Comment on "Asymptotic quantum algorithm for the Toeplitz systems"

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(Received 1 March 2024; accepted 11 June 2024; published 27 August 2024)

In their recent paper [Phys. Rev. A **97**, 062322 (2018)], Wan *et al.* proposed a quantum algorithm to solve systems of linear equations $Ax = b$, where A is an $n \times n$ Toeplitz matrix generated by the Fourier coefficients of a continuous, 2π-periodic, and positive function f. For large enough n, they claimed that the algorithm is valid for all b. In this Comment we prove that, in general, this is not true.

DOI: 10.1103/PhysRevA.110.026401

I. INTRODUCTION

Obtaining the solution of Toeplitz systems of linear equations is very common in many mathematics and engineering problems. Resolution of partial differential equations [1] or signal estimation [2], for instance, are examples of these problems.

Solving a Toeplitz system can be done efficiently on a classical computer. However, quantum algorithms are able to solve some problems faster and more efficiently. Hence, the obtention of a quantum algorithm to solve Toeplitz systems that outperforms its classical counterpart is of great interest.

In this context, in their recent paper [3], Wan *et al.* proposed a quantum algorithm to solve Toeplitz systems of linear equations of the form

$$T_n(f)x = b_n, \tag{1}$$

where b_n is an n-dimensional column vector, f is a continuous, 2π-periodic, and positive function, and $T_n(f)$ is the $n \times n$ Toeplitz matrix defined as

$$T_n(f) := \begin{pmatrix} t_0 & t_{-1} & \cdots & t_{-(n-1)} \\ t_1 & t_0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & t_{-1} \\ t_{n-1} & \cdots & t_1 & t_0 \end{pmatrix}, \tag{2}$$

with

$$t_k := \frac{1}{2\pi}\int_0^{2\pi} f(\omega)\mathrm{e}^{-\mathrm{i}k\omega}\mathrm{d}\omega, \quad k \in \{-(n-1), \dots, n-1\}.$$

The quantum algorithm presented in [3] solves Hermitian circulant systems of linear equations very efficiently. In order to take advantage of such algorithm, the strategy of Wan *et al.* is to replace the Toeplitz matrix $T_n(f)$ in (1) by a Hermitian circulant matrix $C_n(f)$ that somehow approaches $T_n(f)$ as n grows. It is worth mentioning that the strategy of replacing a Toeplitz matrix by a circulant one has been widely used in the literature in a variety of applications (see, e.g., [4–8]).

In their algorithm, Wan *et al.* claimed that the solution of the system obtained after replacing $T_n(f)$ by a particular circulant matrix converges to the solution of the original Toeplitz system regardless of the column b_n (see Corollary 2 in [3]). In this Comment we prove that, in general, this is not true.

The remainder of this paper is organized as follows. In Sec. II we give a counterexample to prove that the solution of the Toeplitz system obtained with the algorithm proposed by Wan *et al.* does not always converge to the correct solution. In Sec. III we compare the relative error between the solution obtained with the algorithm proposed by Wan *et al.* and the correct solution for two different Toeplitz systems. In Sec. IV we present some conclusions.

II. COUNTEREXAMPLE

First, we will introduce some notation. Here $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$, and $\mathbb{C}$ denote the set of positive integer numbers, the set of integer numbers, the set of real numbers, and the set of complex numbers, respectively. The dagger stands for conjugate transpose, $\|\cdot\|_2$ is the spectral norm, and $\|\cdot\|_F$ denotes the Frobenius norm. If $m, n \in \mathbb{N}$, then $\mathbb{C}^{m\times n}$ is the set of all $m \times n$ complex matrices, $0_{m\times n}$ is the $m \times n$ zero matrix, I_n is the $n \times n$ identity matrix, and $\mathrm{diag}(a_1, a_2, \dots, a_n)$ is the $n \times n$ diagonal matrix with $a_1, a_2, \dots, a_n$ the entries in the main diagonal.

Consider the Toeplitz system in (1). Here $x = [T_n(f)]^{-1}b_n$ and we denote by x^* the solution of the corresponding circulant system, that is, $x^* = [C_n(f)]^{-1}b_n$.

