(\Phi)$ as the quantum mutual information (see Table I). However, $I_c(\Phi)$ is nonadditive and there is no definite order between $I_c(\Phi)$ and $(\log_2 d + I(\Phi))/2$. Therefore, models I, II, and IV form a subhierarchy, and models I, III, and IV also form a subhierarchy. Moreover, there could also be more models in this coding family based on our study of the gaps among those capacities.

This paper contains the following parts. Section II provides the necessary background and Sec. III introduces our definition of coding models. The quantum capacities of these four models are studied in Sec. IV, and the resource theory of them are established in Sec. V. We also carry out numerical simulation to study the gaps among these quantum capacities for the qubit case in Sec. VI. We conclude in Sec. VII.

II. PRELIMINARIES

A. Quantum channels and superchannels

A quantum channel Φ acting on a finite-dimensional Hilbert space $\mathcal{X}$ is a completely positive, trace-preserving (CPTP) map of the form

$$\Phi(\rho) = \sum_i K_i \rho K_i^\dagger, \tag{1}$$

with K_i known as Kraus operators satisfying $\sum_i K_i^\dagger K_i = \mathbb{1}_d$ (identity operator of dimension $d = \dim\mathcal{X}$), and states $\rho \in \mathcal{D}(\mathcal{X})$ as trace-class nonnegative semidefinite operators. We often ignore the subscript of $\mathbb{1}_d$ if there is no confusion. Also, we will use π_d to denote the completely mixed state of dimension d, and simply as π if the dimension is implicit. The above formalism can also be extended to describe channels $\Phi \in C(\mathcal{X}, \mathcal{Y})$ that do not necessarily preserve dimension.

A different representation of channel Φ is an isometry V with $\Phi(\rho) = \text{tr}_{\text{E}} V \rho V^\dagger$ for E denoting an environment (or Eve). The isometry V is formed by Kraus operators $V = \sum_i |i\rangle \otimes K_i$ for $|i\rangle$ as orthogonal states of E. The isometry V forms a part of a unitary circuit U with $V = U|0\rangle$ for $|0\rangle$ as a state of E. A complementary channel Φ^c can be defined as the map from the input to E, and the state of E is

$$\rho_{\text{E}} = \sum_{ij} \text{tr}(\rho K_i^\dagger K_j)|j\rangle\langle i|. \tag{2}$$

The notable channel-state duality [27–30] also enables the representation of Φ as a state

$$\omega_\Phi := (\Phi \otimes \mathbb{1})(\omega), \tag{3}$$

usually known as a Choi state, for $\omega := |\omega\rangle\langle\omega|$, and $|\omega\rangle := \sum_i |ii\rangle/\sqrt{d}$ is known as the ebit. The rank of the channel $\text{r}(\Phi)$ is the rank of ω_Φ.

The operations on Choi states are further known as quantum superchannels [25]. For notation, we use a hat on the symbols for superchannels. The circuit representation of a superchannel is

$$\hat{\mathcal{S}}(\Phi)(\rho) = \text{tr}_a V\ \Phi\ U(\rho \otimes |0\rangle\langle 0|). \tag{4}$$

Here U and V are unitary operators, and a is an ancilla. See Fig. 1 for an illustration. The dimension of V can be larger than U [31], but we do not need the details here. It is clear that higher-order superchannels can also be defined by an iterative use of the channel-state duality, and this will be used for our resource theory of superchannels.

B. Quantum resource theory

We follow standard QRT [16] and our recent framework [18–20]. Given a set $\mathcal{D}$, a resource theory on it is defined by a tuple

$$(\mathcal{F}, \mathcal{O}, \mathcal{R}), \tag{5}$$

with $\mathcal{F} \subset \mathcal{D}$ as a set of free elements, $\mathcal{O} : \mathcal{F} \to \mathcal{F}$ the set of free operations, and $\mathcal{R} := \mathcal{D} \backslash \mathcal{F}$ the set of resources [16]. The framework of QRT is quite broad. For instance, set $\mathcal{D}$ can be the set of density operators acting on a Hilbert space or a set of unitary operators. A measure of resource can be defined but here we do not need to review it.

A universal resource theory is further defined as

$$(\mathcal{F}, \mathcal{O}, \mathcal{R}, \mathcal{U}), \tag{6}$$

with an additional set $\mathcal{U} \subset \mathcal{R}$ as the set of universal resource compared with a usual resource theory. The universality means that $\mathcal{O}(\mathcal{F} \otimes \mathcal{U})$ can simulate any other process $\mathcal{O}(\mathcal{F} \otimes \mathcal{R})$ efficiently. Here, efficiency means that the costs for the free operations $\mathcal{O}$, free elements $\mathcal{F}$, and universal resources $\mathcal{U}$ all do not grow exponentially fast with the size of the given

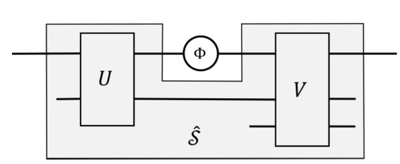

FIG. 1. A schematic of quantum superchannel $\hat{\mathcal{S}}$ (grey area) containing a pre U and a post V unitary operation (boxes) acting on a channel Φ (circle)

process. Another way to see this is that universal resource $\mathcal{U}$ optimizes the resource measure in $\mathcal{R}$.

