# Tensor rank and other multipartite entanglement measures of graph states

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Graph states play an important role in quantum information theory through their connection to measurement-based computing and error correction. Prior work revealed elegant connections between the graph structure of these states and their multipartite entanglement. We continue this line of investigation by identifying additional entanglement properties for certain types of graph states. From the perspective of tensor theory, we tighten both upper and lower bounds on the tensor rank of odd ring states ($|R_{2n+1}\rangle$) to read $2^n + 1 \leqslant \text{rank}(|R_{2n+1}\rangle) \leqslant 3 \times 2^{n-1}$. Next we show that several multipartite extensions of bipartite entanglement measures are dichotomous for graph states based on the connectivity of the corresponding graph. Finally, we give a simple graph rule for computing the $n$-tangle $\tau_n$.

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## I. INTRODUCTION

Entanglement is one of the defining properties of quantum systems [1] and has been recognized as a fundamental resource for quantum information processing [2,3]. A pure state is considered to be entangled if it cannot be written in the form $|\psi\rangle = \bigotimes_{i=1}^{n} |\psi^{(i)}\rangle$. Similarly, a mixed state is entangled if it cannot be written as $\rho = \sum_k p_k \bigotimes_{i=1}^{n} \rho_k^{(i)}$.

Quantifying the amount of entanglement in a quantum state is not always straightforward. For pure bipartite systems, the Schmidt decomposition and resulting spectra fully characterize the entanglement properties and transformations under local operations and classical communication (LOCC) [2,4,5]. The Schmidt decomposition of a pure state takes the form $|\psi\rangle = \sum_i \sqrt{\mu_i}|u_i\rangle|v_i\rangle$, where $\mu = \{\mu_i\}$ are the Schmidt coefficients and $\{|u_i\rangle\}$ and $\{|v_i\rangle\}$ are sets of orthogonal states. Further, a variety of entanglement measures are known, i.e., functionals $E(\rho)$ that are nonincreasing (on average) under LOCC and $E(\rho) = 0$ if $\rho$ is a separable state [6]. Examples include the entanglement of formation [7], distillable entanglement [8], negativity [9,10], geometric measure [11], and concurrence [12,13]. However, the picture grows significantly more complicated when considering multipartite entanglement, as we discuss below.

In this work we consider the amount and form of multipartite entanglement that arises in a class of quantum states known as graph states. These are of particular interest due to their applications in measurement-based quantum computing [14,15], error correction codes [16], secret sharing [17], and stabilizer computation simulation [18]. Further, by studying the entanglement properties of graph states, we actually quantify the entanglement of the larger set of stabilizer states. This follows from the fact that every stabilizer state is equivalent under local unitaries to at least one graph state [19,20]. As entanglement measures are invariant under local unitaries, one need only consider graph states to analyze all stabilizer states.

In this work we focus on the tensor rank and the GME concurrence, negativity, and geometric measures of entanglement. Our main contributions to the study of graph states are twofold. First, we consider ring states of $2n + 1$ qubits and sharpen the bound on tensor rank from $2^n \leqslant \text{rank}(|R_{2n+1}\rangle) \leqslant 2^{n+1}$ to $2^n + 1 \leqslant \text{rank}(|R_{2n+1}\rangle) \leqslant 3 \times 2^{n-1}$. This demonstrates that existing upper and lower bounds on tensor rank based on bipartitions of the associated graphs [21] are not tight for all stabilizer states. While this may seem like incremental progress, we stress that computing the tensor rank is a very challenging problem, and any progress in this direction is noteworthy. Indeed, the analysis we employ goes beyond the bipartite bounding techniques of previous approaches. This work thus contributes to the steadily growing research on the tensor rank of multipartite entangled states [22–30]. Operationally, the improved bounds better characterize the amount of entanglement needed to generate ring states using LOCC. Second, we study the GME concurrence, negativity, and geometric measure for general graph states. These are shown to be sharply dichotomous and take on constant value for all connected graphs. Thus, our results can be taken to show that considering bipartite cuts fails to capture the multipartite entanglement of stabilizer states. Before presenting our results, we briefly review the main concepts considered in this paper.

### A. Schmidt measure and tensor decomposition

Any $n$-party pure state $|\psi\rangle \in \mathcal{H}^{(1)} \otimes \cdots \otimes \mathcal{H}^{(n)}$ can be represented as

$$|\psi\rangle = \sum_{i=1}^{R} \mu_i \big|\psi_i^{(1)}\big\rangle \otimes \cdots \otimes \big|\psi_i^{(n)}\big\rangle, \tag{1}$$

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where each $|\psi_i^{(j)}\rangle \in \mathcal{H}^{(j)}$. When $|\psi\rangle$ is viewed as an $n$-dimensional tensor, Eq. (1) is also known as a canonical polyadic (CP) tensor decomposition [31,32]. The CP rank $r = \text{rank}(|\psi\rangle)$ of a tensor is defined as the smallest $R$ such that (1) can be satisfied. The CP rank is also known as the tensor rank or Schmidt measure, and we will use these terms interchangeably throughout this paper. In general, finding the CP rank of a tensor is NP-hard [33].

Tensor rank plays an important role in algebraic complexity theory, namely, the number of nonscalar multiplications performed by a multilinear program can be recast as the rank of a certain tensor, a famous case being matrix multiplication [34]. Explicitly, the asymptotic coefficient for matrix multiplication, i.e., the smallest $\tau$ such that matrix multiplication can be performed in time $O(n^\tau)$, is equal to the tightest bound on rank for a family of tensors.

Unlike the matrix case, the best rank-$R$ approximation of a tensor may not exist. There exist tensors that can be approximated arbitrarily well by rank-$R$ tensors where $R < \text{rank}(|\psi\rangle)$. In this case, border rank [35,36] is defined as the minimum number of rank-1 tensors that are sufficient to approximate the given tensor with arbitrarily small error.

The tensor rank is a *bona fide* entanglement measure [22] that is particularly useful for studying transformations under stochastic local operations and classical communication (SLOCC). These are transformations such that $|\psi\rangle \xrightarrow{\text{SLOCC}} |\phi\rangle$ with some nonzero probability (and is thus a generalization of LOCC). It is known that if $|\psi\rangle \xrightarrow{\text{SLOCC}} |\phi\rangle$, then $\text{rank}|\psi\rangle \geqslant \text{rank}|\phi\rangle$ [23]. If two states are equivalent under SLOCC, i.e., $|\psi\rangle \xrightarrow{\text{SLOCC}} |\phi\rangle$ and $|\phi\rangle \xrightarrow{\text{SLOCC}} |\psi\rangle$, then there exist invertible linear operators $\{A_i\}_{i=1}^n$ such that

$$|\psi\rangle = A_1 \otimes A_2 \otimes \cdots \otimes A_n |\phi\rangle, \tag{2}$$

implying that $\text{rank}(|\psi\rangle) = \text{rank}(|\phi\rangle)$. Finally, we note that the tensor rank relates to entanglement cost. In particular, a generalized $d$-dimensional Greenberger-Horne-Zeilinger (GHZ) state (or any equivalent state) can be converted to an arbitrary state $|\psi\rangle$ via SLOCC if and only if $d \geqslant \text{rank}(|\psi\rangle)$ [24]. This provides an operational meaning to the tensor rank in terms of the entanglement resources needed to build $|\psi\rangle$ using GHZ states in the distributed setting.

### B. Measures of genuine multipartite entanglement

An $n$-partite pure state $|\psi\rangle$ is said to have genuine multipartite entanglement (GME) if it is not a product state under any bipartition $A|\bar{A}$ [37,38], i.e., $|\psi\rangle \neq |\alpha\rangle \otimes |\beta\rangle$, where $|\alpha\rangle$ is held by parties in $A$ and $|\beta\rangle$ is held by parties $\bar{A}$. States for which $|\psi\rangle = |\alpha\rangle \otimes |\beta\rangle$ are called biseparable, and in general it may be desirable for a multipartite entanglement measure to capture how close a state is to being biseparable. Accordingly, one can define measures via minimization over all possible bipartitions. More specifically, given some bipartite entanglement measure $E(|\phi\rangle)$, $|\phi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$, we take the multipartite extension to be $\min_A E_A(|\psi\rangle)$, where $E_A$ is the measure $E$ evaluated according to partition $A|\bar{A}$. Note that $E(|\psi\rangle) = 0$ if $|\psi\rangle$ is biseparable according to some partition. Thus, this multipartite extension is faithful with respect to GME. In this work we consider, beyond tensor rank, GME concurrence [39]

$$\mathcal{C}_{\text{GME}}(\rho) = \min_A \sqrt{2\left[1 - \text{Tr}\left(\rho_A^2\right)\right]}, \tag{3}$$

GME negativity [9]

$$\mathcal{N}_{\text{GME}}(\rho) = \min_A \tfrac{1}{2}(\|\rho^{T_A}\|_1 - 1), \tag{4}$$

and the GME geometric measure [38]

$$\mathcal{G}_{\text{GME}}(\rho) = \min_A \left(1 - \max_i \mu_i\right), \tag{5}$$

where $\|\cdot\|_1$ denotes the Schatten 1-norm, $\rho^{T_A} = \mathbb{I}_{\bar{A}} \otimes T_A(\rho)$ is the partial transpose, and $\mu_i$ are the Schmidt coefficients from the Schmidt decomposition according to partition $A|\bar{A}$. The geometric measure for bipartite systems takes this form, but this is different than the general definition [11].

Finally, we also evaluate the $n$-tangle $\tau_n$ [40,41] on graph states. For pure states of even numbers of qubits, this is defined as

$$\tau_n(|\psi\rangle) = |\langle\psi|\tilde{\psi}\rangle|^2, \tag{6}$$

where $|\tilde{\psi}\rangle = \sigma_y^{\otimes n}|\psi^*\rangle$ and $|\psi^*\rangle$ indicates the complex conjugate. The $n$-tangle is the square of a quadratic SLOCC invariant and can thus be used to distinguish between types of multipartite entangled states [42].

### C. Graph states

A graph state corresponds to some graph $G = (V, E)$, where $V$ is the vertex set and $E$ is the edge set with corresponding adjacency matrix $\Gamma$ [14,43]. There are two equivalent ways to think of graph states. The first is operational in the sense that it provides a formula for preparing the state given a graph

$$|G\rangle = \prod_{(a,b)\in E} U^{(a,b)} |+\rangle^{\otimes|V|}, \tag{7}$$

where

$$U^{(a,b)} = |0\rangle\langle 0|^{(a)} \otimes \mathbb{I}^{(b)} + |1\rangle\langle 1|^{(a)} \otimes \sigma_z^{(b)} \tag{8}$$

is a controlled-$Z$ operation on qubits $a$ and $b$ and

$$|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle), \quad |-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

form the Hadamard basis. To prepare the graph state corresponding to a graph $G$, simply initialize $|V|$ qubits in the state $|+\rangle^{\otimes|V|}$ and, for each edge, apply a controlled-$Z$ operation between the corresponding qubits.