In their recent work (see p. 062322-4 in [3]), Wan *et al.* claimed that if $f : \mathbb{R} \to \mathbb{R}$ is continuous, 2π-periodic, and positive, then

$$\lim_{n\to\infty}\frac{\|x^* - x\|_2}{\|x\|_2} = \lim_{n\to\infty}\frac{\|[C_n(f)]^{-1}b_n - [T_n(f)]^{-1}b_n\|_2}{\|[T_n(f)]^{-1}b_n\|_2} = 0 \tag{3}$$

for all $b_n \in \mathbb{C}^{n\times 1}$ with $\|b_n\|_2 = 1$. We here prove that (3) is not true for all b_n by giving a counterexample. Specifically, we show that

$$\frac{\|x^* - x\|_2}{\|x\|_2} = \frac{\|[C_n(f)]^{-1}b_n - [T_n(f)]^{-1}b_n\|_2}{\|[T_n(f)]^{-1}b_n\|_2} \geqslant \frac{1}{5}\ \forall\, n \in \mathbb{N}\backslash\{1, 2\} \tag{4}$$

when

$$f(\omega) = \frac{3}{\sqrt{10}} + \frac{2}{\sqrt{10}}\cos(\omega)\,\forall\,\omega \in \mathbb{R}$$

and

$$b_n = \begin{pmatrix} 0_{n-2\times 1} \\ \frac{1}{\sqrt{10}} \\ \frac{3}{\sqrt{10}} \end{pmatrix}.$$

Observe that

$$\begin{aligned} f(\omega) &= \frac{3}{\sqrt{10}} + \frac{2}{\sqrt{10}}\cos(\omega) \\ &= \frac{1}{\sqrt{10}}e^{-i\omega} + \frac{3}{\sqrt{10}}e^{i0\omega} + \frac{1}{\sqrt{10}}e^{i\omega}\,\forall\,\omega \in \mathbb{R}. \end{aligned}$$

Here f is a trigonometric polynomial of degree 1, and consequently the sequence of Fourier coefficients of f, $\{t_k\}_{k\in\mathbb{Z}}$, is given by

$$t_k = \begin{cases} \frac{3}{\sqrt{10}} & \text{if } k = 0 \\ \frac{1}{\sqrt{10}} & \text{if } k \in \{-1, 1\} \\ 0 & \text{if } k \in \mathbb{Z}\backslash\{-1, 0, 1\}. \end{cases}$$

Therefore, according to (2), the Toeplitz matrix $T_n(f)$ is symmetric for all $n \in \mathbb{N}$ and

$$x = \begin{pmatrix} 0_{n-1\times 1} \\ 1 \end{pmatrix}$$

is the unique solution of the Toeplitz system $T_n(f)x = b_n$. From p. 728 in [4] or p. 197 in [5],

$$C_n(f) = T_n(f) + \begin{pmatrix} 0 & 0_{1\times n-2} & \frac{1}{\sqrt{10}} \\ 0_{n-2\times 1} & 0_{n-2\times n-2} & 0_{n-2\times 1} \\ \frac{1}{\sqrt{10}} & 0_{1\times n-2} & 0 \end{pmatrix}$$
$$\forall\, n \in \mathbb{N}\backslash\{1, 2\}. \tag{5}$$

Applying p. 5674 in [6] or Lemma 5.4 in [7] yields

$$C_n(f) = F_n \mathrm{diag}\left(f(0), f\left(\frac{2\pi}{n}\right), \ldots, f\left(\frac{2\pi(n-1)}{n}\right)\right)(F_n)^\dagger$$
$$\forall\, n \in \mathbb{N}\backslash\{1, 2\},$$

where F_n is the $n \times n$ Fourier unitary matrix given by

$$[F_n]_{j,k} = \frac{1}{\sqrt{n}}e^{-i2\pi(j-1)(k-1)/n}, \quad j, k \in \{1, \ldots, n\}. \tag{6}$$