For a given total set $\mathcal{D}$, we can define a hierarchy of resource theories. If $\mathcal{F}_1 \subset \mathcal{F}_2$ for two resource theories, then $\mathcal{O}_1 \subset \mathcal{O}_2$ and $\mathcal{R}_1 \supset \mathcal{R}_2$, i.e., more elements are treated as resourceful in the first theory. However, to achieve universality, more resource power is needed for the first theory, denoted as $\mathcal{U}_1 \succ \mathcal{U}_2$. Then there exist the following conversions between two universal resources:

$$(\mathcal{O}_2 \backslash \mathcal{O}_1) u_2 = u_1, \quad \mathcal{O}_1 u_1 = u_2 \tag{7}$$

for universal resource $u_{1,2} \in \mathcal{U}_{1,2}$, modulo free elements.

C. Entropy and distance measures

The von Neumann entropy is defined as

$$H(\rho) = \log_2 d - R(\rho \| \pi_d), \tag{8}$$

for the quantum relative entropy $R(\rho\|\sigma) = \text{tr}\rho \log_2 \rho - \text{tr}\rho \log_2 \sigma$, $\forall \rho, \sigma \in \mathcal{D}(\mathcal{X})$ (see Ref. [7] for more details). It is monotonic $H(\Phi(\rho)) \geqslant H(\rho)$ under unital channels Φ but not for nonunital ones. Actually, this is a simple example of a QRT, with π_d as the free set, unital channels as free operations, and $R(\rho\|\pi_d)$ as a measure of resource for all nonidentity states ρ.

Given a channel Φ, the coherent information $I_c(\rho, \Phi)$ is defined as [4]

$$I_c(\rho, \Phi) := H(\Phi(\rho)) - H(\Phi \otimes \mathbb{1}_d(|\varphi_\rho\rangle)) \tag{9}$$

for $|\varphi_\rho\rangle$ as a purification of ρ. The quantity

$$J(\rho, \Phi) := H(\rho) + I_c(\rho, \Phi)$$

is the quantum mutual information, which is always nonnegative. Let

$$I(\Phi) := I_c(\pi_d, \Phi), \tag{10}$$

which, plus $\log_2 d$, is the quantum mutual information contained in the Choi state ω_Φ. It is clear it is additive. Also, $J(\Phi) := \max_\rho J(\rho, \Phi)$ is additive [32] but $I_c(\Phi) := \max_\rho I_c(\rho, \Phi)$ is not. These quantities are used to define quantum capacities.

To quantify the distance between channels, we use the fidelity between Choi states,

$$F_E(\Phi, \Psi) := F(\Phi \otimes \mathbb{1}(\omega), \Psi \otimes \mathbb{1}(\omega)), \tag{11}$$

for the fidelity $F(\rho, \sigma) := \|\sqrt{\rho}\sqrt{\sigma}\|_1^2$, with $\|\cdot\|_1$ denoting the trace norm.

III. CODING MODELS

A coding task refers to the conversion of n uses of a noise channel Φ into k approximate uses of the identity channel. The noise channel may depend on some parameters, μ, but we will simply denote it as Φ. The value $r := k/n$ is called the coding rate for the encoding of k qubits into $n \geqslant k$ qubits. The operations on channels are, in general, superchannels; see Fig. 2.

Definition 1 (Quantum coding). A coding scheme for a channel Φ is a superchannel $\hat{\mathcal{S}}$ satisfying

$$F_E(\mathbb{1}^{\otimes k}, \hat{\mathcal{S}}(\Phi^{\otimes n})) \geqslant 1 - \epsilon, \tag{12}$$

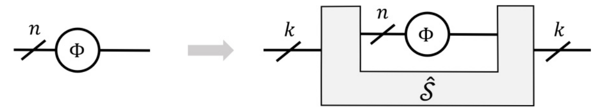

FIG. 2. A schematic of quantum coding with a superchannel $\hat{\mathcal{S}}$ serving as the coding operation converting n uses of a channel Φ into k uses which approximates the identity channel.

with $\epsilon \in [0, 1]$ and positive integers n and k.

Note a primary requirement on the coding is that the tight error bound ϵ should be smaller than $1 - F_E(\Phi^{\otimes k}, \mathbb{1}^{\otimes k})$, which we name the bare error of the channel. The coding scheme $\hat{\mathcal{S}}$ may also contain tunable parameters, λ. A code is approximate, in general, since the coding error ϵ is not zero but is called quasiexact [33] if $\epsilon(\mu, \lambda, k, n) \to 0$ occurs in the parameter space of (μ, λ, k, n). It becomes exact when ϵ is exactly zero, corresponding to the exact error-correction condition [34].

An important fact is that realizing superchannels requires an ancillary system which is assumed to be noise-free [31]. Therefore, to justify a coding scheme, it should fit into the physical settings properly. This actually leads to different types of channel capacities and the use of QRT to characterize them. Here we define four coding models via two notions of locality.

Definition 2 (Transmission locality). A quantum communication task from Alice to Bob is transmissionally local if there is no preshared entanglement between them.

When Alice and Bob both are multipartite and there is a separable partition of their subsystems, we can define a logical locality.

Definition 3 (Logical locality). A quantum communication task from Alice to Bob is logically local if there is no entanglement among the subsystems of Alice and the subsystems of Bob.

