Steady-state tripartite non-Gaussian entanglement and steering in the output field from intracavity triple-photon parametric down-conversion

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Nondegenerate triple-photon parametric down-conversion (NTPD) is a potential source for unconditional tripartite non-Gaussian entangled states of continuous variables. Recent experiment has demonstrated strong third-order correlations among bright photon triplets via microwave NTPD in a superconducting cavity [C. W. Sandbo Cheng *et al.*, Phys. Rev. X **10**, 011011 (2020)]. Previous theoretic works have revealed that only short-time genuine tripartite non-Gaussian entanglement can be generated in NTPD even in the absence of dissipation. In this paper, we investigate the properties of tripartite non-Gaussian entanglement and steering in the cavity output field by taking into account the cavity dissipation. We first derive experimentally detectable criteria for fully inseparable and genuine tripartite non-Gaussian entanglement and steering. With the criteria, we then find that steady-state tripartite non-Gaussian entanglement and steering can be generated in the output field, although they merely exist in the short-time regime inside the cavity. We also find that the initial cavity-field coherent states can obviously enhance the steady-state and transient tripartite entanglement and steering, in comparison to the case of initial vacuum states. We finally show that the output tripartite non-Gaussian steerable correlations can be applied to the remote generation of negative Wigner-function quantum states by homodyne detection.

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I. INTRODUCTION

Entanglement, a fundamental property within the domain of quantum mechanics, describes the inseparability inherent in composite quantum systems consisting of multiple constituent elements and is a vital resource in quantum information science [1]. The concept of quantum steering traces its origin back to 1935, originally termed by Schrödinger in his response to the well-known Einstein-Podolsky-Rosen (EPR) paradox [2] to critique the nonlocal aspects of quantum mechanics. It has been verified that EPR steering is intermediate between Bell nonlocality and entanglement [3] and useful in, e.g., one-sided device-independent quantum cryptography [4], subchannel discrimination [5,6], and secure quantum teleportation [7]. Recent studies have further shown that Gaussian steering is a sufficient and necessary condition for remotely creating negative Wigner nonclassicality on certain conditions [8,9]. Steering has nowadays been realized in a variety of systems of discrete and continuous variables [10,11].

Non-Gaussian entanglement of continuous variables is of paramount importance in various aspects of quantum science [12–15]. Non-Gaussian entangled states feature diverse high-order moments of field quadrature operators, beyond second-order moments statistics in Gaussian states, resulting in its advantages in the aspects, e.g., fundamental test of quantum mechanics such as loophole-free Bell test [16], quantum error correction [17], entanglement distillation [18], and especially universal quantum computation [19]. Non-Gaussian entangled states have also proven to be more efficient in quantum communication [20–24] and quantum sensing and metrology [25–27]. Over the past decades, the generation of non-Gaussian entangled states via photon-addition or photon-subtraction operation on Gaussian states has been extensively studied theoretically and experimentally [28–34], but this approach is probabilistic and the target states are conditioned on the detection results. Alternative way is to employ intrinsic nonlinearity of systems to achieve unconditional non-Gaussian states [35–40].

Nondegenerate triple-photon parametric down-conversion (NTPD) describes a nonlinear process in which a pump photon is down-converted into photon triplets of different frequencies and is considered as a potential source for deterministically generating tripartite non-Gaussian highly entangled states directly [41–48]. So far, NTPD process has been demonstrated in different three-order optical nonlinear mediums but with low rates of triple photon generation, which makes it difficult to certify quantum features [49–52]. Very recently, microwave NTPD in a superconducting cavity has been achieved and strong third-order correlations among bright photon triplets has been demonstrated [53]. This achievement immediately attracts much interest in exploring the properties of tripartite non-Gaussian entanglement of continuous variables [54–58]. It has been revealed that genuine tripartite non-Gaussian entanglement can be directly generated but it just appears in the short-time regime, even without the consideration of dissipation. In view that the NTPD process operates in the cavity in the experiment [53], quantum steering is stronger than entanglement and steady-state entanglement is more desirable, a question naturally arises:

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Does the cavity output field exhibit steady-state tripartite non-Gaussian entanglement and even steering? As we know, the cavity output field is a continuum of frequency modes and indeed subject to realistic detection and various applications, with different behaviors from the intracavity field [59,60]. In this paper, we intend to investigate in detail the properties of tripartite non-Gaussian entanglement and steering in the cavity output field in the system of intracavity NTPD. To this end, we treat the output field with specific modes as virtual cavities connected to the NTPD cavity in a way of a quantum input-output (cascade) network [61,62]. We also derive experimentally detectable criteria for fully inseparable and genuine tripartite non-Gaussian entanglement and steering. With the criteria, we find that the steady-state tripartite non-Gaussian entanglement and steering can be generated in the output field, although they merely exist in the short-time regime inside the cavity. Moreover, the initial coherent cavity-field states can effectively enhance the output steady-state and intracavity transient entanglement and steering. We also show that the output tripartite non-Gaussian steerable nonlocality can be used to remotely generate negative Wigner-function non-Gaussian states by homodyne detection. Our findings further unravel the novel non-Gaussian nonclassical characteristics in the nonlinear NTPD process.

This paper is arranged as follows. In Sec. II, the system is introduced and the master equation is given. In Sec. III, the criteria for fully inseparable and genuine tripartite non-Gaussian entanglement and steering are derived in detail. In Sec. IV, the numerical results are presented. In Sec. V, the summary is given.

II. SYSTEM

In this paper, we consider an intracavity nondegenerate three-photon down-conversion process, which can be described by the Hamiltonian ($\hbar = 1$) [52,53]

$$\hat{H}_s = \omega_0 \hat{b}_0^\dagger \hat{b}_0 + \sum_{k=1}^{3} \omega_k \hat{b}_k^\dagger \hat{b}_k + g_0(\hat{b}_0^\dagger \hat{b}_1 \hat{b}_2 \hat{b}_3 + \hat{b}_0 \hat{b}_1^\dagger \hat{b}_2^\dagger \hat{b}_3^\dagger), \quad (1)$$

where g_0 represents the three-order nonlinear coupling constant, and the annihilation operators $\hat{b}_0$ and $\hat{b}_k$ ($k = 1, 2, 3$, and similarly hereinafter) describe the the pump and three down-converted modes, respectively. By choosing the frequencies $\omega_0 = \omega_1 + \omega_2 + \omega_3$ and treating the pump classically (assuming it in a large-amplitude coherent state), the above interaction Hamiltonian reduces to

$$\hat{H}_s = g(e^{i\theta} \hat{b}_1 \hat{b}_2 \hat{b}_3 + e^{-i\theta} \hat{b}_1^\dagger \hat{b}_2^\dagger \hat{b}_3^\dagger), \quad (2)$$

where $g = g_0|\beta_0|$ represents the NTPD interaction strength proportional to the pump amplitude $\beta_0 \equiv |\beta_0| e^{i\theta}$. Here, we take $\theta = \pi/2$ for simplicity. Note that the phase θ can be canceled via the local transformation $\hat{b}_j e^{-i\theta} \to \hat{b}_j$, which does not alter the tripartite correlations. Microscopically, it describes that the medium absorbs a high-frequency pump photon and then emits three low-frequency photons simultaneously into the cavity modes, during which strong non-Gaussian quantum correlations are therefore be established among the down-converted photons. The NTPD process has been demonstrated in optical nonlinear mediums [52] and in a superconducting device [53].