Graph states can be equivalently thought of as stabilizer states [21]. Here the stabilizers are $S_a = \sigma_x^{(a)} \prod_{b\in N_a} \sigma_z^{(b)}$, where $N_a$ is the neighborhood of vertex $a$. As there are $|V|$ qubits and stabilizers, $|G\rangle$ is the unique state stabilized by all $S_a$.

Also note that a basis for $\mathcal{H} = \bigotimes_{i=1}^n \mathcal{H}_2^{(i)}$ can be constructed given a graph $G$ [15]. Let $\mathbf{s} \in \mathbb{Z}_2^n$ be a bit string of length $n$. Then define the state

$$|G_{\mathbf{s}}\rangle = \sigma_z^{\mathbf{s}}|G\rangle = \prod_{(a,b)\in E} U^{(a,b)} \bigotimes_{i=1}^n (\sigma_z^{(i)})^{s_i} |+\rangle^{(i)}. \tag{9}$$

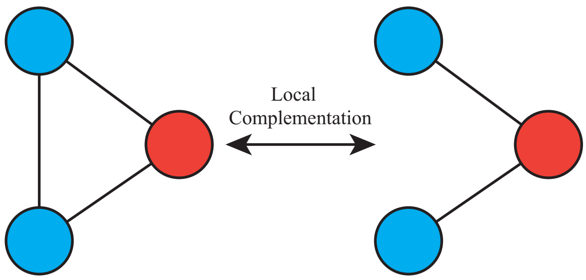


FIG. 1. Example of local complementation. Here the rule is applied to the red vertex, adding or removing the edge connecting the two other vertices.

It is clear that there are $2^n$ such orthogonal states and thus they form a basis. Further, one can think of $\mathbf{s}$ as flipping the eigenvalues of stabilizers $S_a$ from $+1$ to $-1$. Going forward, we will refer to this basis as graph basis states. We will later use a result from [21] that the partial trace of a graph state can be expressed in the graph basis

$$\mathrm{Tr}_A(|G\rangle\langle G|) = \frac{1}{2^{|A|}} \sum_{\mathbf{z}\in\mathbb{Z}_2^{|A|}} U(\mathbf{z})|G-A\rangle\langle G-A|U(\mathbf{z})^\dagger, \quad (10)$$

where $\mathbf{z}$ sums over all binary strings of length $|A|$, $U(\mathbf{z}) = \prod_{a\in A}(\prod_{b\in N_a}\sigma_z^{(b)})^{z_a}$, and $|G-A\rangle$ denotes the state corresponding to deleting all vertices in $A$ from $G$. Further analysis of this state leads to the useful structural fact.

*Lemma 1 (from [21]).* The states $U(\mathbf{z})|G-A\rangle$ satisfy the orthogonality condition

$$\langle G-A|U^\dagger(\mathbf{z}')U(\mathbf{z})|G-A\rangle = \begin{cases} 0 & \text{if } \mathbf{z}-\mathbf{z}' \in \ker(\Gamma_{A\bar{A}}) \\ 1 & \text{if } \mathbf{z}-\mathbf{z}' \notin \ker(\Gamma_{A\bar{A}}), \end{cases} \quad (11)$$

where $\Gamma_{A\bar{A}}$ is the submatrix of $\Gamma_G$ restricted to edges from $A$ to $\bar{A}$. Hence, $\rho^{(\bar{A})} = \mathrm{Tr}_A|G\rangle\langle G|$ is maximally mixed over a subspace of dimension $2^d$, where $d = \mathrm{rank}(\Gamma_{A\bar{A}})$.

### D. Existing tensor rank bounds for graph states

Here we briefly review existing results on the graph-state CP rank. From [21] we have that

$$\mathrm{rank}(|\psi\rangle) \geqslant 2^{(\mathrm{rank}\Gamma_{A\bar{A}})/2}, \quad (12)$$

where $\Gamma_{A\bar{A}}$ is the subset of the adjacency matrix restricted to edges from $\bar{A}$ to $A$. This bound essentially comes from taking a bipartite cut of the state. The authors also give a general case upper bound

$$\mathrm{rank}(|\psi\rangle) \leqslant 2^{\tau(G)}, \quad (13)$$

where $\tau(G)$ is the size of the smallest vertex cover of $G$.

While these bounds may not be tight, it is often possible to use complementation rules to find locally equivalent graphs for which these bounds improve. It is known that the full orbit of any graph state under local Clifford operations can be found via local complementations [19,21], that is, for some vertex $a \in V$, complement the subgraph given by the neighborhood $N_a$ (Fig. 1). These rules have been used to classify all graph states of up to eight qubits [21,44,45]. Further, classes of two-colorable graphs corresponding to states of maximal Schmidt measure are known [46]. However, odd rings, corresponding to non-two-colorable graphs, lead to loose bounds.

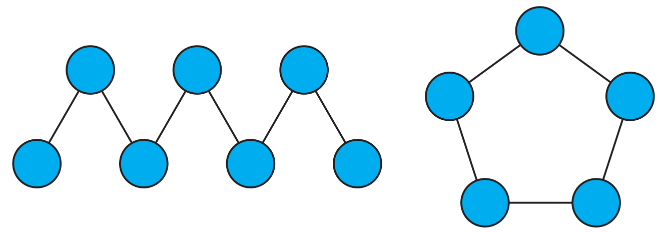

FIG. 2. Line and ring graphs. The left graph is a line (one-dimensional cluster state) on seven qubits, which we denote by $|L_7\rangle$. The right graph is an odd ring on five qubits, which we denote by $|R_5\rangle$.

We will often concern ourselves with two forms of graphs: lines (also known as one-dimensional cluster states) and rings (Fig. 2), which we denote by $|L_n\rangle$ and $|R_n\rangle$, where $n$ is the number of qubits. Explicitly,

$$|L_n\rangle = \left(\prod_{i=1}^{n-1} U^{(i,i+1)}\right)|+\rangle^{(1,\ldots,n)}, \quad (14)$$

$$|R_n\rangle = U^{(1,n)}|L_n\rangle. \quad (15)$$

An explicit construction of a minimal CP decomposition for line states is given in the Appendix.

*Lemma 2.* We have

$$\mathrm{rank}(|L_n\rangle) = 2^{\lfloor n/2\rfloor}. \quad (16)$$

*Proof.* This readily follows from the mentioned graph-theoretic tools. See [21] for details. ■

For any even ring $|R_{2n}\rangle$, it is known that the lower bound equals the upper bound, and thus the CP rank is $2^n$ [21]. For any odd ring $|R_{2n+1}\rangle$, it is known that $2^n \leqslant \mathrm{rank}(|R_{2n+1}\rangle) \leqslant 2^{n+1}$, coming from the rank of the adjacency matrix and minimal vertex cover. Any tightening of these bounds will therefore require a new type of analysis not based on the latter graph-theoretic concepts.

## II. TENSOR RANK OF RING STATES

In this paper we improve the odd ring CP rank bounds as stated in the following theorem.

*Theorem 1.* The CP rank of the graph state corresponding to any odd ring $|R_{2n+1}\rangle$ is bounded by

$$2^n + 1 \leqslant \mathrm{rank}(|R_{2n+1}\rangle) \leqslant 3 \times 2^{n-1}. \quad (17)$$

We will break the proof into two propositions corresponding to the upper and lower bounds.

### A. Upper bound analysis

In this section we provide a CP rank upper bound of $3 \times 2^{n-1}$ for odd ring graph states $|R_{2n+1}\rangle$. To elucidate our argument, an explicit construction for $|R_7\rangle$ is given in the Appendix.

Throughout the proof, we let

$$P_0 = |0\rangle\langle 0|, \quad P_1 = |1\rangle\langle 1|$$

and let $U^{(a,b)}$ denote a controlled-$Z$ operation on qubits $a$ and $b$. We have

$$U^{(a,b)} = P_0^{(a)} \otimes I^{(b)} + P_1^{(a)} \otimes \sigma_z^{(b)} = I^{(a)} \otimes \sigma_z^{(b)} + 2P_0^{(a)} \otimes P_1^{(b)}. \tag{18}$$

Below we show the main statement.

*Proposition 1 (CP rank upper bound for odd ring graph states).* The CP rank of any odd ring state $|R_{2n+1}\rangle$ is upper bounded by

$$\text{rank}(|R_{2n+1}\rangle) \leqslant 3 \times 2^{n-1}. \tag{19}$$

*Proof.* For the case with $n = 1$, we can easily verify that

$$|R_3\rangle = |++-\rangle + \frac{1}{\sqrt{2}}|001\rangle - \frac{1}{\sqrt{2}}|110\rangle,$$

thus satisfying the upper bound. Below we show the cases with $n \geqslant 2$.

Based on (18) and the fact that

$$|R_{2n+1}\rangle = U^{(1,2n+1)}|L_{2n+1}\rangle,$$

we have

$$|R_{2n+1}\rangle = U^{(1,2n+1)}|L_{2n+1}\rangle = I^{(1)} \otimes \sigma_z^{(2n+1)}|L_{2n+1}\rangle + 2P_0^{(1)} \otimes P_1^{(2n+1)}|L_{2n+1}\rangle.$$

The CP rank of the term $I^{(1)} \otimes \sigma_z^{(2n+1)}|L_{2n+1}\rangle$ is $2^n$ since the CP rank of $|L_{2n+1}\rangle$ is $2^n$. Define the (unnormalized) state

$$|\phi_{2n+1}\rangle = P_0^{(1)}|L_{2n+1}\rangle. \tag{20}$$

Based on Lemmas 3 and 4 below, the CP rank of the term $P_1^{(2n+1)}|\phi_{2n+1}\rangle$ for all integers $n \geqslant 2$ is upper bounded by $2^{n-1}$, thus proving the statement. ■

Below we present Lemmas 3 and 4, which upper bound the CP rank of $P_0^{(2n+1)}|\phi_{2n+1}\rangle$ and $P_1^{(2n+1)}|\phi_{2n+1}\rangle$ for all integers $n \geqslant 2$. In our analysis below, we define a generalized controlled gate

$$CZZ^{(i,j,k)} := U^{(i,j)}U^{(i,k)} = P_0^{(i)} \otimes I^{(j)} \otimes I^{(k)} + P_1^{(i)} \otimes \sigma_z^{(j)} \otimes \sigma_z^{(k)},$$

whose CP rank is also 2. The line state $|L_{2n+1}\rangle$ can be expressed as

$$|L_{2n+1}\rangle = \prod_{i=1}^{n} U^{(2i,2i-1)}U^{(2i,2i+1)}|+\rangle^{(1,\dots,2n+1)} = \prod_{i=1}^{n} CZZ^{(2i,2i-1,2i+1)}|+\rangle^{(1,\dots,2n+1)}. \tag{21}$$

*Lemma 3.* When $n = 2$, the ranks of both $P_0^{(2n+1)}|\phi_{2n+1}\rangle$ and $P_1^{(2n+1)}|\phi_{2n+1}\rangle$ with $|\phi_{2n+1}\rangle$ defined in (20) are bounded by 2.