These localities can be well understood by treating them as variations of the locality or separability to define entanglement [17]. We can then introduce four coding models shown in Fig. 3, forming a coding family. Roughly speaking, model I is both spatially and logically local, model II is logically or, equivalently, spatially, semilocal, model III is spatially local but logically nonlocal, model IV is both spatially and logically nonlocal. There can also be other models and, in principle, there could be an indefinite number of models in the family. For instance, there are settings when model III is assisted by one-way or two-way classical communication [13].

Models I and II are defined in our recent paper [26], and was called the refined setting [26], but here we call it local for model I, and semilocal for model II. They can be treated as the local versions of models III and IV, respectively. Model III is the standard setting for quantum capacity [6] and model IV is the standard entanglement-assisted setting [5].

IV. QUANTUM CAPACITIES

In this section, we study quantum capacities in these four models. A quantum capacity can, in general, be defined as follows.

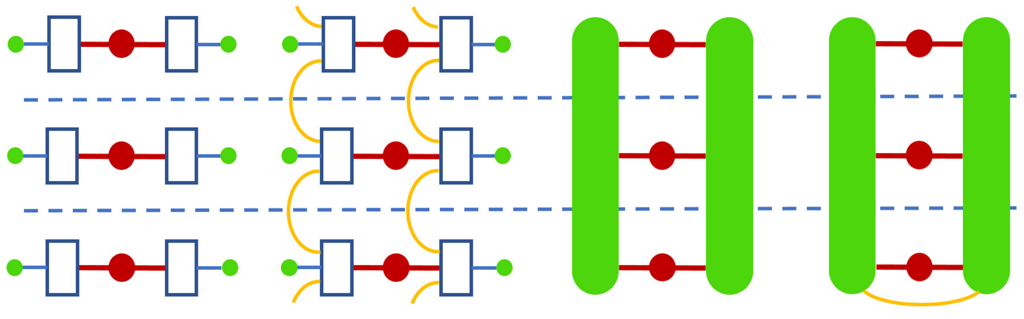

FIG. 3. Four models of quantum coding. (a) Model I: Spatially and logically local. The quantum capacity is given by $I(\Phi)$. (b) Model II: Bipartite correlation between code blocks are allowed, and both the encoding and decoding are logically semilocal. The correlation between code blocks within a player can simulate spatial correlation between corresponding code blocks for the two players. Here the correlation can be simply understood as ebits. The quantum capacity is given by $(I(\Phi) + \log_2 d)/2$. (c) Model III is the standard setting for quantum capacity. It is logically nonlocal. (d) Model IV is the standard setting for entanglement-assisted quantum capacity. It is both spatially and logically nonlocal. Notation: Red balls are a few parallel use of noise channels, blue boxes are local coding, the giant green boxes are nonlocal coding, and yellow curves are entanglement assistance.

Definition 4 (Quantum capacity of a channel). Let $\Phi \in C(\mathcal{X}, \mathcal{Y})$ be a channel and $k = \lfloor \alpha n \rfloor$ for all but finitely many positive integers n and an achievable rate $\alpha \geqslant 0$, there exists a coding superchannel $\hat{\mathcal{S}}$ defined in a coding model such that

$$F_E(\mathbb{1}^{\otimes k}, \hat{\mathcal{S}}(\Phi^{\otimes n})) \geqslant 1 - \epsilon \tag{13}$$

for every choice of a positive real number ϵ and the quantum capacity of Φ, denoted $Q(\Phi)$, is defined as the supremum of all α.

A. Models I and III

For models I and III, the coding splits into a pair of encoding $\mathcal{E}$ and decoding $\mathcal{D}$ operations. The quantum capacity in model III is the standard notion of quantum capacity and has been proven to equal to the entanglement generation capacity [6]. Model I is a special case of model III, and its capacity has been proven to be equal to an ebit-distribution capacity [26]. A feature of model I is that the encoding preserves identity $\mathcal{E}(\mathbb{1}^{\otimes k}) = \mathbb{1}^{\otimes n}$. This can be realized by a mixture of encoding isometry but with a random ancillary state, namely, $\mathrm{tr}_a U(\omega^{\otimes(n-k)} \otimes \mathbb{1}^{\otimes k})$ with $(n-k)$ pair of ebits, half of which are acted upon by U while the other half are traced out. As has been shown [26], this leads to a single-letter capacity for model I. Below we present the proofs of the two quantum capacities following a unified method. We will follow Ref. [7] and Theorem 2 below is Theorem 8.55 in it.

Theorem 1 (Quantum capacity theorem: Model I [26]). The quantum capacity of a channel Φ defined in model I is $Q_{\rm I}(\Phi) = I(\Phi)$.

Theorem 2 (Quantum capacity theorem: Model III [7]). The quantum capacity of a channel Φ defined in model III is $Q_{\rm III}(\Phi) = \lim_{n\to\infty} \frac{I_c(\Phi^{\otimes n})}{n}$.