As $T_n(f)$ is symmetric, according to (5), $C_n(f)$ is also symmetric. Furthermore, from (6), F_n is symmetric too. Thus,

$$\begin{aligned} C_n(f) = [C_n(f)]^\top &= \left[F_n \mathrm{diag}\left(f(0), f\left(\frac{2\pi}{n}\right), \ldots, f\left(\frac{2\pi(n-1)}{n}\right)\right)(F_n)^\dagger\right]^\top \\ &= [(F_n)^\dagger]^\top \left[\mathrm{diag}\left(f(0), f\left(\frac{2\pi}{n}\right), \ldots, f\left(\frac{2\pi(n-1)}{n}\right)\right)\right]^\top (F_n)^\top \\ &= (F_n)^\dagger \mathrm{diag}\left(f(0), f\left(\frac{2\pi}{n}\right), \ldots, f\left(\frac{2\pi(n-1)}{n}\right)\right)F_n\ \forall\, n \in \mathbb{N}\backslash\{1, 2\}, \end{aligned}$$

which is the form in which the matrix $C_n(f)$ is expressed in [3]. Hence, since $f(\omega) \in \left[\frac{1}{\sqrt{10}}, \frac{5}{\sqrt{10}}\right]$ for all $\omega \in \mathbb{R}$, we conclude that

$$\begin{aligned} \frac{\|x^* - x\|_2}{\|x\|_2} &= \frac{\|[C_n(f)]^{-1}b_n - [T_n(f)]^{-1}b_n\|_2}{\|[T_n(f)]^{-1}b_n\|_2} \\ &= \frac{\left\|[C_n(f)]^{-1}T_n(f)\binom{0_{n-1\times 1}}{1} - \binom{0_{n-1\times 1}}{1}\right\|_2}{\left\|\binom{0_{n-1\times 1}}{1}\right\|_2} \\ &= \left\|[C_n(f)]^{-1}T_n(f)\begin{pmatrix} 0_{n-1\times 1} \\ 1 \end{pmatrix} - \begin{pmatrix} 0_{n-1\times 1} \\ 1 \end{pmatrix}\right\|_2 \\ &= \left\|[C_n(f)]^{-1}\left[C_n(f) - \begin{pmatrix} 0 & 0_{1\times n-2} & \frac{1}{\sqrt{10}} \\ 0_{n-2\times 1} & 0_{n-2\times n-2} & 0_{n-2\times 1} \\ \frac{1}{\sqrt{10}} & 0_{1\times n-2} & 0 \end{pmatrix}\right]\begin{pmatrix} 0_{n-1\times 1} \\ 1 \end{pmatrix} - \begin{pmatrix} 0_{n-1\times 1} \\ 1 \end{pmatrix}\right\|_2 \end{aligned}$$