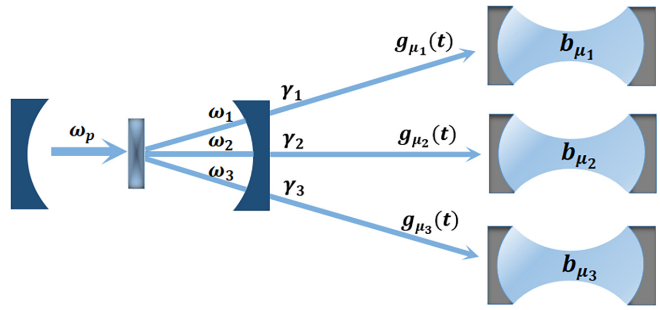

FIG. 1. Schematic diagram for intracavity NTPD in which a high-frequency pump photon (with frequency ω_p) is down-converted into a triplet [(denoted by $\hat{b}_k$ ($k = 1, 2, 3$)] in the cavity modes of frequencies ω_k. The parameters γ_k denote the dissipation rates of the cavity modes. Three virtual cavities, with modes denoted by $\hat{b}_{\mu_k}$, are employed to study the specific temporal modes $\mu_k(t)$ in the continuous cavity output fields $\hat{b}_k^{\rm out}(t)$ via the cascaded couplings to the (master) cavity modes $\hat{b}_k$, with the time-dependent coupling rates $g_{\mu_k}(t)$ dependent on the modes $\mu_k(t)$.

For the cavity mode $\hat{b}_k$ coupled to external environment, one is interested in quantum properties of its output field which is indeed subject to detection and various realistic applications. The output field $\hat{b}_k^{\rm out}$ is related to the cavity mode $\hat{b}_k$ and input field $\hat{b}_k^{\rm in}$ via the input-output relation $\hat{b}_k^{\rm out}(t) = \sqrt{\gamma_k}\hat{b}_k(t) - \hat{b}_k^{\rm in}(t)$, where γ_k denote the dissipation rate of the cavity mode and $\hat{b}_k^{\rm in}$ is the vacuum input. Since the cavity output field has continuous spectra, from which one can define a temporal mode $\mu_k(t)$ with the annihilation operator

$$\hat{b}_{\mu_k} = \int \mu_k^*(t')\hat{b}_k^{\rm out}(t')dt', \quad (3)$$

which satisfies the commutation relation $[\hat{b}_{\mu_k}, \hat{b}_{\mu_k}^\dagger] = 1$, leading to $\int |\mu_k(t)|^2 dt = 1$. The mode $\hat{b}_{\mu_k}$ filtered from the output field $\hat{b}_k^{\rm out}(t)$ can be considered as a virtual cavity (filter) which is directionally driven by the output field, as shown in Fig. 1, with the coupling strength between the output field and the virtual cavity [61]

$$g_{\mu_k}(t) = -\frac{\mu_k^*(t)}{\sqrt{\int_0^t dt' |\mu_k(t')|^2}}. \quad (4)$$

In this description, the master equation for the whole cascaded system consisting of the (master) cavity mode $\hat{b}_k$ in the NTPD and the corresponding (slave) virtual cavity field $\hat{b}_{\mu_k}$ can be obtained as [61,62]

$$\frac{d\hat{\rho}}{dt} = -i[\hat{H}_s + \hat{H}_{\rm ex}, \hat{\rho}] + \sum_{k=1}^{3} \mathcal{L}_k[\hat{J}_k]\hat{\rho}, \quad (5)$$

where the unidirectional-coupling resulted coherent exchange couplings

$$\hat{H}_{\rm ex} = \frac{i}{2}\sum_{k=1}^{3}(\sqrt{\gamma_k} g_{\mu_k}^* \hat{b}_k^\dagger \hat{b}_{\mu_k} - \text{H.c.}), \quad (6)$$

and the collective decay $\mathcal{L}[\hat{J}_k]\hat{\rho} = \hat{J}_k\hat{\rho}\hat{J}_k^\dagger - \frac{1}{2}(\hat{J}_k^\dagger\hat{J}_k\hat{\rho} + \hat{\rho}\hat{J}_k^\dagger\hat{J}_k)$, with the jump operators