*Proof.* For $n = 2$,

$$\begin{aligned}|L_{2n+1}\rangle &= |L_5\rangle \\ &= CZZ^{(4,3,5)}CZZ^{(2,1,3)}|+\rangle^{\otimes 5} \\ &= (I \otimes P_0 \otimes I \otimes P_0 \otimes I \\ &\quad + I \otimes P_0 \otimes \sigma_z \otimes P_1 \otimes \sigma_z)|+\rangle^{\otimes 5} \\ &\quad + (\sigma_z \otimes P_1 \otimes \sigma_z \otimes P_0 \otimes I \\ &\quad + \sigma_z \otimes P_1 \otimes I \otimes P_1 \otimes \sigma_z)|+\rangle^{\otimes 5} \\ &= \tfrac{1}{2}|+0+0+\rangle + \tfrac{1}{2}|+0-1-\rangle + \tfrac{1}{2}|-1-0+\rangle \\ &\quad + \tfrac{1}{2}|-1+1-\rangle,\end{aligned}$$

and thus we have

$$\begin{aligned}|\phi_5\rangle &= P_0^{(1)}|L_5\rangle \\ &= \frac{1}{2\sqrt{2}}|0\rangle(|0+0+\rangle + |0-1-\rangle + |1-0+\rangle \\ &\quad + |1+1-\rangle) \\ &= \underbrace{\frac{1}{4}|0\rangle(|0+0\rangle + |0-1\rangle + |1-0\rangle + |1+1\rangle)|0\rangle}_{P_0^{(5)}|\phi_5\rangle} \\ &\quad + \underbrace{\frac{1}{4}|0\rangle(|0+0\rangle - |0-1\rangle + |1-0\rangle - |1+1\rangle)|1\rangle}_{P_1^{(5)}|\phi_5\rangle}.\end{aligned}$$

The above expressions for $P_0^{(5)}|\phi_5\rangle$ and $P_1^{(5)}|\phi_5\rangle$ can be rewritten as

$$P_0^{(5)}|\phi_5\rangle = \frac{1}{2\sqrt{2}}|0\rangle(|+0+\rangle + |-1-\rangle)|0\rangle,$$

$$P_1^{(5)}|\phi_5\rangle = \frac{1}{2\sqrt{2}}|0\rangle(|+0-\rangle + |-1+\rangle)|1\rangle,$$

and thus the CP ranks are bounded by 2. ■

*Lemma 4.* When $n \geqslant 2$, the CP ranks of both $P_0^{(2n+1)}|\phi_{2n+1}\rangle$ and $P_1^{(2n+1)}|\phi_{2n+1}\rangle$ are bounded by $2^{n-1}$.

*Proof.* We argue by induction on $n$. Assume that the ranks of both $P_0^{(2n+1)}|\phi_{2n+1}\rangle$ and $P_1^{(2n+1)}|\phi_{2n+1}\rangle$ are bounded by $2^{n-1}$. We will show that the CP ranks of both vectors $P_0^{(2n+3)}|\phi_{2n+3}\rangle$ and $P_1^{(2n+3)}|\phi_{2n+3}\rangle$ are bounded by $2^n$.

The $|\phi_{2n+3}\rangle$ can be rewritten as follows:

$$\begin{aligned}|\phi_{2n+3}\rangle &= P_0^{(1)}|L_{2n+3}\rangle \\ &= P_0^{(1)}CZZ^{(2n+2,2n+1,2n+3)}|L_{2n+1}\rangle|++\rangle \\ &= CZZ^{(2n+2,2n+1,2n+3)}P_0^{(1)}|L_{2n+1}\rangle|++\rangle \\ &= CZZ^{(2n+2,2n+1,2n+3)}P_0^{(2n+1)}|\phi_{2n+1}\rangle \otimes |++\rangle \\ &\quad + CZZ^{(2n+2,2n+1,2n+3)}P_1^{(2n+1)}|\phi_{2n+1}\rangle \otimes |++\rangle \\ &= \frac{1}{\sqrt{2}}P_0^{(2n+1)}|\phi_{2n+1}\rangle \otimes (|0+\rangle + |1-\rangle) \\ &\quad + \frac{1}{\sqrt{2}}P_1^{(2n+1)}|\phi_{2n+1}\rangle \otimes (|0+\rangle - |1-\rangle).\end{aligned} \tag{22}$$

Note that the third equality comes from the commutativity of $CZZ^{(2n+2,2n+1,2n+3)}$ and $P_0^{(1)}$. Based on the identity

$$|0+\rangle + |1-\rangle = |+0\rangle + |-1\rangle,$$

$$|0+\rangle - |1-\rangle = |+1\rangle + |-0\rangle,$$

Eq. (22) can be rewritten as

$$\begin{aligned}|\phi_{2n+3}\rangle &= \frac{1}{\sqrt{2}}P_0^{(2n+1)}\phi_{2n+1}\otimes(|{+}0\rangle+|{-}1\rangle)\\ &\quad+\frac{1}{\sqrt{2}}P_1^{(2n+1)}|\phi_{2n+1}\rangle\otimes(|{+}1\rangle+|{-}0\rangle)\\ &=\underbrace{\frac{1}{\sqrt{2}}\big(P_0^{(2n+1)}|\phi_{2n+1}\rangle\otimes|+\rangle+P_1^{(2n+1)}|\phi_{2n+1}\rangle\otimes|-\rangle\big)|0\rangle}_{P_0^{(2n+3)}|\phi_{2n+3}\rangle}\\ &\quad+\underbrace{\frac{1}{\sqrt{2}}\big(P_0^{(2n+1)}|\phi_{2n+1}\rangle\otimes|-\rangle+P_1^{(2n+1)}|\phi_{2n+1}\rangle\otimes|+\rangle\big)|1\rangle}_{P_1^{(2n+3)}|\phi_{2n+3}\rangle}.\end{aligned}$$

It can be easily seen that the CP ranks of both tensors $P_0^{(2n+3)}|\phi_{2n+3}\rangle$ and $P_1^{(2n+3)}|\phi_{2n+3}\rangle$ are bounded by $2^n$. Since the rank upper bound for the base case ($n=2$) has been shown in Lemma 3, the lemma is proved. ■

### B. Lower bound analysis

We now turn to the lower bound and begin by giving an outline of the proof. The following lemma holds in general for CP decompositions.

*Lemma 5.* Suppose that

$$|\psi\rangle=\sum_{i=1}^{R}\mu_i|\psi_i\rangle^{(1)}\otimes\cdots\otimes|\psi_i\rangle^{(n)} \tag{23}$$

is a CP decomposition of $|\psi\rangle$. For any subset of parties $\bar{A}$, the span of $\{\bigotimes_{c\in\bar{A}}|\psi_i\rangle^{(c)}\}_{i=1}^{R}$ contains the support of $\rho^{(\bar{A})}=\mathrm{Tr}_A|\psi\rangle\langle\psi|$. Moreover, if $\rho^{(\bar{A})}$ has matrix rank $R$, then conversely the states $\{\bigotimes_{c\in\bar{A}}|\psi_i\rangle^{(c)}\}_{i=1}^{R}$ must belong to the support of $\rho^{(\bar{A})}$.

*Proof.* Given Eq. (23), the reduced density matrix of $|\psi\rangle$ on $\bar{A}$ is

$$\rho^{(\bar{A})}=\sum_{i=1}^{R}\sum_{j=1}^{R}\underbrace{\mu_i\mu_j\prod_{k\in A}\langle\psi_i^{(k)}|\psi_j^{(k)}\rangle}_{M_{ij}}\prod_{c\in\bar{A}}|\psi_i^{(c)}\rangle\langle\psi_j^{(c)}|.$$

Hence, $\rho^{(\bar{A})}=UMU^T$, where

$$U=\big[\bigotimes_{c\in\bar{A}}|\psi_1^{(c)}\rangle\quad\cdots\quad\bigotimes_{c\in\bar{A}}|\psi_R^{(c)}\rangle\big].$$

Consequently, the support (column span) of this reduced density matrix is contained in $\mathrm{span}(\{\bigotimes_{c\in\bar{A}}|\psi_i^{(c)}\rangle\}_{i=1}^{R})$. Further, if $\mathrm{rank}(\rho^{(\bar{A})})=R$, the rank of $M$ is also $R$, and the column span of $\rho^{(\bar{A})}$ is the same as that of $U$. ■

Armed with this lemma, the proof goes through the following steps.

(i) Select a subset of qubits $A$ such that $\rho^{(\bar{A})}$ has rank $2^n$.

(ii) Determine all of the product states in the support of $\rho^{(\bar{A})}$.

(iii) Show that any CP decomposition of $|R_{2n+1}\rangle$ into these product states requires more than $2^n$ terms.

Via the prior lemma, this then implies that $|R_{2n+1}\rangle$ must have CP rank strictly greater than $2^n$.

#### 1. Rank $2^n$ reduced density matrix

Recall that for any graph $G$ and subset of vertices $A\subset V$, the graph state $|G\rangle$ can be expressed via Eq. (10) as

$$|G\rangle=\frac{1}{\sqrt{2^{|A|}}}\sum_{\mathbf{z}\in\mathbb{Z}_2^{|A|}}(-1)^{|\mathbf{z}|}|\mathbf{z}\rangle^{(A)}U(\mathbf{z})|G-A\rangle^{(\bar{A})}, \tag{24}$$

where $\mathbf{z}$ sums over all binary strings of length $|A|$ and $U(\mathbf{z})=\prod_{a\in A}(\prod_{b\in N_a}\sigma_z^{(b)})^{z_a}$. Consider splitting the ring into $n$ even and $n+1$ odd vertices, defining $A=\{2,4,\dots,2n\}$ and $\bar{A}=\{1,3,\dots,2n+1\}$. For such partitions we write $\mathbf{z}$ as $\mathbf{z}=(z_2,z_4,\dots,z_{2n})$. Now the density matrix of $\bar{A}$ in $|R_{2n+1}\rangle$ is, using Eq. (24),

$$\rho^{(\bar{A})}=\frac{1}{2^n}\sum_{\mathbf{z}\in\mathbb{Z}_2^n}U(\mathbf{z})|R_{2n+1}-A\rangle\langle R_{2n+1}-A|U(\mathbf{z})^\dagger,$$

since for each $a\in A$, $N_a\subseteq\bar{A}$. The above density matrix decomposition is an eigendecomposition as a consequence of the lemma below.

Regarding notation going forward, for convenience we will write the pure states on the reduced system as $|\phi\rangle|\varphi\rangle$, where the first factor is on parties 1 and $2n+1$ and the second is on parties $\{3,5,\dots,2n-1\}$. Qubit labels will be used when we wish to refer to other factorizations.

*Lemma 6.* The states

$$|e_{\mathbf{z}}\rangle:=U(\mathbf{z})|R_{2n+1}-A\rangle\quad\forall\,\mathbf{z}\in\mathbb{Z}_2^n$$

are orthonormal and thus eigenvectors of $\rho^{(\bar{A})}$, which is consequently of rank $2^n$.