$$
\begin{aligned}
&= \left\| -[C_n(f)]^{-1} \begin{pmatrix} 0 & 0_{1\times n-2} & \frac{1}{\sqrt{10}} \\ 0_{n-2\times 1} & 0_{n-2\times n-2} & 0_{n-2\times 1} \\ \frac{1}{\sqrt{10}} & 0_{1\times n-2} & 0 \end{pmatrix} \begin{pmatrix} 0_{n-1\times 1} \\ 1 \end{pmatrix} \right\|_2 \\
&= \frac{1}{\sqrt{10}} \left\| [C_n(f)]^{-1} \begin{pmatrix} 1 \\ 0_{n-1\times 1} \end{pmatrix} \right\|_2 = \frac{1}{\sqrt{10}} \sqrt{\sum_{j=1}^{n} |\{[C_n(f)]^{-1}\}_{j,1}|^2} \\
&\geqslant \frac{1}{\sqrt{10}} |\{[C_n(f)]^{-1}\}_{1,1}| \\
&= \frac{1}{\sqrt{10}} \left| \left[(F_n)^{\dagger} \operatorname{diag}\left(\frac{1}{f(0)}, \frac{1}{f\left(\frac{2\pi}{n}\right)}, \dots, \frac{1}{f\left(\frac{2\pi(n-1)}{n}\right)} \right) F_n \right]_{1,1} \right| \\
&= \frac{1}{\sqrt{10}} \left| \sum_{j=1}^{n} [(F_n)^{\dagger}]_{1,j} \left[\operatorname{diag}\left(\frac{1}{f(0)}, \frac{1}{f\left(\frac{2\pi}{n}\right)}, \dots, \frac{1}{f\left(\frac{2\pi(n-1)}{n}\right)} \right) F_n \right]_{j,1} \right| \\
&= \frac{1}{\sqrt{10}} \left| \sum_{j=1}^{n} [(F_n)^{\dagger}]_{1,j} \sum_{k=1}^{n} \left[\operatorname{diag}\left(\frac{1}{f(0)}, \frac{1}{f\left(\frac{2\pi}{n}\right)}, \dots, \frac{1}{f\left(\frac{2\pi(n-1)}{n}\right)} \right) \right]_{j,k} [F_n]_{k,1} \right| \\
&= \frac{1}{\sqrt{10}} \left| \sum_{j=1}^{n} [(F_n)^{\dagger}]_{1,j} \frac{1}{f\left(\frac{2\pi(j-1)}{n}\right)} [F_n]_{j,1} \right| = \frac{1}{\sqrt{10}} \left| \sum_{j=1}^{n} \frac{1}{f\left(\frac{2\pi(j-1)}{n}\right)} \overline{[F_n]_{j,1}} [F_n]_{j,1} \right| \\
&= \frac{1}{\sqrt{10}} \left| \sum_{j=1}^{n} \frac{1}{f\left(\frac{2\pi(j-1)}{n}\right)} \left| [F_n]_{j,1} \right|^2 \right| = \frac{1}{\sqrt{10}} \sum_{j=1}^{n} \frac{1}{f\left(\frac{2\pi(j-1)}{n}\right)} \left| [F_n]_{j,1} \right|^2 \\
&= \frac{1}{\sqrt{10}} \frac{1}{n} \sum_{j=1}^{n} \frac{1}{f\left(\frac{2\pi(j-1)}{n}\right)} \geqslant \frac{1}{\sqrt{10}} \frac{1}{n} \sum_{j=1}^{n} \frac{1}{\frac{5}{\sqrt{10}}} = \frac{1}{5} \; \forall n \in \mathbb{N} \backslash \{1, 2\}.
\end{aligned}
$$

Equation (4) proves that Corollary 2 in [3] is not true. Therefore, the proof of Corollary 2 in [3] presented by Wan *et al.* is not correct. The error is in Eq. (11) in [3]. They claimed that Eq. (11) in [3] is obtained by using the inequality given in Sec. 5.8 in [9],

$$
\frac{\|B^{-1}b - A^{-1}b\|_2}{\|A^{-1}b\|_2} \leqslant \frac{\kappa(A)\frac{\|B-A\|_F}{\|A\|_F}}{1-\kappa(A)\frac{\|B-A\|_F}{\|A\|_F}} \quad \text{if } \|A^{-1}\|_F \|B-A\|_F < 1,
$$

where $\kappa(A)$ is the condition number of A. However, they did not realize that, in general, $A = T_n(f)$ and $B = C_n(f)$ do not satisfy the assumption $\|A^{-1}\|_F \|B - A\|_F < 1$ (see Eq. (5.8.5) in [9]).

Although Corollary 2 in [3] is not true, the quantum algorithm presented by Wan *et al.* can be applied to solve the Toeplitz systems considered in Corollary 3 in [3] and to solve any Hermitian circulant system.

III. NUMERICAL SIMULATION OF TWO TOEPLITZ SYSTEMS

In this section we perform a numerical simulation to study the relative error between the solution obtained with the algorithm proposed in [3] and the correct solution for two Toeplitz systems. First, we consider the Toeplitz system of the counterexample presented in Sec. II. Second, we analyze a Toeplitz system for which this algorithm is valid.

First, we consider the Toeplitz system in (1), with

$$
T_n(f) = \begin{pmatrix} \frac{3}{\sqrt{10}} & \frac{1}{\sqrt{10}} & 0 & \cdots & 0 \\ \frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}} & \frac{1}{\sqrt{10}} & \ddots & \vdots \\ 0 & \frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}} & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \frac{1}{\sqrt{10}} \\ 0 & \cdots & 0 & \frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}} \end{pmatrix} \tag{7}
$$

and

$$
b_n = \begin{pmatrix} 0_{n-2\times 1} \\ \frac{1}{\sqrt{10}} \\ \frac{3}{\sqrt{10}} \end{pmatrix}.
$$