Proof. Theorem 8.53 in Ref. [7] proves $I(\Phi) \leqslant Q_{\rm I}(\Phi)$, and together with Theorem 8.54 in Ref. [7] gives $I_c(\Phi^{\otimes n}) \leqslant nQ(\Phi)$. To prove $I_c(\Phi^{\otimes n}) \geqslant nQ_{\rm III}(\Phi)$, Theorem 8.55 in Ref. [7] refers to the entanglement generation scheme, namely, for any rate $\alpha \leqslant Q_{\rm III}(\Phi)$, there exists a state $|u\rangle \in \mathcal{X}^{\otimes n} \otimes \mathcal{Z}^{\otimes k}$ and a decoding channel $\mathcal{D} \in C(\mathcal{Y}^{\otimes n}, \mathcal{Z}^{\otimes k})$ such that

$$F(\omega^{\otimes k}, (\mathcal{D}\Phi^{\otimes n} \otimes \mathbb{1}^{\otimes k})(|u\rangle)) \geqslant 1 - \epsilon. \tag{14}$$

Then the result follows from

$$H(\mathcal{D}\Phi^{\otimes n} \otimes \mathbb{1}^{\otimes k}(|u\rangle)) \leqslant 2\delta m + 1 \tag{15}$$

and

$$H(\mathcal{D}\Phi^{\otimes n}(\rho)) \geqslant m - \delta m - 1, \tag{16}$$

and the data-processing inequality [4], for ρ as the reduced state of $|u\rangle$ on $\mathcal{X}^{\otimes n}$. To prove $I(\Phi) \geqslant Q_{\rm I}(\Phi)$, it specifies to ebit distribution with $|u\rangle$ replaced by $\omega^{\otimes k}$ and encoding with $\mathcal{E}(\mathbb{1}^{\otimes k}) = \mathbb{1}^{\otimes n}$. The above two inequalities become

$$H(\mathcal{D}\Phi^{\otimes n} \otimes \mathbb{1}^{\otimes k}(\omega^{\otimes k})) \leqslant 2\delta m + 1 \tag{17}$$

and

$$H(\mathcal{D}\Phi^{\otimes n}(\pi^{\otimes k})) \geqslant m - \delta m - 1, \tag{18}$$

which implies $I(\Phi) \geqslant Q_{\rm I}(\Phi)$. ■

By comparing the above two theorems, we see in model III it allows more general $|u\rangle$ and its reduced state ρ, instead of copies of ebit ω and the completely mixed state π. This reflects the difference between ebit distribution and entanglement generation. For the former, only products of ebits are allowed, corresponding to local coding schemes, while for the latter the state $|u\rangle$ can be multipartite entangled, corresponding to general nonlocal coding schemes.

Also, the preservation of identity is important for the proof of model-I capacity, which was implicitly used [26] but not emphasized. This relies on random encoding, which is not isometric but isometric encoding will suffice [35]. If a mixture of isometric encodings guarantees a high entanglement fidelity F_E, then each of the isometric encodings also works. This also applies to model III, but for models II and IV below, the encoding is not isometric if ignoring the noiseless entanglement assistance.

B. Models II and IV

Models II and IV are EA. In model II, due to the logical locality, only bipartite entanglement is allowed and the perfect resource is ebit. For model IV, any preshared entangled state is allowed. Note that in model II, the shared ebits are within each player, Alice or Bob. These preshared states can be understood as being generated by a preround of coding on it, and then we do not need to consider noises on it anymore. For notation, model II was simply denoted as EA in our previous paper [26]. A simple but important fact to verify is that this can be used to generate remote ebits between the two players by quantum teleportation. Also, remote ebits can be used to generate ebits at Alice's or Bob's side. Therefore, the logical semilocality is equivalent to spatial semilocality.

In the EA settings, an important phenomenon is that the quantum capacity is half of its classical capacity of a channel based on quantum teleportation and dense coding [7]. A technical part is the usage of EA Holevo quantity χ_{EA} for the standard EA setting [3], which is restricted to an orthogonal case, denoted EAO, due to the usage of ebit assistance [26]. The classical capacities are first expressed as Holevo quantities,

$$C_{\mathrm{IV}}(\Phi) = \lim_{n\to\infty} \frac{\chi_{\mathrm{EA}}(\Phi^{\otimes n})}{n},$$

and $C_{\mathrm{II}}(\Phi) = \chi_{\mathrm{EAO}}(\Phi)$, and then related to coherent information. For simplicity, we recall the two theorems below and reproduce a unified proof. Theorem 4 below is Theorem 8.41 in Ref. [7].

Theorem 3 (Quantum capacity theorem: Model II [26]). The quantum capacity of a channel Φ defined in model II is $Q_{\mathrm{II}}(\Phi) = (\log_2 d + I(\Phi))/2$.

Theorem 4 (Quantum capacity theorem: Model IV [5]). The quantum capacity of a channel Φ defined in model IV is $Q_{\mathrm{IV}}(\Phi) = J(\Phi)/2$.

Proof. In Lemma 8.39 of Ref. [7], it proves that

$$\chi_{\mathrm{EA}}(\Phi) \geqslant H(\pi_d) + I_c(\pi_d, \Phi) \tag{19}$$

for $\pi_d = \Pi/d$ and $d = \mathrm{tr}(\Pi)$, and Π is a projector. The proof actually proves the stronger result

$$\chi_{\mathrm{EAO}}(\Phi) \geqslant H(\pi_d) + I_c(\pi_d, \Phi), \tag{20}$$

as it uses a completely uniform ensemble of Bell states, η_*, which is an EAO ensemble. Applying Lemma 8.36 in Ref. [7] leads to $C_{\mathrm{II}}(\Phi) \geqslant \log_2 d + I(\Phi)$ and $C_{\mathrm{IV}}(\Phi) \geqslant J(\Phi)$.