*Proof.* After removing vertices in $A$, the graph is composed of a two-qubit line state and product states

$$|R_{2n+1}-A\rangle=|L_2\rangle|+\rangle^{\otimes n-2},$$

where $|L_2\rangle=\frac{1}{\sqrt{2}}(|0+\rangle+|1-\rangle)$. From the definition of $U(\mathbf{z})$, for any $(z_2,z_4,\dots,z_{2n})\in\mathbb{Z}_2^n$ we can then write

$$|e_{\mathbf{z}}\rangle=\big(\sigma_z^{z_2}\otimes\sigma_z^{z_{2n}}|L_2\rangle\big)\otimes|\varphi_{\mathbf{z}}\rangle,$$

where

$$|\varphi_{\mathbf{z}}\rangle=\bigotimes_{k=2,4,\dots,2(n-1)}\sigma_z^{z_k\oplus z_{k+2}}|+\rangle. \tag{25}$$

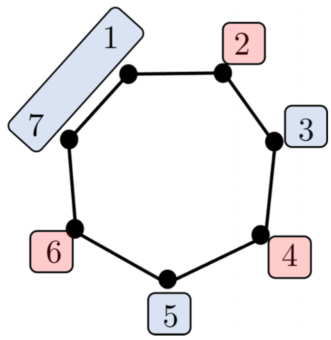


FIG. 3. Lower bounding the rank of $|R_{2n+1}\rangle$ for $n = 3$. For the seven-qubit ring state, we trace out the subset of qubits $A = \{2, 4, 6\}$, depicted as the red nodes in the ring. This set is judiciously chosen such that after tracing out qubits belonging to $A$, the remaining qubits $\bar{A} = \{1, 3, 5, 7\}$ (shown in blue) consists of just one entangled pair (1,7) and the rest are completely uncoupled. This relatively simple structure allows us to characterize all the product states in the support of $\rho^{(\bar{A})}$ (Lemma 7). Using this characterization, the desired lower bound on the tensor rank is proven (Theorem 2).

We first show that the product states $|\varphi_{\mathbf{z}}\rangle$ and $|\varphi'_{\mathbf{z}}\rangle$ are orthogonal except when $\mathbf{z}' = \mathbf{z}$ or $\mathbf{z}' = \bar{\mathbf{z}}$, where $\bar{\mathbf{z}}$ denotes the bitwise conjugate of $\mathbf{z}$. Observe that

$$\langle\varphi_{\mathbf{z}}|\varphi_{\mathbf{z}'}\rangle = \prod_{k=2,4,\ldots,2(n-1)} \langle +|\sigma_z^{z_k \oplus z_{k+2} \oplus z'_k \oplus z'_{k+2}}|+\rangle;$$

hence $\langle\varphi_{\mathbf{z}}|\varphi_{\mathbf{z}'}\rangle = 0$, unless $z_k \oplus z'_k = z_{k+2} \oplus z'_{k+2}$ for all $k = 2, 4, \ldots, 2(n-1)$. If each $z_k \oplus z'_k = 0$, then $\mathbf{z}' = \mathbf{z}$; otherwise each $z_k \oplus z'_k = 1$, so $\mathbf{z}' = \bar{\mathbf{z}}$. Hence, we have established

$$\langle\varphi_{\mathbf{z}}|\varphi_{\mathbf{z}'}\rangle = \begin{cases} 0 & \text{if } \mathbf{z}' \neq \bar{\mathbf{z}} \\ 1 & \text{if } \mathbf{z}' = \bar{\mathbf{z}} \text{ or } \mathbf{z}' = \mathbf{z}. \end{cases} \tag{26}$$

We now complete the proof of the lemma by showing that if $\mathbf{z}' = \bar{\mathbf{z}}$, the state of the first and $(2n+1)$th qubits (in the line state) are orthogonal:

$$\begin{aligned} &\langle L_2|\sigma_z^{z_2 \oplus \bar{z}_2} \otimes \sigma_z^{z_{2n} \oplus \bar{z}_{2n}}|L_2\rangle \\ &\quad = \tfrac{1}{2}(\langle 0+| + \langle 1-|)(\sigma_z \otimes \sigma_z)(|0+\rangle + |1-\rangle) \\ &\quad = \tfrac{1}{2}(\langle 0+| + \langle 1-|)(|0-\rangle - |1+\rangle) = 0. \end{aligned} \tag{27}$$

■

#### 2. *Product states in the support of* $\rho^{\bar{A}}$

As the following lemma shows, there are a finite number of product states in the support of $\rho^{(\bar{A})}$ (see also Fig. 3).

*Lemma 7.* Let $S^n_{0,0} \subset \mathbb{Z}_2^n$ be the collection of sequences $\mathbf{z} = (z_2, z_4, \ldots, z_{2n})$ with $z_2 = 0$ and $z_{2n} = 0$, and let $S^n_{0,1}$ be the collection of sequences $\mathbf{z} = (z_2, z_4, \ldots, z_{2n})$ with $z_2 = 0$ and $z_{2n} = 1$. Then the support of $\rho^{(\bar{A})}$ contains only $2^n$ product states given by

$$\begin{aligned} \frac{1}{\sqrt{2}}(e^{i\pi/4}|e_{\mathbf{z}}\rangle + e^{-i\pi/4}|e_{\bar{\mathbf{z}}}\rangle) &= |\tilde{+}\tilde{+}\rangle|\varphi_{\mathbf{z}}\rangle, \\ \frac{1}{\sqrt{2}}(e^{-i\pi/4}|e_{\mathbf{z}}\rangle + e^{i\pi/4}|e_{\bar{\mathbf{z}}}\rangle) &= |\tilde{-}\tilde{-}\rangle|\varphi_{\mathbf{z}}\rangle \end{aligned}$$

for all $\mathbf{z} \in S_{0,0}$ and

$$\begin{aligned} \frac{1}{\sqrt{2}}(e^{i\pi/4}|e_{\mathbf{z}}\rangle + e^{-i\pi/4}|e_{\bar{\mathbf{z}}}\rangle) &= |\tilde{+}\tilde{-}\rangle|\varphi_{\mathbf{z}}\rangle, \\ \frac{1}{\sqrt{2}}(e^{-i\pi/4}|e_{\mathbf{z}}\rangle + e^{i\pi/4}|e_{\bar{\mathbf{z}}}\rangle) &= |\tilde{-}\tilde{+}\rangle|\varphi_{\mathbf{z}}\rangle \end{aligned} \tag{28}$$

for all $\mathbf{z} \in S^n_{0,1}$, where $|\tilde{\pm}\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm i|1\rangle)$.

*Proof.* By Lemma 6, the support of $\rho^{(\bar{A})}$ is spanned by the orthonormal states $|e_{\mathbf{z}}\rangle$. Then, since $|\varphi_{\mathbf{z}}\rangle = |\varphi_{\bar{\mathbf{z}}}\rangle$, we can organize the $2^n$ eigenstates $|e_{\mathbf{z}}\rangle$ into four sets as follows:

$$\begin{aligned} &\left\{\frac{1}{\sqrt{2}}(|0+\rangle + |1-\rangle)|\varphi_{\mathbf{z}}\rangle \mid \mathbf{z} \in S^n_{0,0}\right\} && \text{for } z_2 = 0,\ z_{2n} = 0, \\ &\left\{\frac{1}{\sqrt{2}}(|0-\rangle - |1+\rangle)|\varphi_{\mathbf{z}}\rangle \mid \mathbf{z} \in S^n_{0,0}\right\} && \text{for } z_2 = 1,\ z_{2n} = 1, \\ &\left\{\frac{1}{\sqrt{2}}(|0-\rangle + |1+\rangle)|\varphi_{\mathbf{z}}\rangle \mid \mathbf{z} \in S^n_{0,1}\right\} && \text{for } z_2 = 0,\ z_{2n} = 1, \\ &\left\{\frac{1}{\sqrt{2}}(|0+\rangle - |1-\rangle)|\varphi_{\mathbf{z}}\rangle \mid \mathbf{z} \in S^n_{0,1}\right\} && \text{for } z_2 = 1, z_{2n} = 0. \end{aligned} \tag{29}$$

We wish to find product states that span all of these states. The crucial observation is that the states in the first two sets can be written using only two product states for qubits 1 and $2n+1$ and similarly for the states in the last two sets:

$$\begin{aligned} \frac{1}{\sqrt{2}}(|0+\rangle + |1-\rangle) &= \frac{1}{\sqrt{2}}(e^{-i\pi/4}|\tilde{+}\tilde{+}\rangle + e^{i\pi/4}|\tilde{-}\tilde{-}\rangle), \\ \frac{1}{\sqrt{2}}(|0-\rangle - |1+\rangle) &= \frac{1}{\sqrt{2}}(e^{i\pi/4}|\tilde{+}\tilde{+}\rangle + e^{-i\pi/4}|\tilde{-}\tilde{-}\rangle), \\ \frac{1}{\sqrt{2}}(|0-\rangle + |1+\rangle) &= \frac{1}{\sqrt{2}}(e^{-i\pi/4}|\tilde{+}\tilde{-}\rangle + e^{i\pi/4}|\tilde{-}\tilde{+}\rangle), \\ \frac{1}{\sqrt{2}}(|0-\rangle - |1+\rangle) &= \frac{1}{\sqrt{2}}(e^{i\pi/4}|\tilde{+}\tilde{-}\rangle + e^{-i\pi/4}|\tilde{-}\tilde{+}\rangle). \end{aligned} \tag{30}$$

Thus it suffices to consider $\mathbf{z} \in S^n_{0,0}$ and $\mathbf{z} \in S^n_{0,1}$. Explicitly, we conclude that the support of $\rho^{\bar{A}}$ is spanned by $2^n$ orthogonal product states

$$\begin{aligned} &\left\{|\tilde{+}\tilde{+}\rangle|\varphi_{\mathbf{z}}\rangle, |\tilde{-}\tilde{-}\rangle|\varphi_{\mathbf{z}}\rangle \mid \mathbf{z} \in S^n_{0,0}\right\} \\ &\cup\left\{|\tilde{+}\tilde{-}\rangle|\varphi_{\mathbf{z}}\rangle,\ |\tilde{-}\tilde{+}\rangle|\varphi_{\mathbf{z}}\rangle \mid \mathbf{z} \in S^n_{0,1}\right\}. \end{aligned} \tag{31}$$

We next show that these are the only product states in the support of $\rho^{(\bar{A})}$.