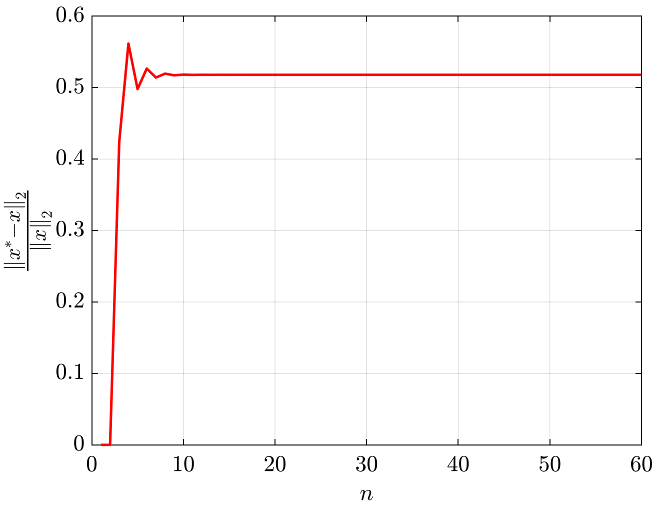

FIG. 1. Relative error between x^* and x for the Toeplitz system of the counterexample.

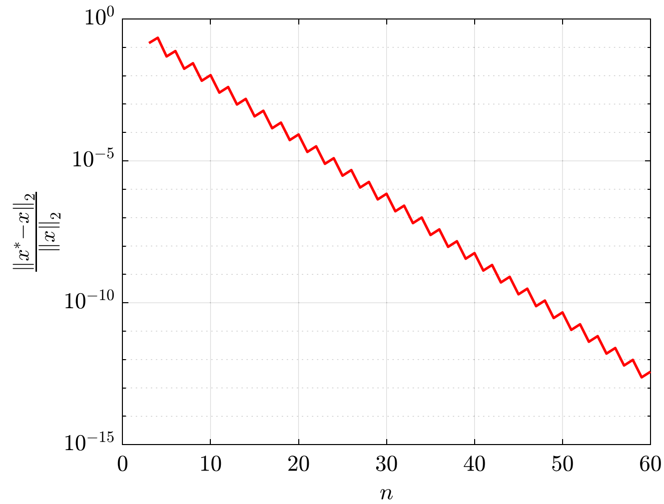

FIG. 2. Relative error between x^* and x for a Toeplitz system of the type considered in Corollary 3 in [3].

Figure 1 shows the relative error between the solution obtained with the algorithm proposed by Wan *et al.*, denoted by x^*, and the actual solution of the Toeplitz system, denoted by x, for different values of n. It can be observed that the relative error does not vanish and

$$\frac{\|x^* - x\|_2}{\|x\|_2} \geqslant \frac{1}{5}$$

when n grows, as expected according to (4). Consequently, x^* does not converge to x.

Second, we consider the Toeplitz system in (1) with $T_n(f)$ the one given in (7) and

$$b_n = \begin{pmatrix} 0_{\lfloor (n-1)/2 \rfloor \times 1} \\ 1 \\ 0_{\lceil (n-1)/2 \rceil \times 1} \end{pmatrix},$$

where $\lfloor\cdot\rfloor$ and $\lceil\cdot\rceil$ denote the floor and ceiling functions, respectively. This system is an example of the Toeplitz systems considered in Corollary 3 in [3]. Figure 2 shows the relative error between the solution obtained with the algorithm proposed by Wan *et al.*, x^*, and the actual solution of the Toeplitz system, x, for different values of n. It can be observed that the relative error vanishes as n grows. Therefore, x^* converges to x.

IV. CONCLUSIONS

In this Comment we have shown that Corollary 2 in [3] is not correct and consequently the algorithm proposed in [3] is not valid for solving the Toeplitz systems considered therein. However, there are some particular Toeplitz systems for which this algorithm is valid, for instance, the Toeplitz systems considered in Corollary 3 in [3] and any Hermitian circulant system.

ACKNOWLEDGMENTS

This work was partially supported by the Department of Education of the Basque Government through a predoctoral research grant, as well as by the Spanish Ministry of Science and Innovation through the MADDIE project (No. PID2022-137099NB-C44).

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