Lemma 8.40 [7] applied to EA ensemble η gives

$$\chi(\Phi \otimes \mathbb{1}(\eta)) \leqslant H(\sigma) + I_c(\sigma, \Phi). \tag{21}$$

It also applies to any EAO ensemble η_{EAO} which becomes

$$\chi_{\mathrm{ORT}}(\Phi \otimes \mathbb{1}(\eta_{\mathrm{EAO}})) \leqslant \log_2 d + I(\Phi). \tag{22}$$

This leads to $C_{\mathrm{II}}(\Phi) \leqslant \log_2 d + I(\Phi)$ and $C_{\mathrm{IV}}(\Phi) \leqslant J(\Phi)$. This completes the proof. ■

Therefore, we established the quantum capacities for the four models above, with only the standard model III having a nonadditive measure of capacity. Due to the additivity of capacities for models I, II, and IV, the converse quantum Shannon theorem in these models are easy to prove by following well-established methods [15,36–38]. The capacity in each of the three models also serves as the strong converse capacity, meaning that once a rate is larger than a capacity, the coding error would converge exponentially fast to 1 in the asymptotic limit for all possible codings.

Our framework also applies to classical capacities and private capacities [8]. Here we will not reproduce the details [26]. The channel capacities studied in this paper are summarized in Table I. The quantity χ is the Holevo quantity [3] and χ_{ORT} here is the Holevo quantity when the classical-to-quantum encoding is restricted to being isometric [26]. The private capacities are equal to the quantum ones for the entanglement-assisted cases, and otherwise they are from the Holevo quantity of a channel minus its complementary channel, hence the notation Δ in the table.

V. QRT OF CODING

We now use QRT to characterize the coding models. As codings are superchannels, the QRT of codings is a QRT of superchannels. Due to the channel-state duality [28], it is not hard to formulate it by referring to QRTs of other kinds, especially the QRT of channels [16]. Here, the set of objects we consider are superchannels that are used for codings.

To define a family, we need to make sure the goal of each model is the same. For coding, the goal is indeed the same, which is to convert a channel into the perfect identity channel with high accuracy. The universal resources are the codings that achieve the capacity of a channel.

Definition 5 (QRT of codings). A QRT of codings $(\mathcal{F}, \mathcal{O}, \mathcal{R}, \mathcal{U})$ for a channel Φ is defined by a proper set of free superchannels $\mathcal{F}$ used in a coding model, which are transmissionally local and can only preserve or increase its bare error, a free set $\mathcal{O}: \mathcal{F} \to \mathcal{F}$, and the resource $\mathcal{R}$ is formed by all allowed superchannels that can decrease the bare error. The universal set $\mathcal{U} \subset \mathcal{R}$ contains the codings that achieve the capacity of a channel Φ in a coding model.

Furthermore, a hierarchy can be defined based on a subset structure of free sets. We say such coding models belong to a coding family. Model I has the largest $\mathcal{F}$ while mode IV has the smallest, but model I has the least powerful $\mathcal{U}$ while model IV has the most powerful $\mathcal{U}$. To prepare for the theorem below, we clarify a few points. All one-way classical communication from Alice to Bob is free in all the models. For model II, we say an operation is semilocal when a local operation can be assisted by bipartite entanglement. For model III, the back classical communication from Bob to Alice is not allowed, however, since it will change its capacity [13]. Although model III is logically nonlocal but spatially local, it is not comparable with model II, and this leads to two subhierarchies.

Theorem 5 (Hierarchy of coding models). Models I, III, and IV form a hierarchy and models I, II, and IV also form a hierarchy in the coding family.

Proof. We prove the theorem by the explicit construction of QRT for each coding model. Relative to a logical locality and the transmission locality, model I is defined by the free set $\mathcal{F}_{\mathrm{I}}$ which can only preserve or increase the bare error. For instance, all one-side processing at Alice's or Bob's side is free. The set $\mathcal{F}_{\mathrm{II}} \subset \mathcal{F}_{\mathrm{I}}$ is logically biseparable or semilocal, which preserves or increases the bare error. This selects some ebit-assisted semilocal codings as resources. The set $\mathcal{F}_{\text{III}} \subset \mathcal{F}_{\text{I}}$ is transmissionally local but without back classical communication, and logically being l local for $l \in o(n)$ as a small value compared with n. Finally, $\mathcal{F}_{\text{IV}} \subset \mathcal{F}_{\text{II}}, \mathcal{F}_{\text{III}}$ only allows transmissionally local and logically product operations. It is also clear there is no set relation between $\mathcal{F}_{\text{II}}$ and $\mathcal{F}_{\text{III}}$.

Meanwhile, $\mathcal{R}_{\text{I}} \subset \mathcal{R}_{\text{II}} \subset \mathcal{R}_{\text{IV}}$ and $\mathcal{R}_{\text{I}} \subset \mathcal{R}_{\text{III}} \subset \mathcal{R}_{\text{IV}}$. The subset structures can be shown case by case. For instance, while $\hat{\mathcal{S}} \in \mathcal{R}_1$ are the case of $Q_{\text{I}}(\Phi) \geqslant 0$, the case of $Q_{\text{I}}(\Phi) = 0$ leads to the existence of multipartite entangled codings $\hat{\mathcal{S}} \in \mathcal{R}_3 \cap \mathcal{F}_1$ or ebit-assisted codings $\hat{\mathcal{S}} \in \mathcal{R}_2 \cap \mathcal{F}_1$. Similar arguments also apply to other cases.