Suppose that $|\Psi\rangle$ is a product state in the support of $\rho^{(\bar{A})}$. Then we can find coefficients such that

$$\begin{aligned} |\Psi\rangle &= \sum_{\mathbf{z} \in S_{0,0}}^{n} (a_{\mathbf{z}}|\tilde{+}\tilde{+}\rangle|\varphi_{\mathbf{z}}\rangle + b_{\mathbf{z}}|\tilde{-}\tilde{-}\rangle|\varphi_{\mathbf{z}}\rangle) \\ &\quad + \sum_{\mathbf{z} \in S_{0,1}}^{n} (c_{\mathbf{z}}|\tilde{+}\tilde{-}\rangle|\varphi_{\mathbf{z}}\rangle + d_{\mathbf{z}}|\tilde{-}\tilde{+}\rangle|\varphi_{\mathbf{z}}\rangle) \\ &= |\tilde{+}\tilde{+}\rangle|\alpha\rangle + |\tilde{-}\tilde{-}\rangle|\beta\rangle + |\tilde{+}\tilde{-}\rangle|\gamma\rangle + |\tilde{-}\tilde{+}\rangle|\delta\rangle, \end{aligned} \tag{32}$$

where $|\alpha\rangle = \sum_{\mathbf{z} \in S^n_{0,0}} a_{\mathbf{z}}|\varphi_{\mathbf{z}}\rangle$, $|\beta\rangle = \sum_{\mathbf{z} \in S^n_{0,0}} b_{\mathbf{z}}|\varphi_{\mathbf{z}}\rangle$, $|\gamma\rangle = \sum_{\mathbf{z} \in S^n_{0,1}} c_{\mathbf{z}}|\varphi_{\mathbf{z}}\rangle$, and $|\delta\rangle = \sum_{\mathbf{z} \in S^n_{0,1}} d_{\mathbf{z}}|\varphi_{\mathbf{z}}\rangle$. Here, in an abuse of notation, $\{|\alpha\rangle, |\beta\rangle, |\gamma\rangle, |\delta\rangle\}$ may not be properly normalized unit vectors. By Eq. (26), the states $\{|\alpha\rangle, |\beta\rangle\}$ are orthogonal to the states $\{|\gamma\rangle, |\delta\rangle\}$.

Suppose first that both $|\alpha\rangle$ and $|\beta\rangle$ are nonzero. Then we can find a vector $|v\rangle$ in the linear span of $\{|\alpha\rangle, |\beta\rangle\}$ that has nonzero overlap with both $|\alpha\rangle$ and $|\beta\rangle$. Partially contracting both sides of Eq. (32) by $|v\rangle$ yields

$$(\mathrm{id} \otimes \langle v|)|\Psi\rangle = x|\tilde{+}\tilde{+}\rangle + y|\tilde{-}\tilde{-}\rangle,$$

with $x, y \neq 0$. However, since $|\Psi\rangle$ is a product state, it remains a product state under partial contraction and so the right-hand side must be a product state. However, the only product states contained in the linear span of $|\tilde{+}\tilde{+}\rangle$ and $|\tilde{-}\tilde{-}\rangle$ are these states themselves. We thus have a contradiction and so it is not possible for both $|\alpha\rangle$ and $|\beta\rangle$ to be nonzero. A similar argument shows that both $|\gamma\rangle$ and $|\delta\rangle$ cannot be nonzero. Hence, there are only four possible forms of $|\Psi\rangle$, each pairing an element in $\{|\alpha\rangle, |\beta\rangle\}$ with an element in $\{|\gamma\rangle, |\delta\rangle\}$. For example, we have could have

$$|\Psi\rangle = |\tilde{+}\tilde{+}\rangle|\alpha\rangle + |\tilde{+}\tilde{-}\rangle|\delta\rangle = |\tilde{+}\rangle(|\tilde{+}\rangle|\alpha\rangle + |\tilde{-}\rangle|\delta\rangle),$$

which is not a product state unless either $|\alpha\rangle$ or $|\delta\rangle$ is zero, since $\langle\alpha|\delta\rangle = 0$. A similar argument applies for the other three possible forms of $|\Psi\rangle$. Therefore, any product state in $|\Psi\rangle$ must have the form $|\tilde{+}\tilde{+}\rangle|\alpha\rangle$, $|\tilde{-}\tilde{-}\rangle|\beta\rangle$, $|\tilde{+}\tilde{-}\rangle|\gamma\rangle$, or $|\tilde{-}\tilde{-}\rangle|\delta\rangle$, where $|\alpha\rangle$ and $|\beta\rangle$ are product states in the span of $\{|\varphi_{\mathbf{z}}\rangle \mid \mathbf{z} \in S^n_{0,0}\}$ and $|\gamma\rangle$ and $|\delta\rangle$ are product states in the span of $\{|\varphi_{\mathbf{z}}\rangle \mid \mathbf{z} \in S^n_{0,1}\}$.

Then it finally remains to be shown that the only product states in the span of $\{|\varphi_{\mathbf{z}}\rangle \mid \mathbf{z} \in S^n_{0,0}\}$ are the states $|\varphi_{\mathbf{z}}\rangle$ themselves; likewise, the only product states in the span of $\{|\varphi_{\mathbf{z}}\rangle \mid \mathbf{z} \in S^n_{0,1}\}$ are the states $|\varphi_{\mathbf{z}}\rangle$ themselves.

Suppose that $|\alpha\rangle = \sum_{\mathbf{z}\in S^n_{0,0}} a_{\mathbf{z}}|\varphi_{\mathbf{z}}\rangle$ and $|\gamma\rangle = \sum_{\mathbf{z}\in S^n_{0,1}} a_{\mathbf{z}}|\varphi_{\mathbf{z}}\rangle$ are product states. If $n = 2$, then there is only a single party in $|\varphi_{\mathbf{z}}\rangle$. If $n = 3$, then there are two terms

$$|\alpha\rangle = a_0|{+}{+}\rangle + a_1|{-}{-}\rangle$$

and

$$|\gamma\rangle = c_0|{+}{-}\rangle + c_1|{-}{+}\rangle,$$

which require that either $a_0 = 0$ or $a_1 = 0$ and either $c_0 = 0$ or $c_1 = 0$ in order for $|\alpha\rangle$ and $|\gamma\rangle$ to be product states. We now prove the claim for arbitrary $n$ via induction, for which we have just shown the base case. Assume that the only product states in the span of $\{|\varphi_{\mathbf{z}} \mid \mathbf{z} \in S^n_{0,0})\}$ and in the span on $\{|\varphi_{\mathbf{z}} \mid \mathbf{z} \in S^n_{0,1})\}$ are the states themselves. Now consider some $|\alpha\rangle = \sum_{\mathbf{z}\in S^{n+1}_{0,0}} a_{\mathbf{z}}|\varphi_{\mathbf{z}}\rangle$. Performing a partial contraction with $\langle +|$ on the last party yields

$$(\mathbb{I} \otimes \langle +|^{(2n-1)})|\alpha\rangle = \sum_{\mathbf{z}'\in S^n_{0,0}} a_{\mathbf{z}'}|\varphi_{\mathbf{z}'}\rangle. \tag{33}$$

By the inductive assumption, the only way this can be a product state is if at most one $a_{\mathbf{z}'}$ is nonzero. Similarly,

$$(\mathbb{I} \otimes \langle -|^{(2n-1)})|\alpha\rangle = \sum_{\mathbf{z}'\in S^n_{0,1}} a_{\mathbf{z}'}|\varphi_{\mathbf{z}'}\rangle, \tag{34}$$

implying that at most one term in this summation is nonzero as well. We could have thus that

$$|\alpha\rangle = a_{\mathbf{z}_1}|\varphi_{\mathbf{z}'_1}\rangle|+\rangle + a_{\mathbf{z}_2}|\varphi_{\mathbf{z}'_2}\rangle|-\rangle, \tag{35}$$

where $\mathbf{z}'_1 \in S^n_{0,0}$ and $\mathbf{z}'_1 \in S^n_{0,1}$. However, as we have previously argued, by Eq. (26) we must have that $\langle\varphi_{\mathbf{z}'_1}|\varphi_{\mathbf{z}'_2}\rangle = 0$. We are left with $|\alpha\rangle = |\varphi_{\mathbf{z}}\rangle$ for some $\mathbf{z} \in S^{n+1}_{0,0}$. A similar line of reasoning yields the analogous result for $|\gamma\rangle$. This concludes the proof. ∎

At this point we have identified a subsystem of rank $2^n$ and characterized all product states in its support. Using Lemma 5, we can piece the parts together to obtain the following lower bound.

*Theorem 2.* We have $\mathrm{rank}(|R_{2n+1}\rangle) > 2^n$.

*Proof.* Lemma 5 constructs a reduced density matrix from $|R_{2n+1}\rangle$ of rank $2^n$, hence $\mathrm{rank}(|R_{2n+1}\rangle) \geqslant 2^n$. Now suppose for the sake of contradiction that $\mathrm{rank}(|R_{2n+1}\rangle) = 2^n$. Since $\rho^{(\bar{A})}$ has matrix rank $2^n$ for the subset $A = \{2, 4, \ldots, 2n\}$, Lemma 5 says that any CP decomposition of $|R_{2n+1}\rangle$ of minimal length must contain product states $\{\bigotimes_{c\in\bar{A}} |\psi_i\rangle^{(c)}\}_{i=1}^R$ belonging to the support of $\rho^{(\bar{A})}$. However, Lemma 7 then implies that these product states must be the ones given in (28), that is, we must be able to write

$$\begin{aligned}
|R_{2n+1}\rangle = &\sum_{\mathbf{z}\in S_{0,0}} |A_{\mathbf{z}}\rangle^{(2,4,\ldots,2n)}|\tilde{+}\tilde{+}\rangle^{(1,2n+1)}|\varphi_{\mathbf{z}}\rangle^{(3,5,\ldots,2n-1)} \\
&+ \sum_{\mathbf{z}\in S_{0,0}} |B_{\mathbf{z}}\rangle^{(2,4,\ldots,2n)}|\tilde{-}\tilde{-}\rangle^{(1,2n+1)}|\varphi_{\mathbf{z}}\rangle^{(3,5,\ldots,2n-1)} \\
&+ \sum_{\mathbf{z}\in S_{0,1}} |C_{\mathbf{z}}\rangle^{(2,4,\ldots,2n)}|\tilde{+}\tilde{-}\rangle^{(1,2n+1)}|\varphi_{\mathbf{z}}\rangle^{(3,5,\ldots,2n-1)} \\
&+ \sum_{\mathbf{z}\in S_{0,1}} |D_{\mathbf{z}}\rangle^{(2,4,\ldots,2n)}|\tilde{-}\tilde{+}\rangle^{(1,2n+1)}|\varphi_{\mathbf{z}}\rangle^{(3,5,\ldots,2n-1)},
\end{aligned} \tag{36}$$

with $|A_{\mathbf{z}}\rangle$, $|B_{\mathbf{z}}\rangle$, $|C_{\mathbf{z}}\rangle$, and $|D_{\mathbf{z}}\rangle$ all being product states. We drop the qubit labels below for readability.