$$\hat{J}_k = \sqrt{\gamma_k}\hat{b}_k + g^*_{\mu_k}\hat{b}_{\mu_k}, \tag{7}$$

describing the collective dissipation due to the couplings of the down-converted cavity and virtual cavity modes to the corresponding common vacuum reservoirs. With Eq. (5), we can study quantum correlations in the intracavity and output fields. Here the six-mode master equation will be solved numerically by using quantum-jump Monte Carlo approach to reduce the Hilbert space dimensions. In this setting, the master equation (5) can be unveiled by considering that the virtual cavity modes are monitored via continuous photon counting and the state of the whole system on one quantum trajectory can be described by the state vector [63]

$$d|\psi(t)\rangle = \sum_{k=1}^{3}\left[dN_k(t)\left(\frac{\hat{J}_k}{\sqrt{\langle\hat{J}_k^\dagger\hat{J}_k\rangle}} - \hat{1}\right) + dt\left(\frac{\langle\hat{J}_k^\dagger\hat{J}_k\rangle(t)}{2} - \frac{\hat{J}_k^\dagger\hat{J}_k}{2} - i\hat{H}\right)\right]|\psi(t)\rangle, \tag{8}$$

where $\hat{H} = \hat{H}_s + \hat{H}_{\rm ex}$ and the stochastic increment $dN_k(t)$ is either one or zero, representing (no) registration of photons of the detector. The density matrix $\hat{\rho}$ is obtained by performing ensemble average on different quantum trajectories. Here, unless otherwise stated, the ensemble average is done with two thousand trajectories. In addition, to solve the equation we consider the initial states of the down-converted modes to be vacuum or coherent states and the virtual cavity modes to be vacuum states.

We consider two kinds of the coupling $g_{\mu_k}(t)$. The first is time dependent and determined by the most populated modes $\mu_k(t)$ via the relation (4). The most populated modes in the output field, which depend on the autocorrelation functions of the cavity modes $\hat{b}_k$, i.e.,

$$\Gamma_k^{(1)}(t_1, t_2) = \gamma_k\langle\hat{b}_k^\dagger(t_1)\hat{b}_k(t_2)\rangle = \sum_i n_{k,i}\mu^*_{k,i}(t_1)\mu_{k,i}(t_2), \tag{9}$$

where $n_{k,i}$ are the mean-photon numbers in each orthogonal (temporal) mode $\mu_{k,i}(t)$. Here we only consider the most populated mode among the modes $\mu_{k,i}$, which is denoted by $\hat{b}_{\mu_k}$ in Eq. (3) with the mode profile μ_k. The two-time correlation function $\langle\hat{b}_k^\dagger(t_1)\hat{b}_k(t_2)\rangle$ can be obtained with the quantum regression theorem and the master equation for the intracavity modes $\hat{b}_k$ [i.e., $g_{\mu k} = 0$ in Eq. (5)]. Figure 2 depicts the modes $\mu_k(t)$ and the time dependence of the mean-photon numbers of $\langle\hat{b}_k^\dagger\hat{b}_k\rangle$ and $\langle\hat{b}_{\mu_k}^\dagger\hat{b}_{\mu_k}\rangle$, respectively, for initial vacuum and coherent states of the down-converted cavity modes. Such a consideration gives rise to the time-dependent coupling g_{μ_k}, as shown in Fig. 2. Besides, we also consider constant coupling, i.e.,

$$g_{\mu_k} = \sqrt{\gamma_{\mu_k}}, \tag{10}$$

which means that the output field is filtered with generic cavities of Lorentz line shapes.