To verify the universal resource conversion (7), it is enough to observe that a coding that works at a higher level also works at a lower level and can be reduced to one that works at a lower level. For instance, by ignoring the assisted entangled state $|\eta\rangle$ in model IV, it reduces to model III. On the contrary, $|\eta\rangle$ can be prepared in model III and further used as the assistance in model IV. ■

A. Applications

The benefit of using QRT is to understand these models systematically, with the capacities as the measure of universal coding resources. It also highlights the role of local computational ability, and the trade-off between local computation and codings. The local computation belongs to their free sets. These models can be chosen for different practical situations. Suppose a communication task is to send a large amount of data, namely, highly entangled states, over a noisy channel. Below we analyze different strategies in these models.

(1) For model I, Alice and Bob have the largest pre-/postcomputational ability. Alice can represent the data $|\psi\rangle$ as a quantum circuit, U, with

$$|\psi\rangle = U|\psi_1, \psi_2, \dots, \psi_k\rangle. \tag{23}$$

Alice can use a classical channel to send the classical representation of the circuit, $[U]$, namely, the type and location of each elementary gate in it, and then only send unentangled qubits over the quantum channel to Bob. Bob has to perform U according to $[U]$ to obtain $|\psi\rangle$. Note U might be of high depths.

This scheme can be applied in many settings. For instance, for channels that are only slightly noisy so no powerful encoding is needed, or in blind quantum computation [39] when Alice wants to hide the input data from Bob but not the algorithm itself, or in teleportation-based models such as the cluster-state model [40] and quantum von Neumann architecture [41] wherein a computation is simulated by a sequence of gate teleportation on initially unentangled states.

(2) If the channel is quite noisy, one can move on to model II, which always has a nonnegative quantum capacity. The ebits play essential roles here. We find, interestingly, Alice can represent the data $|\psi\rangle$ as a matrix-product state (MPS) [42]

$$|\psi\rangle = \sum_{i_1,\dots,i_N} \text{tr}(BA^{i_N} \cdots A^{i_1})|i_1 \dots i_N\rangle, \tag{24}$$

with local tensors A^{i_n} (and a boundary operator B), and then send its circuit representation $[A^{i_n}]$ to Bob, who can then use them and also ebits as resources to obtain $|\psi\rangle$ by only applying constant-depth local operations. As is well-known, MPSs are proper forms to characterize entanglement and play essential roles in many-body physics to describe topological order [43]. Here a MPS can be shared remotely by a few parties, and may have applications in distributed computing [44].

(3) Models III and IV are well known. From our perspective, model III allows any nonlocal encodings and any pre and post l-local operations on the state $|\psi\rangle$. This can indeed describe some nontrivial operations on codes, such as code switching [18,45], which is an important scheme to realize universal set of logical gates. For model IV, Alice would send $|\psi\rangle$ as a whole to Bob, so the only required ability for Bob is to store and manipulate each qubit.

If for a channel $I(\Phi) \geqslant 0$, then all the models would work, with higher-level models achieving larger capacities. When $I(\Phi) \leqslant 0$, one has to move on, e.g., with remote ebits that may be generated by model I for another channel Ψ with $I(\Psi) \geqslant 0$, or choose other models. Also note that the models I and II are not the one-shot versions, since the one-shot setting only allows separable codings, while I and II allow entangling isometry as codings.

TABLE II. A table of the code types induced by the four coding models classified in this paper and some stabilizer code examples in literature. Here the depth refers to the depth of the encoding circuit for a code. Note more refined classification is possible by considering more features of codes.

	Model I	Model II	Model III	Model IV
Type	small block	convolutional	large block	convolutional
Depth	small	small	large	large
Examples	[47–49]	[50–54]	[55–58]	[59–61]

B. Classification of codes

Our classification theory is not only useful for choosing a proper coding model, in practice, but also for the usage of error-correction codes. In the setting of noncooperative communication [46], the classification and recognition of codes are important. When Alice and Bob do not mutually agree upon the code being used, Bob has to recognize the type of code to choose a proper decoding algorithm, which should be adapted to the right type of code.

We know from the classical coding theory [1] that a code is, in general, block or convoluntional. The latter is characterized by a temporal order of data and memory between the encoding of blocks of data, while for block codes, each block of data encodes and decodes separately. For quantum codes, it turns out the memory effect can be simulated by ebits via teleportation, inducing a temporal order of data blocks. It is then easy to see convolutional codes can be described by model II (using teleportation module among the encoding parts) and also model IV (when many of its large blocks are available). But note the EA settings do not have to be convolutional, say, when only one block with EA is used for coding. The difference between models II and IV, and also models I and III, is a block being small or large. This can be characterized by the depth of the encoding circuit for a code; see Table II.

A large depth of the encoding circuit with local gates can lead to large values of entanglement increasing with the system size n, while a small or constant depth circuit cannot do so. There are many ways to characterize entanglement [17,42,43]—one powerful approach is to express states as MPSs (24), and the bipartite entanglement entropy S_E in a state satisfies

$$S_E \propto \log_2 \chi \tag{25}$$

for χ as the bond dimension of the entanglement space, which is used to carry the logical information.