At the same time, from Lemma 6 and Eq. (24) we can express the ring state as

$$|R_{2n+1}\rangle = \frac{1}{\sqrt{2^n}} \sum_{\mathbf{z}\in\mathbb{Z}_2^n} |\hat{\mathbf{z}}\rangle^{(A)}|e_{\mathbf{z}}\rangle^{(\bar{A})}, \tag{37}$$

where $|\hat{\mathbf{z}}\rangle^{(A)} := (-1)^{|\mathbf{z}|}|\mathbf{z}\rangle^{(A)}$. By inverting the equalities in Eq. (28), this can be written as

$$\begin{aligned}
|R_{2n+1}\rangle = &\frac{1}{\sqrt{2^n}} \sum_{\mathbf{z}\in S_{0,0}} \frac{e^{-i\pi/4}|\hat{\mathbf{z}}\rangle + e^{i\pi/4}|\hat{\bar{\mathbf{z}}}\rangle}{\sqrt{2}} |\tilde{+}\tilde{+}\rangle|\varphi_{\mathbf{z}}\rangle \\
&+ \frac{1}{\sqrt{2^n}} \sum_{\mathbf{z}\in S_{0,0}} \frac{e^{i\pi/4}|\hat{\mathbf{z}}\rangle + e^{-i\pi/4}|\hat{\bar{\mathbf{z}}}\rangle}{\sqrt{2}} |\tilde{-}\tilde{-}\rangle|\varphi_{\mathbf{z}}\rangle \\
&+ \frac{1}{\sqrt{2^n}} \sum_{\mathbf{z}\in S_{0,1}} \frac{e^{-i\pi/4}|\hat{\mathbf{z}}\rangle + e^{i\pi/4}|\hat{\bar{\mathbf{z}}}\rangle}{\sqrt{2}} |\tilde{+}\tilde{-}\rangle|\varphi_{\mathbf{z}}\rangle \\
&+ \frac{1}{\sqrt{2^n}} \sum_{\mathbf{z}\in S_{0,1}} \frac{e^{i\pi/4}|\hat{\mathbf{z}}\rangle + e^{-i\pi/4}|\hat{\bar{\mathbf{z}}}\rangle}{\sqrt{2}} |\tilde{-}\tilde{+}\rangle|\varphi_{\mathbf{z}}\rangle.
\end{aligned}$$

Comparing this with Eq. (36) shows that $|A_{\mathbf{z}}\rangle = \frac{1}{\sqrt{2^{n+1}}}(e^{-i\pi/4}|\hat{\mathbf{z}}\rangle + e^{i\pi/4}|\hat{\bar{\mathbf{z}}}\rangle)$, $|B_{\mathbf{z}}\rangle = \frac{1}{\sqrt{2^{n+1}}}(e^{i\pi/4}|\hat{\mathbf{z}}\rangle + e^{-i\pi/4}|\hat{\bar{\mathbf{z}}}\rangle)$, etc., which is a contradiction because these are not product states (which can be seen by noting that these are equivalent to GHZ states under local unitary operations). ■

## III. ENTANGLEMENT MEASURES ON GENERAL GRAPH STATES

### A. Extensions of bipartite measures

Here we evaluate the multipartite extensions of bipartite measures previously discussed. Recall that the multipartite extension of some bipartite entanglement measure $E$ is defined as $\min_A E_A(|\psi\rangle)$, where $E_A$ is the measure $E$ evaluated according to partition $A|\bar{A}$. Surprisingly for graph states, the multipartite extensions of many standard bipartite entanglement measures are dichotomous: one of two values based on if the graph is connected. The GME concurrence, negativity, and geometric measure have been previously calculated for connected graphs [47]. Since these results were derived independently of this work, we provide in the Appendix a self-contained and direct calculation of the following.

*Theorem 3.* We have

$$\mathcal{C}_{\mathrm{GME}}(|G\rangle) = \begin{cases} 0 & \text{if } G \text{ is a disconnected graph} \\ 1 & \text{otherwise,} \end{cases} \tag{38}$$

$$\mathcal{N}_{\mathrm{GME}}(|G\rangle) = \begin{cases} 0 & \text{if } G \text{ is a disconnected graph} \\ \frac{1}{2} & \text{otherwise,} \end{cases} \tag{39}$$

$$\mathcal{G}_{\mathrm{GME}}(|G\rangle) = \begin{cases} 0 & \text{if } G \text{ is a disconnected graph} \\ \frac{1}{2} & \text{otherwise.} \end{cases} \tag{40}$$

### B. Graph state $N$-tangle

We present the following dichotomous result for $\tau_n(|G\rangle)$.

*Theorem 4.* For any graph state $|G\rangle$ on an even number of qubits,

$$\tau_n(|G\rangle) = \begin{cases} 1 & \text{for all } v \in V,\ \delta(v) = 1 \ (\mathrm{mod} 2) \\ 0 & \text{otherwise,} \end{cases} \tag{41}$$

where $\delta(v)$ denotes the degree of vertex $v$, i.e., the number of edges incident on vertex $v$.

*Proof.* As the components of the state vector $|G\rangle$ are all real, $\tau_n(|G\rangle) = |\langle G|\sigma_y^{\otimes n}|G\rangle|^2$. We employ the following $\sigma_x$ rule [48]:

$$\sigma_x^{(a)}|G\rangle = \prod_{b\in N_a} \sigma_z^{(b)}|G\rangle. \tag{42}$$

Thus, applying $\sigma_x^{(a)}$ maps the state to another graph basis state based on the graph's edges. Of course, $\sigma_y = -i\sigma_z\sigma_x$. Note that additional global phases may be picked up in the application of the $\sigma_x$ rule based on the graph basis state. However, global phases do not factor into the calculation of $\tau_n$ and are dropped below. Thus,

$$\sigma_y^{\otimes n}|G\rangle \sim \prod_{a\in V} \sigma_z^{(a)} \prod_{b\in N_a} \sigma_z^{(b)}|G\rangle. \tag{43}$$

If there is a vertex $v$ of even degree, then $\sigma_z^{(v)}$ appears in an even number of the second products above in addition to once in the first product. Thus, $|G\rangle$ is mapped to a different graph basis state and $|\langle G|\sigma_y^{\otimes n}|G\rangle| = 0$.

If all vertices have odd degree, then an even number of $\sigma_z$ operations are applied to each party and the state is an eigenvector of $\sigma_y^{\otimes n}$. Thus, $|\langle G|\sigma_y^{\otimes n}|G\rangle| = 1$. ■

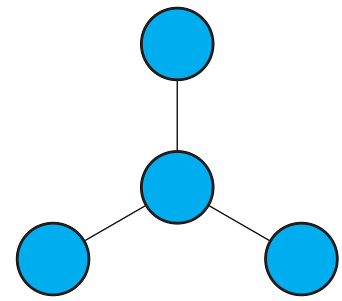

FIG. 4. GHZ state graph. For any system size, the GHZ state is local Clifford equivalent to a star graph state. Through the local complementation rule, this is also local Clifford equivalent to the fully connected graph. Note that all vertices have odd degree and thus $\tau_n = 1$ as we expect.

We illustrate this by recovering the fact that $\tau_n(|\mathrm{GHZ}\rangle) = 1$ for even $n$. The GHZ state is local Clifford equivalent to a star graph (Fig. 4). The central vertex has degree $n-1$, while the others all have degree 1. Thus, $\tau_n(|\mathrm{GHZ}\rangle) = 1$ if and only if $n$ is even, as expected.

*Remark 1.* There are $2^{\binom{n-1}{2}}$ graphs on $n$ (where $n$ is even) vertices such that $\tau_n(|G\rangle) = 1$. There are none for odd $n$.

*Proof.* By the handshake lemma, the number of vertices of odd degree must be even. For $n$ odd this proves the claim.

We claim that, for even $n$, given any graph on $n-1$ vertices, we can construct a graph on $n$ of all odd degree. As $n-1$ is odd, there must be an odd number of vertices of even degree. Simply connect the new $n$th vertex to these originally even degree vertices. As the original graph can be recovered by removing the $n$th vertex, this is a bijection between graphs of $n-1$ vertices and all odd graphs on $n$. This is illustrated in Fig. 5 The claim follows from the number of undirected graphs on $n-1$ labeled vertices. ■

This result readily extends to all stabilizer states. Recall that the weight $w(P)$ of a Pauli string is given by the number of nonidentity terms. For example, $w(X \otimes \mathbb{I}) = 1$. The following remark extends the prior theorem to all stabilizer states.

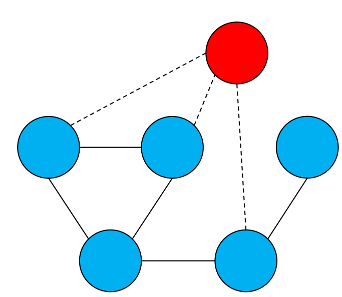

FIG. 5. Forming a graph of entirely odd degree. Starting with a graph on an odd number of vertices, we can always convert it to a new graph on one more vertex such that all vertices have odd degree. To do so, add a new vertex (red) and connect it to all vertices of even degree.

*Remark 2.* The $n$-tangle $\tau_n$ of any stabilizer state is 1 if each stabilizer is of even weight. Otherwise, $\tau_n = 0$.

*Proof.* Since the stabilizers of a graph state are of the form $\sigma_x^{(a)} \prod_{b\in N_a} \sigma_z^{(b)}$ for all vertices $a$, this clearly holds for graph states. A quick proof can then be obtained by recalling that any stabilizer state can be transformed into a graph state via local unitary operations [19]. Local unitary operations leave $\tau_n$ invariant and also do not change the weight of the stabilizers (since if $\pm P$ stabilizes $|\psi\rangle$ then $\pm UPU^\dagger$ stabilizes $U|\psi\rangle$).

We give a more constructive proof in the Appendix which does not rely upon the fact that all stabilizer states are equivalent to graph states under local unitaries. ■

It is worth noting that this property can be verified by checking a set of generators. Let $S$ be a stabilizer group and $\{g_i\}_{i=1}^k$ a generating set. Assume that each $g_i$ has even weight. The following lemma shows that any product of operators in the set $\{g_i\}_{i=1}^k$ is of even weight. Thus, every stabilizer in $S$ is of even weight and it suffices to check the weight of a generating set. Since any stabilizer group has minimal generating sets of size $n$, $\tau_n$ can be efficiently computed starting with generators.

*Lemma 8.* Let $P$ and $T$ be two commuting Pauli strings, each of even weight. Then the weight of $PT$ is even as well.

*Proof.* We refer to each $i \in [n]$ as a site. A Pauli string assigns an operator in the set $\{\mathbb{I}, \sigma_x, \sigma_y, \sigma_z\}$ to each site, which we denote by $P(i)$ and $T(i)$, respectively. The weight of $PT$ is the number of sites where $P(i) \neq T(i)$. We can expand this as

$$w(PT) = w(P) + w(T) - |\{i \in [n] \,|\{P(i), T(i)\} = 0\}| - 2|\{i \in [n] \mid P(i) = T(i)\}|. \tag{44}$$

More specifically, the weight of $PT$ is that of $P$ and $T$ minus the number of anticommuting sites and minus two times the number of matching sites. Since $P$ and $T$ commute, there are an even number of anticommuting sites. Thus, $w(PT)$ must be even. ■

## IV. CONCLUSION

In this work we tightened the bounds on CP rank of odd rings to $2^n + 1 \leqslant \text{rank}(|R_{2n+1}\rangle) \leqslant 3 \times 2^{n-1}$. This indicates that odd rings are, according to the Schmidt measure, more entangled than a line in the same number of qubits. Further, odd rings are thus not of particularly high rank. For $2n+1$ qubits, the maximum CP rank is known to be on the order of $2^{2n-1}$ [49]. Based on numerical CP decomposition, we suspect $\text{rank}(|R_{2n+1}\rangle) = 3 \times 2^n$, but the question remains open.

Beyond CP rank, we considered several multipartite entanglement measures on graph states based on bipartite measures. Surprisingly, these prove dichotomous: either 0 if the graph is disconnected or a fixed value irrespective of graph structure beyond connectivity.