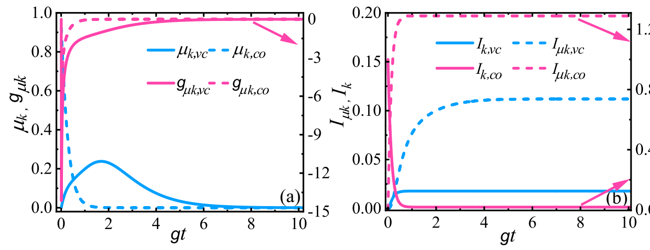

FIG. 2. (a) The most populated output modes $\mu_k(t)$ and the coupling g_{μ_k} for initial vacuum and coherent states of the intracavity modes $\hat{b}_k$, respectively. (b) The mean photon numbers $I_k = \langle\hat{b}_k^\dagger\hat{b}_k\rangle$ and $I_{\mu k} = \langle\hat{b}_{\mu_k}^\dagger\hat{b}_{\mu_k}\rangle$ of the intracavity modes $\hat{b}_k$ and virtual cavity modes $\hat{b}_{\mu_k}$. The parameters $\gamma_k = 9g$ and $\beta_k = 1$. The abbreviations "*vc*" and "*co*" in subscripts stand for initial vacuum and coherent states (similarly in Figs. 5 and 9).

III. DETECTABLE CRITERIA FOR TRIPARTITE NON-GAUSSIAN ENTANGLEMENT AND STEERING

The NTPD process in Eq. (2) is nonlinear and evolves in non-Gaussian states of which quantum characteristics are determined by various high-order moments. To fully capture non-Gaussian correlation nature in the three-mode system, we introduce single-mode high-order quadratures of the operators $\hat{a}_k$ ($a = \{\hat{b}, \hat{b}_\mu\}$ for the present system)

$$\hat{X}_k^n = \hat{a}_k^{\dagger n} + \hat{a}_k^n, \quad \hat{Y}_k^n = i(\hat{a}_k^{\dagger n} - \hat{a}_k^n) \tag{11}$$

for the kth mode and two-mode high-order quadratures

$$\hat{X}_{lm}^n = \hat{a}_l^{\dagger n}\hat{a}_m^{\dagger n} + \hat{a}_l^n\hat{a}_m^n, \quad \hat{Y}_{lm}^n = i(\hat{a}_l^{\dagger n}\hat{a}_m^{\dagger n} - \hat{a}_l^n\hat{a}_m^n) \tag{12}$$

for the lth and mth modes ($\{l, m\} = \{1, 2, 3\}$). The commutation relations for these quadratures $[\hat{X}_k^n, \hat{Y}_k^n] = i\hat{C}_k^n$ and $[\hat{X}_{lm}^n, \hat{Y}_{lm}^n] = i\hat{C}_{lm}^n$. For the present system, $\hat{C}_k^n = 2$ and $8\hat{I}_k + 4$, and $\hat{C}_{lm}^n = 2(\hat{I}_l + \hat{I}_m + 1)$ and $4[2 + \hat{I}_l(3 + \hat{I}_l) + \hat{I}_m^2(1 + 2\hat{I}_l) + \hat{I}_m(3 + 4\hat{I}_l + 2\hat{I}_l^2)]$, for $n = 1$ and 2, respectively, with $\hat{I}_k = \hat{b}_k^\dagger\hat{b}_k$.

To study tripartite entanglement and steering in the system, one can divide the system into a bipartite, i.e., $\{k, (l, m)\}$, and there are three kinds of such bipartition, i.e., $\{1, (2, 3)\}$, $\{2, (1, 3)\}$ and $\{3, (2, 1)\}$. The bipartite entanglement between the kth mode and the subsystem (l, m) falsifies the separable model for the system's density operator

$$\hat{\rho}_{lm-k} = \sum_i \eta_i\hat{\rho}_k^i\hat{\rho}_{lm}^i, \tag{13}$$

where $\sum_i \eta_i = 1$ and $\hat{\rho}_{k(l,m)}$ is the density operator of the subsystem $k(l, m)$. By defining the linear combinations