Small block codes and convolutional codes will have small values of entanglement entropy S_E. Examples of small block codes are those with small k and small encoding circuit depths, including some well-known small-size codes [47,48] and symmetry-protected codes [49], and also some convolutional codes [50–53] and quantum Turbo codes, as interleaved convoluntional codes [54].

Meanwhile, large codes are those with a large value of k, and most likely also the rate $r = k/n$. Notable examples are some high-rate LDPC codes [56], Polar codes [57], the MERA codes [58], etc. Some of them are entanglement-assisted [59,60] and hence can be expanded to convolutional ones when many blocks are used.

We can also consider the *reverse* problem: Given a code that is promised to be one of the types we know, how to determine its type? Such a quantum code recognition task is an example of quantum hypothesis testing [7], but here the goal is to test its type not its formula. In general, the quantum machine learning algorithm [62] can be employed to serve as a classifier, which, however, is resource intensive. This task is also hard for the classical case; see Ref. [63] for a latest study. Different from the classical case, a quantum test is a measurement that will consume many samples of the state $|\psi\rangle$. Here we lay out an entanglement-based scheme but more advanced method is necessary.

There are ways to measure S_E in experiments [64,65]. Once the whole state $|\psi\rangle$ is obtained by Bob sent from Alice, Bob can do a few binary tests. First, it is easy to see if it is EA or not since the EA side channel is noise-free and has to be established beforehand. Then the value of S_E can tell small codes from large ones. Due to the state form (23) for block codes, it is with high probability that far-apart sites have no entanglement, but not the case for convolutional codes. We can use the Bell test [66] to distinguish them, but it is not easy to distinguish convolutional from Turbo codes, and LDPC from Polar codes by entanglement entropy. Therefore, more quantities are needed, which could be other entanglement measures or machine-learned features that deserve further study.

VI. NUMERICAL SIMULATION

To further understand quantum capacities and the non-additivity, here we numerically explore the gaps among the quantum capacities for the case of qubit channels. A general qubit channel contains 12 parameters. This can be seen in the so-called affine representation $\mathcal{T}$ of the form

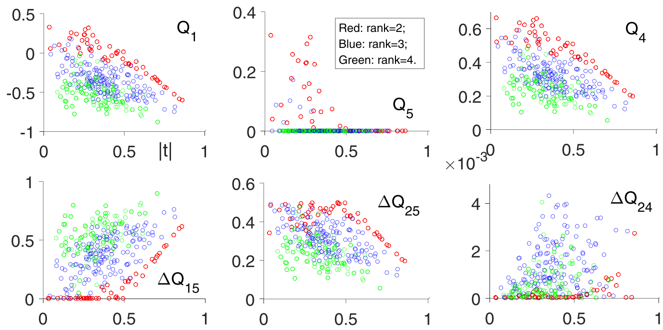

FIG. 4. The quantum capacity gaps for qubit channels of rank two (red), rank three (blue), and rank four (green). Each panel is for a capacity or capacity gap. The horizontal axes are all $|t|$, which is the size of the shift vector in the affine representation of a channel.

$$\mathcal{T} = \begin{pmatrix} 1 & 0 \\ \vec{t} & T \end{pmatrix}, \tag{26}$$

which is a 4×4 real matrix. The vector $\vec{t}$ is the shift of the center of the Bloch ball, and the matrix T enables the distortion of the ball. To represent a channel succinctly, we use $|t|$ and the Frobenius norm $\|T\|_F$ to represent a channel. A larger $|t|$ means larger nonunitality while a smaller $\|T\|_F$ means larger distortion of the ball. In our simulations, we do not observe a clear dependence on $\|T\|_F$, so we focus on the behaviors of capacities as functions of $|t|$. In our algorithm, given a random qubit channel Φ [67], we use an optimization algorithm from MATLAB to compute the capacity quantities. The rank of Φ is an input parameter, and for each rank we randomly sample hundreds of qubit channels.

A. Model I

From the general relation between Holevo quantity and coherent information, it is easy to see that

$$Q_{\mathrm{I}}(\Phi) \leqslant C_{\mathrm{I}}(\Phi) \leqslant I_c(\Phi). \tag{27}$$

This means the one-shot quantity $Q_{\mathrm{V}}(\Phi) := I_c(\Phi) = \max_\sigma I_c(\sigma, \Phi)$ serves as a good upper bound in model I, however, it does not serve as a quantum capacity in general.

It is shown that $Q_{\mathrm{V}}(\Phi)$ is almost surely positive if $\mathrm{r}(\Phi) \leqslant d$; otherwise, it is almost surely zero [68]. Here we numerically confirmed this for qubit channels in Fig. 4. Note we use subscript numbers to simplify the capacity quantities. For qubit rank-two channels, we find $Q_{\mathrm{V}}(\Phi)$ are not only positive but also there is a clear dependence on $|t|$. There appears to be a transition region at about $|t| \sim 0.5$, beyond which $Q_{\mathrm{V}}(\Phi)$ are mostly zero, $Q_{\mathrm{I}}(\Phi)$ are mostly negative, and the upper envelope is almost linear with $|t|$. For rank-three and rank-four channels, we see that $Q_{\mathrm{I}}(\Phi)$ are mostly negative while $Q_{\mathrm{V}}(\Phi)$ are mostly zero. When $Q_{\mathrm{V}}(\Phi) = 0$, the optimal input state is pure. The gap $\Delta Q_{15} = Q_{\mathrm{V}} - Q_{\mathrm{I}}$, and similarly for $\Delta Q_{25} = Q_{\mathrm{II}} - Q_{\mathrm{V}}$, shows a clear transition region for the rank-two case.