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## APPENDIX

### 1. Minimal decomposition of line states

We find it informative to give a recursive minimal CP decomposition for line states $|L_n\rangle$. Note that this can be used to construct the $3 \times 2^n$ term CP decomposition for $|R_{2n+1}\rangle$.

*Remark 3.* A minimal decomposition for any $|L_n\rangle$ can be found via a simple recursive method. Define the following two qubit states:

$$|a\rangle = |0+\rangle, \quad |b\rangle = |1-\rangle, \quad |c\rangle = |0-\rangle, \quad |d\rangle = |1+\rangle. \tag{A1}$$

Any line state can be written as some sum over tensor products of these states, i.e., $|L_n\rangle = \frac{1}{2^{\lfloor n/2\rfloor}} \sum_{|\psi\rangle\in\mathcal{L}_n} |\psi\rangle$, where all terms $|\psi\rangle$ are some string such as $|aaa\ldots a\rangle$, $|acb\ldots d\rangle$, etc. By $\mathcal{L}_n$ we denote the set of such strings in the decomposition of an $n$-qubit line state. In particular, the line state on $2n$ qubits takes the form

$$|L_{2n}\rangle = \frac{1}{2^{n/2}}\left( (|a\rangle + |b\rangle) \sum_{\substack{|\psi\rangle\in\mathcal{L}_{2(n-1)} \\ |\psi\rangle_1 = |0\rangle}} |\psi\rangle + (|c\rangle + |d\rangle) \sum_{\substack{|\phi\rangle\in\mathcal{L}_{2(n-1)} \\ |\phi\rangle_1 = |1\rangle}} |\phi\rangle \right). \tag{A2}$$

Here by $|\psi\rangle_1 = |0\rangle$ we mean that the first party in $|\psi\rangle$ is in state $|0\rangle$. The line state on $2n+1$ qubits takes the form

$$|L_{2n+1}\rangle = |+\rangle P_0^{(1)}|L_{2n}\rangle + |-\rangle P_1^{(1)}|L_{2n}\rangle. \tag{A3}$$

The base case is $|L_2\rangle = \frac{1}{\sqrt{2}}(|a\rangle + |b\rangle)$.

*Proof.* We can write a line state as

$$|L_n\rangle = \frac{1}{2^{\lfloor n/2\rfloor}} \sum_{x\in\mathbb{F}_2^n} c(x)|x\rangle, \quad c(x) = \prod_{i=0}^{n-2} (-1)^{x_i x_{i+1}}. \tag{A4}$$

This follows from the action of $U^{(i,i+1)}$. Thus, we are looking to find a decomposition that contains all binary strings with equal weight, but with signs based off of the number of consecutive ones. It is clear that $|a\rangle$ and $|b\rangle$ satisfy these requirements. From inspection it is clear that $|c\rangle$ and $|d\rangle$ satisfy this property when following a party in the state $|1\rangle$. Further, $|a\rangle/|b\rangle$ combined and $|c\rangle/|d\rangle$ combined generate every binary string when expanded in the computational basis. When $n$ is even, we simply append new characters such that the sign properties are maintained. When $n$ is odd we do the same but with an addition of $|\pm\rangle$ chosen such that the sign property is maintained.

The recursive structure above doubles the number of terms in the decomposition between $|L_{2n}\rangle$ ($|L_{2n+1}\rangle$) and $|L_{2(n+1)}\rangle$ ($|L_{2(n+1)+1}\rangle$). As the base case is rank 2, these are rank $2^{\lfloor n/2\rfloor}$ decompositions, which is optimal. ■

### 2. Upper bound for the seven-qubit ring

Here we explicitly construct the rank $3\times 2^2 = 12$ decomposition for $|R_7\rangle$. From Remark 3 the seven-qubit line state can be written as

$$|L_7\rangle = \frac{1}{2\sqrt{2}}|+\rangle(|aaa\rangle + |acb\rangle + |cba\rangle + |cdb\rangle) + \frac{1}{2\sqrt{2}}|-\rangle(|baa\rangle + |bcb\rangle + |dba\rangle + |ddb\rangle), \quad \text{(A5)}$$

$$P_0^{(1)}P_1^{(7)}|L_7\rangle = \frac{1}{4\sqrt{2}}|0\rangle(|aa0\rangle - |ac1\rangle + |cb0\rangle - |cd1\rangle + |ba0\rangle - |bc1\rangle + |db0\rangle - |dd1\rangle)|1\rangle. \quad \text{(A6)}$$

Following the proof in the main text, we can find the desired decomposition from that of $|R_5\rangle$, $|a^{(4)}\rangle = \frac{1}{2\sqrt{2}}|0\rangle(|{+}0{+}\rangle + |{-}1{-}\rangle)$ and $|b^{(4)}\rangle = \frac{1}{2\sqrt{2}}|0\rangle(|{+}0{-}\rangle + |{-}1{+}\rangle)$,

$$P_1^{(7)}|\phi^{(7)}\rangle = \frac{1}{\sqrt{2}}(|a^{(4)}\rangle|0-\rangle + |b^{(4)}\rangle|1+\rangle)|1\rangle = \frac{1}{4}|0\rangle[(|{+}0{+}\rangle + |{-}1{-}\rangle)|0-\rangle + (|{+}0{-}\rangle + |{-}1{+}\rangle)|1+\rangle]|1\rangle. \quad \text{(A7)}$$

It can be verified that these are the same states via expanding into the computational basis. As $|R_7\rangle = U^{(1,7)}|L_7\rangle = (\sigma_z^{(2n+1)} + 2P_0^{(1)}\otimes P_1^{(7)})|L_7\rangle$, we can thus write $|R_7\rangle$ in the 12-term decomposition

$$|R_7\rangle = \frac{1}{2\sqrt{2}}|+\rangle(|0+0+0+\rangle + |0+0-1-\rangle + |0-1-0+\rangle + |0-1+1-\rangle) + \frac{1}{2\sqrt{2}}|-\rangle(|1-0+0+\rangle + |1-0-1-\rangle + |1+1-0+\rangle + |1+1+1-\rangle) + \frac{1}{2}|0\rangle[(|{+}0{+}\rangle + |{-}1{-}\rangle)|0-\rangle + (|{+}0{-}\rangle + |{-}1{+}\rangle)|1+\rangle]|1\rangle. \quad \text{(A8)}$$

### 3. Calculation of GME entanglement for graph states

To show Theorem 3, we first observe the following.

*Corollary 1.* The reduced density matrix for any individual qubit party $i$ corresponding to vertex $v$ is

$$\rho_i = \begin{cases} \frac{1}{2}\mathbb{I}, & \delta(v) > 0 \\ |+\rangle\langle+|, & \delta(v) = 0, \end{cases} \quad \text{(A9)}$$

where $\delta(v)$ is the degree of vertex $v$.

*Proof.* This follows readily from Lemma 1. If $v$ is not an isolated vertex, there is at least one nonzero value in $\Gamma_{A\bar{A}}$, where $A = \{v\}$, and thus $\text{rank}(\Gamma_{A\bar{A}}) = 1$. If $v$ is an isolated vertex, then $\text{rank}(\Gamma_{A\bar{A}}) = 0$. ■

With this corollary and Lemma 1, we now show that the measures previously introduced are either 0 or a fixed constant based on if the graph is connected.

*Theorem 5.* We have

$$\mathcal{C}_{\text{GME}}(|G\rangle) = \begin{cases} 0 & \text{if } G \text{ is a disconnected graph} \\ 1 & \text{otherwise.} \end{cases} \quad \text{(A10)}$$

*Proof.* From Lemma 1 we know that any reduced density matrix is maximally mixed on a certain subspace of dimension $2^d = 2^{\text{rank}(\Gamma_{A\bar{A}})}$. By finding $\max_A \text{Tr}(\rho_A^2)$ we minimize GME concurrence. The purity of a $k$-dimensional maximally mixed state is $\frac{1}{k}$. If there is a disconnected component $A$, $\text{rank}(\Gamma_{A\bar{A}}) = 0$ and $\mathcal{C}_{\text{GME}}(|G\rangle) = 0$. Otherwise, the maximal purity is $\frac{1}{2}$, which is achieved by considering any single vertex. Thus, $\mathcal{C}_{\text{GME}}(|G\rangle) = 1$. ■

*Theorem 6.* We have

$$\mathcal{N}_{\text{GME}}(|G\rangle) = \begin{cases} 0 & \text{if } G \text{ is a disconnected graph} \\ \frac{1}{2} & \text{otherwise.} \end{cases} \quad \text{(A11)}$$

*Proof.* Before continuing, we note that negativity can be equivalently written as a summation of the absolute value of the negative eigenvalues of the partial transpose

$$\mathcal{N}(\rho^{AB}) = \tfrac{1}{2}(\|\rho^{T_A}\|_1 - 1) = \sum_{\lambda<0}|\lambda|, \quad \text{(A12)}$$

where $\lambda$ are the eigenvalues of $\rho^{T_A}$. Next we use Lemma 1 to write $|G\rangle$ in a Schmidt decomposition $|G\rangle = 2^{-d/2}\sum_{i=1}^{2^d}|u_i\rangle|v_i\rangle$, where $d = \text{rank}(\Gamma_{A:\bar{A}})$. Thus, the partial transpose with respect to $\bar{A}$ is

$$\mathbb{I}\otimes T(|G\rangle\langle G|) = 2^{-d}\sum_{i,j=1}^{2^d}|u_i\rangle\langle u_j|\otimes|v_j\rangle\langle v_i|. \quad \text{(A13)}$$

This has a negative eigenvalue $-2^{-d}$ with multiplicity $\binom{2^d}{2}$, corresponding to eigenvectors $\frac{1}{\sqrt{2}}(|u_i\rangle|v_j\rangle - |u_j\rangle|v_i\rangle)$. Thus, the negativity according to partition $A|\bar{A}$ is

$$\mathcal{N}(\rho^{A\bar{A}}) = \binom{2^d}{2}2^{-d} = \frac{1}{2}(2^d - 1). \quad \text{(A14)}$$

Clearly this is increasing with $d$. Thus, the GME negativity will be minimized by a partition with the smallest $\text{rank}(\Gamma_{A\bar{A}})$. If $G$ is disconnected, there exists a partition $A$ such that $\text{rank}(\Gamma_{A\bar{A}}) = 0$. Otherwise, $d = 1$ for any partition into a single vertex, for which $\mathcal{N}(\rho^{A\bar{A}}) = \frac{1}{2}$. ■

*Theorem 7.* We have

$$\mathcal{G}_{\text{GME}}(|G\rangle) = \begin{cases} 0 & \text{if } G \text{ is a disconnected graph} \\ \frac{1}{2} & \text{otherwise.} \end{cases} \quad \text{(A15)}$$

*Proof.* If $G$ is disconnected there is a partition with Schmidt coefficient 1 (Corollary 1). Otherwise, the largest Schmidt coefficient possible is always $\frac{1}{2}$ via Lemma 1. ■

### 4. Alternative proof of Remark 2

In this section we give a more explicit proof for Remark 2 on $\tau_n$ evaluated on stabilizer states. We restate the remark here: The $n$-tangle $\tau_n$ of any stabilizer state is 1 if each stabilizer is of even weight. Otherwise, $\tau_n = 0$.