$$\hat{U}_{k,lm}^n = \hat{X}_k^n + g_{k,n}\hat{X}_{lm}^n, \quad \hat{V}_{k,lm}^n = \hat{Y}_k^n + h_{k,n}\hat{Y}_{lm}^n, \tag{14}$$

with the gain parameters $g_{k,n}$ and $h_{k,n}$, the sum of their variances satisfies

$$\langle(\Delta\hat{U}_{k,lm}^n)^2\rangle + \langle(\Delta\hat{V}_{k,lm}^n)^2\rangle \geqslant C_k^n + |g_{k,n}h_{k,n}|C_{lm}^n \tag{15}$$

for the bipartite separable model, with $C_{k(lm)}^n = \langle\hat{C}_{k(lm)}^n\rangle$. The violation of the above inequality verifies the corresponding bipartite entanglement. The violation of all three inequalities for the three bipartitions, i.e.,

$$\langle(\Delta\hat{U}^n_{1,23})^2\rangle + \langle(\Delta\hat{V}^n_{1,23})^2\rangle \geqslant C^n_1 + |g_{1,n}h_{1,n}|C^n_{23}, \quad (16a)$$

$$\langle(\Delta\hat{U}^n_{2,13})^2\rangle + \langle(\Delta\hat{V}^n_{2,13})^2\rangle \geqslant C^n_2 + |g_{2,n}h_{2,n}|C^n_{13}, \quad (16b)$$

$$\langle(\Delta\hat{U}^n_{3,12})^2\rangle + \langle(\Delta\hat{V}^n_{3,12})^2\rangle \geqslant C^n_3 + |g_{3,n}h_{3,n}|C^n_{12}, \quad (16c)$$

demonstrates fully inseparable tripartite entanglement. When the whole system is symmetric with respect to the three modes [i.e., the master equation (5) is invariant by exchanging the operators $\hat{a}_1$, $\hat{a}_2$, $\hat{a}_3$], the criteria of fully inseparable tripartite entanglement can be simplified as (see the Appendix)

$$\begin{aligned} &|\langle\hat{a}^n_1\hat{a}^n_2\hat{a}^n_3\rangle - \langle\hat{a}^n_1\rangle\langle\hat{a}^n_2\hat{a}^n_3\rangle| \\ &> \sqrt{\left[\frac{\langle\hat{I}_1!\rangle}{\langle\hat{I}^{-n}_1!\rangle} - \langle\hat{a}^n_1\rangle^2\right]\left[\frac{\langle\hat{I}_2!\hat{I}_3!\rangle}{\langle\hat{I}^{-n}_2!\hat{I}^{-n}_3!\rangle} - \langle\hat{a}^n_2\hat{a}^n_3\rangle^2\right]} \\ &\equiv \mathcal{F}^n_e, \end{aligned} \quad (17)$$

where the sign $\hat{I}^{-n}_k = (\hat{I}_k - n)$, with the gain parameters $g_{k,n} = -h_{k,n}$.

The genuine tripartite entanglement is confirmed if the state can not be written as a more general state mixed by the three bipartitions, i.e.,

$$\begin{aligned} \hat{\rho}_{123} = {} & P_1\sum_{i_1}\eta_{i_1}\rho^{i_1}_1\rho^{i_1}_{23} + P_2\sum_{i_2}\eta_{i_2}\rho^{i_2}_2\rho^{i_2}_{13} \\ & + P_3\sum_{i_3}\eta_{i_3}\rho^{i_3}_3\rho^{i_3}_{12}, \end{aligned} \quad (18)$$

where $\sum_i P_i = 1$ and $\sum_i \eta_i = 1$. For the variances in Eq. (16), the inequality for confirming the genuine tripartite non-Gaussian entanglement is derived in detail in the Appendix. Again, when the present system is symmetric, the criterion of genuine tripartite non-Gaussian entanglement reduces to (see the Appendix)