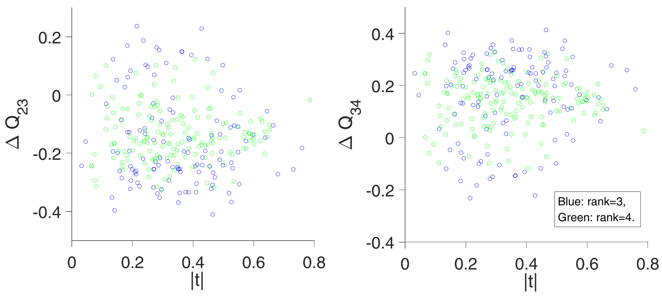

FIG. 5. The quantum capacity gaps for qubit channels of rank three (blue) and rank four (green). Each panel is for a capacity gap. The $|t|$ is the size of the shift vector in the affine representation of a channel.

B. Model II vs model IV

In model IV, achieving $Q_{\text{IV}}(\Phi)$ is to achieve the regularized EA Holevo capacity, which shall require a highly entangled state $|\eta\rangle$ as a resource which is not a product of ebits. The gap

$$\Delta Q_{24}(\Phi) = Q_{\text{IV}}(\Phi) - Q_{\text{II}}(\Phi) \tag{28}$$

is a measure of the nonlocal assistance effect of an optimal resource state $|\eta\rangle$. The Q_{IV} is upper bounded by $(\log_2 d + Q_{\text{V}})/2$, so $\Delta Q_{24} \leqslant \Delta Q_{15}/2$. We can see from the result in Fig. 4 that the gap $\Delta Q_{24}(\Phi)$ is quite small (in the order 10^{-3}) and is more apparent for larger shift $|t|$ for rank-two channels. For higher-rank cases, the gap becomes larger and there is no obvious dependence on $|t|$. Based on primary simulation tests not reported here, we expect this gap will get more apparent for higher dimensional channels.

C. Model III

The quantum capacity $Q_{\text{III}}(\Phi)$ cannot be explicitly computed, in general, but it can be upper bounded. A nice method is to use the convex decomposition of channels [69]. For the qubit case, it is well-known that a qubit channel can be decomposed as the convex sum

$$\Phi = p\Phi_1^{\text{g}} + (1-p)\Phi_2^{\text{g}}, \tag{29}$$

for two so-called generalized extreme channels which are channels with a rank up to two [67,70]. A qubit generalized extreme channel is either degradable or antidegradable [71], for which its quantum capacity is additive. Therefore, it has been proposed to use the following as an upper bound of the quantum capacity:

$$Q_{\text{III}}(\Phi) \leqslant \inf\left(pQ_{\text{III}}(\Phi_1^{\text{g}}) + (1-p)Q_{\text{III}}(\Phi_2^{\text{g}})\right). \tag{30}$$

Any such decomposition would serve as a looser upper bound for its capacity, and we denote such a value as $Q_{\text{III}}^{\text{UB}}(\Phi)$. Also, there is a lower bound $Q_{\text{III}}(\Phi) \geqslant Q_{\text{V}}(\Phi)$, with equality holds for degradable channels [72]. We also observe that $Q_{\text{V}}(\Phi) \leqslant Q_{\text{II}}(\Phi)$. This is explained by the behavior of Q_{I}: when it is positive, $Q_{\text{II}} \geqslant \frac{1}{2}$ while $Q_{\text{V}} \geqslant 0$, when it is negative, $Q_{\text{II}} \geqslant 0$ while $Q_{\text{V}} = 0$.

Here we plot the quantity $\Delta Q_{23} = Q_{\text{III}}^{\text{UB}} - Q_{\text{II}}$ and $\Delta Q_{34} = Q_{\text{IV}} - Q_{\text{III}}^{\text{UB}}$ for random qubit channels of ranks three (blue) and four (green) in Fig. 5. We see that both ΔQ_{23} and ΔQ_{34} can be either positive or negative. There is no clear dependence on the rank and $|t|$ of a noise channel. Most of ΔQ_{34} is positive while most of ΔQ_{23} is negative. A negative ΔQ_{34} means the bound is not tight, while a negative ΔQ_{23} means Q_{III} is even smaller than Q_{II} for some channels. This indeed confirms that model III does not perfectly lie in between models II and IV for arbitrary channels.

VII. CONCLUSION

In this paper, we establish a quantum resource theory approach to describe a family of coding models that are of importance to quantum communication and error correction. By treating codings as superchannels, our approach is broad to describe a few important quantum capacities and types of codes, and may also be used to discover new ones.

Along that line, as we have mentioned, there are other types of models or quantities, including back classical communication [13], the simultaneous classical and quantum communication [72], reverse coherent information [14], the entanglement cost of channels [73], and the Rains information [15]. Whether proper coding models relating to them can be defined and put in the hierarchy of coding family is unclear. It is also worth mentioning the codings with infinite-dimensional systems [74,75], which require further investigation to generalize our approach.

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