*Proof.* Since $\tau_n(|\psi\rangle) = |\langle\psi|\tilde{\psi}\rangle|^2$, we proceed by finding $|\tilde{s}\rangle\langle\tilde{s}| = \sigma_y^{\otimes n}|s^*\rangle\langle s^*|\sigma_y^{\otimes n}$ for a stabilizer state $|s\rangle$. Any stabilizer state can be written in the form

$$|s\rangle\langle s| = \frac{1}{2^n}\sum_{P\in S} a_P P, \tag{A16}$$

where $S$ is the stabilizer subgroup, $P$ is a Pauli string, and $a_P \in \{\pm 1\}$. Let $w_y(P)$ denote the number of $\sigma_y$ terms in $P$. For example, $w_y(\sigma_y \otimes \sigma_x \otimes \sigma_y) = 2$. Then

$$|s^*\rangle\langle s^*| = \frac{1}{2^n}\sum_{P\in S} a_P(-1)^{w_y(P)}P, \tag{A17}$$

since complex conjugation leaves $\sigma_z$ and $\sigma_x$ unchanged. Next we consider conjugation by $\sigma_y^{\otimes n}$:

$$\sigma_y^{\otimes n}|s^*\rangle\langle s^*|\sigma_y^{\otimes n} = \frac{1}{2^n}\sum_{P\in S} a_P(-1)^{w_y(P)}\sigma_y^{\otimes n}P\sigma_y^{\otimes n}. \tag{A18}$$

Since the Pauli strings either commute or anticommute, $\sigma_y^{\otimes n}P\sigma_y^{\otimes n}$ is equal to $P$ or $-P$. In particular, $\sigma_y^{\otimes n}P\sigma_y^{\otimes n} = (-1)^{w_z(P)+w_x(P)}P$, where $w_z(P)$ and $w_x(P)$ are the numbers of $\sigma_z$ and $\sigma_x$ terms, respectively. In total, we have that

$$\sigma_y^{\otimes n}|s^*\rangle\langle s^*|\sigma_y^{\otimes n} = \frac{1}{2^n}\sum_{P\in S} a_P(-1)^{w(P)}P, \tag{A19}$$

where $w(P)$ is the weight of $P$. If $a_P(-1)^{w(P)} \neq a_P$ for any $P \in S$, then $|\tilde{s}\rangle$ must be orthogonal to $|s\rangle$ (since it then necessarily belongs to a different eigenspace of a stabilizer). This proves the claim. ■


[1] A. Einstein, B. Podolsky, and N. Rosen, Can quantum-mechanical description of physical reality be considered complete? Phys. Rev. **47**, 777 (1935).

[2] R. Horodecki, P. Horodecki, M. Horodecki, and K. Horodecki, Quantum entanglement, Rev. Mod. Phys. **81**, 865 (2009).

[3] A. Ekert and R. Jozsa, Quantum algorithms: Entanglement-enhanced information processing, Philos. Trans. R. Soc. A **356**, 1769 (1998).

[4] A. Ekert and P. L. Knight, Entangled quantum systems and the Schmidt decomposition, Am. J. Phys. **63**, 415 (1995).

[5] M. Walter, D. Gross, and J. Eisert, Multipartite entanglement, in *Quantum Information: From Foundations to Quantum Technology Applications*, edited by D. Bruß and G. Leuchs (Wiley, New York, 2016), pp. 293–330.

[6] G. Vidal, Entanglement monotones, J. Mod. Opt. **47**, 355 (2000).

[7] C. H. Bennett, D. P. DiVincenzo, J. A. Smolin, and W. K. Wootters, Mixed-state entanglement and quantum error correction, Phys. Rev. A **54**, 3824 (1996).

[8] E. M. Rains, Rigorous treatment of distillable entanglement, Phys. Rev. A **60**, 173 (1999).

[9] G. Vidal and R. F. Werner, Computable measure of entanglement, Phys. Rev. A **65**, 032314 (2002).

[10] S. Lee, D. P. Chi, S. D. Oh, and J. Kim, Convex-roof extended negativity as an entanglement measure for bipartite quantum systems, Phys. Rev. A **68**, 062304 (2003).

[11] T.-C. Wei and P. M. Goldbart, Geometric measure of entanglement and applications to bipartite and multipartite quantum states, Phys. Rev. A **68**, 042307 (2003).

[12] W. K. Wootters, Entanglement of formation and concurrence, Quantum Inf. Comput. **1**, 27 (2001).

[13] F. Mintert, M. Kuś, and A. Buchleitner, Concurrence of mixed multipartite quantum states, Phys. Rev. Lett. **95**, 260502 (2005).

[14] R. Raussendorf and H. J. Briegel, A one-way quantum computer, Phys. Rev. Lett. **86**, 5188 (2001).

[15] M. Hein, W. Dür, J. Eisert, R. Raussendorf, M. Van den Nest, and H. J. Briegel, Entanglement in graph states and its applications, in *Quantum Computers, Algorithms and Chaos, Proceedings of the International School of Physics "Enrico Fermi," Course CLXII, Varenna, 2005*, edited by G. Casati, D. Shepelyansky, P. Zoller, and G. Benenti (IOS, Amsterdam, 2006), pp. 115–218.

[16] D. Schlingemann and R. F. Werner, Quantum error-correcting codes associated with graphs, Phys. Rev. A **65**, 012308 (2001).

[17] D. Markham and B. C. Sanders, Graph states for quantum secret sharing, Phys. Rev. A **78**, 042309 (2008).

[18] S. Anders and H. J. Briegel, Fast simulation of stabilizer circuits using a graph-state representation, Phys. Rev. A **73**, 022334 (2006).

[19] M. Van den Nest, J. Dehaene, and B. De Moor, Graphical description of the action of local Clifford transformations on graph states, Phys. Rev. A **69**, 022316 (2004).

[20] D. Schlingemann, Stabilizer codes can be realized as graph codes, Quantum Inf. Comput. **2**, 307 (2002).

[21] M. Hein, J. Eisert, and H. J. Briegel, Multiparty entanglement in graph states, Phys. Rev. A **69**, 062311 (2004).

[22] J. Eisert and H. J. Briegel, Schmidt measure as a tool for quantifying multiparticle entanglement, Phys. Rev. A **64**, 022306 (2001).

[23] E. Chitambar, R. Duan, and Y. Shi, Tripartite entanglement transformations and tensor rank, Phys. Rev. Lett. **101**, 140502 (2008).

[24] L. Chen, E. Chitambar, R. Duan, Z. Ji, and A. Winter, Tensor rank and stochastic entanglement catalysis for multipartite pure states, Phys. Rev. Lett. **105**, 200501 (2010).

[25] N. Yu, C. Guo, and R. Duan, Obtaining a W state from a Greenberger-Horne-Zeilinger state via stochastic local operations and classical communication with a rate approaching unity, Phys. Rev. Lett. **112**, 160401 (2014).

[26] P. Vrana and M. Christandl, Asymptotic entanglement transformation between W and GHZ states, J. Math. Phys. **56**, 022204 (2015).

[27] P. Vrana and M. Christandl, Entanglement distillation from Greenberger–Horne–Zeilinger shares, Commun. Math. Phys. **352**, 621 (2017).

[28] L. Chen and S. Friedland, The tensor rank of tensor product of two-qubit W states is eight, Linear Algebra Its Appl. **543**, 1 (2018).

[29] M. Christandl and J. Zuiddam, Tensor surgery and tensor rank, Comput. Complex. **28**, 27 (2019).

[30] W. Bruzda, S. Friedland, and K. Życzkowski, Rank of a tensor and quantum entanglement, Linear Multilinear Algebra, **72**, 1796 (2023).

[31] F. L. Hitchcock, The expression of a tensor or a polyadic as a sum of products, Stud. Appl. Math. **6**, 164 (1927).

[32] R. A. Harshman, Foundations of the PARAFAC procedure: Models and conditions for an "explanatory" multimodal factor analysis, UCLA Working Papers in Phonetics **16**, 1 (1970).

[33] J. Håstad, Tensor rank is NP-complete, in *Proceedings of the 16th International Colloquium on Automata, Languages, and Programming*, edited by G. Ausiello, M. Dezani-Ciancaglini, and S. R. Della Rocca (Springer, Berlin, 1989), pp. 451–460.

[34] V. Strassen, Gaussian elimination is not optimal, Numer. Math. **13**, 354 (1969).

[35] D. Bini, in *Proceedings of the Sixth International Conference on Industrial and Applied Mathematics, Zürich, 2007* (European Mathematical Society, Helsinki, 2007).

[36] D. Bini, M. Capovani, F. Romani, and G. Lotti, $O(n^{2.7799})$ complexity for $n \times n$ approximate matrix multiplication, Inf. Process. Lett. **8**, 234 (1979).

[37] J. I. de Vicente and M. Huber, Multipartite entanglement detection from correlation tensors, Phys. Rev. A **84**, 062306 (2011).

[38] Y. Dai, Y. Dong, Z. Xu, W. You, C. Zhang, and O. Gühne, Experimentally accessible lower bounds for genuine multipartite entanglement and coherence measures, Phys. Rev. Appl. **13**, 054022 (2020).

[39] Z.-H. Ma, Z.-H. Chen, J.-L. Chen, C. Spengler, A. Gabriel, and M. Huber, Measure of genuine multipartite entanglement with computable lower bounds, Phys. Rev. A **83**, 062325 (2011).

[40] V. Coffman, J. Kundu, and W. K. Wootters, Distributed entanglement, Phys. Rev. A **61**, 052306 (2000).

[41] A. Wong and N. Christensen, Potential multiparticle entanglement measure, Phys. Rev. A **63**, 044301 (2001).

[42] X. Li and D. Li, Relationship between the *n*-tangle and the residual entanglement of even *n* qubits, Quantum Inf. Comput. **10**, 1018 (2010).

[43] H. J. Briegel and R. Raussendorf, Persistent entanglement in arrays of interacting particles, Phys. Rev. Lett. **86**, 910 (2001).

[44] J. C. Adcock, S. Morley-Short, A. Dahlberg, and J. W. Silverstone, Mapping graph state orbits under local complementation, Quantum **4**, 305 (2020).

[45] A. Cabello, A. J. López-Tarrida, P. Moreno, and J. R. Portillo, Entanglement in eight-qubit graph states, Phys. Lett. A **373**, 2219 (2009).

[46] S. Severini, Two-colorable graph states with maximal Schmidt measure, Phys. Lett. A **356**, 99 (2006).

[47] P. Contreras-Tejada, C. Palazuelos, and J. I. de Vicente, Resource theory of entanglement with a unique multipartite maximally entangled state, Phys. Rev. Lett. **122**, 120503 (2019).

[48] S. Y. Looi, L. Yu, V. Gheorghiu, and R. B. Griffiths, Quantum-error-correcting codes using qudit graph states, Phys. Rev. A **78**, 042303 (2008).

[49] T. Sumi, T. Sakata, and M. Miyazaki, Rank of tensors with size $2 \times \cdots \times 2$, arXiv:1306.0708.