$$\begin{aligned} &|\langle\hat{a}^n_1\hat{a}^n_2\hat{a}^n_3\rangle - \langle\hat{a}^n_1\rangle\langle\hat{a}^n_2\hat{a}^n_3\rangle| \\ &> 3\sqrt{\left[\frac{\langle\hat{I}_1!\rangle}{\langle\hat{I}^{-n}_1!\rangle} - \langle\hat{a}^n_1\rangle^2\right]\left[\frac{\langle\hat{I}_2!\hat{I}_3!\rangle}{\langle\hat{I}^{-n}_2!\hat{I}^{-n}_3!\rangle} - \langle\hat{a}^n_2\hat{a}^n_3\rangle^2\right]} \\ &\equiv \mathcal{G}^n_e. \end{aligned} \quad (19)$$

We see that the fully inseparable and genuine tripartite non-Gaussian entanglement depends on the high-order self and cross correlations, i.e., $\langle\hat{a}^n_k\rangle$ and $\langle\hat{a}^n_l\hat{a}^n_m\rangle$. For the present system, when the cavity mode $\hat{b}_k$ is initially seeded with coherent states, these terms have nonzero values and have obvious effects on the non-Gaussian tripartite entanglement and steering, as will be shown later. When the system starts from vacuum or thermal states, $\langle\hat{a}^n_k\rangle = 0$ and $\langle\hat{a}^n_l\hat{a}^n_m\rangle = 0$, and the above criteria of Eqs. (17) and (19) for $n = 1$ are further simplified into

$$|\langle\hat{a}_1\hat{a}_2\hat{a}_3\rangle| > \sqrt{\langle\hat{I}_1\rangle\langle\hat{I}_2\hat{I}_3\rangle} \quad (20)$$

and

$$|\langle\hat{a}_1\hat{a}_2\hat{a}_3\rangle| > 3\sqrt{\langle\hat{I}_1\rangle\langle\hat{I}_2\hat{I}_3\rangle}, \quad (21)$$

respectively, which can also been derived directly with the Hillery-Zubairy entanglement criterion [64].

We next derive the criterion for tripartite steering in the system. Different from the entanglement, the steering of the kth mode by the subsystem (l, m) is confirmed by violating the model of local hidden state (LHS), i.e.,

$$\hat{\rho}_{lm\to k} = \sum_i \eta_i\hat{\rho}^i_{kQ}\rho^i_{lm}, \quad (22)$$

where we utilize $\hat{\rho}^i_{kQ}$ and $\hat{\rho}^i_{lm}$ to replace $\hat{\rho}^i_k$ and $\hat{\rho}^i_{lm}$ in Eq. (13), respectively, since for the LHS model no explicit assumption is made that $\hat{\rho}^i_{lm}$ would necessarily be a quantum state described by a quantum density operator. According to the LHS model, the sum of the variances of the operators $\hat{U}^n_{k,lm}$ and $\hat{V}^n_{k,lm}$ satisfies the inequality

$$\langle[\Delta\hat{U}^n_{k,lm}]^2\rangle + \langle[\Delta\hat{V}^n_{k,lm}]^2\rangle \geqslant C^n_k, \quad (23)$$

whose violation means the bipartite steering from the subsystem (l, m) to the kth mode. The violation of all three inequalities for the three bipartitions

$$S^n_1 = \langle(\Delta\hat{U}^n_{1,23})^2\rangle + \langle(\Delta\hat{V}^n_{1,23})^2\rangle \geqslant C^n_1, \quad (24a)$$

$$S^n_2 = \langle(\Delta\hat{U}^n_{2,13})^2\rangle + \langle(\Delta\hat{V}^n_{2,13})^2\rangle \geqslant C^n_2, \quad (24b)$$

$$S^n_3 = \langle(\Delta\hat{U}^n_{3,12})^2\rangle + \langle(\Delta\hat{V}^n_{3,12})^2\rangle \geqslant C^n_3 \quad (24c)$$

for any n is sufficient to confirm fully inseparable tripartite steering for the present three-mode system [65]. For the symmetric system, the fully inseparable tripartite steering becomes (see the Appendix)

$$\begin{aligned} &|\langle\hat{a}^n_1\hat{a}^n_2\hat{a}^n_3\rangle - \langle\hat{a}^n_1\rangle\langle\hat{a}^n_2\hat{a}^n_3\rangle| \\ &> \frac{1}{2}\sqrt{\frac{\langle\hat{I}_2!\hat{I}_3!\rangle}{\langle\hat{I}^{-n}_2!\hat{I}^{-n}_3!\rangle} + \frac{\langle\hat{I}^{+n}_2!\hat{I}^{+n}_3!\rangle}{\langle\hat{I}_2!\hat{I}_3!\rangle} - 2\langle\hat{a}^n_2\hat{a}^n_3\rangle^2} \\ &\times\sqrt{\frac{\langle\hat{I}_1!\rangle}{\langle\hat{I}^{-n}_1!\rangle} + \frac{\langle\hat{I}^{+n}_1!\rangle}{\langle\hat{I}_1!\rangle} - \frac{1}{2}C^n_1 - 2\langle\hat{a}^n_1\rangle^2} \\ &\equiv \mathcal{F}^n_s. \end{aligned} \quad (25)$$

with $\hat{I}^{+n}_k = (\hat{I}_k + n)$.

Similarly, the genuine tripartite steering is achieved if one can exclude more general LHS models that are constructed from convex combinations of LHS models across the three bipartitions [66], i.e.,

$$\begin{aligned} \hat{\rho}_{123} = {} & P_1\sum_{i_1}\eta_{i_1}\hat{\rho}^{i_1}_{1Q}\rho^{i_1}_{23} + P_2\sum_{i_2}\eta_{i_2}\hat{\rho}^{i_2}_{2Q}\rho^{i_2}_{13} \\ & + P_3\sum_{i_3}\eta_{i_3}\hat{\rho}^{i_3}_{3Q}\rho^{i_3}_{12}, \end{aligned} \quad (26)$$

where $\sum_i P_i = 1$, and $\sum_i \eta_i = 1$. With Eqs. (24), the violation of the inequality $S^n_1 + S^n_2 + S^n_3 \geqslant \min\{C^n_1, C^n_2, C^n_3\}$ for any n is sufficient to certify genuine tripartite non-Gaussian steering. In our fully symmetric system, the criteria of genuine tripartite non-Gaussian steering can be derived as

(see the Appendix)

$$\begin{aligned}&|\langle \hat{a}_1^n \hat{a}_2^n \hat{a}_3^n\rangle - \langle \hat{a}_1^n\rangle\langle \hat{a}_2^n \hat{a}_3^n\rangle| \\ &\quad > \frac{1}{2}\sqrt{\frac{\langle \hat{I}_2!\hat{I}_3!\rangle}{\langle \hat{I}_2^{-n}!\hat{I}_3^{-n}!\rangle} + \frac{\langle \hat{I}_2^{+n}!\hat{I}_3^{+n}!\rangle}{\langle \hat{I}_2!I_3!\rangle} - 2\langle \hat{a}_2^n \hat{a}_3^n\rangle^2} \\ &\quad\quad \times \sqrt{\frac{\langle \hat{I}_1!\rangle}{\langle \hat{I}_1^{-n}!\rangle} + \frac{\langle \hat{I}_1^{+n}!\rangle}{\langle \hat{I}_1!\rangle} - \frac{1}{6}C_1^n - 2\langle \hat{a}_1^n\rangle^2} \\ &\quad \equiv \mathcal{G}_s^n. \end{aligned} \quad (27)$$

For the case of initial thermal or vacuum states, the above criteria of Eqs. (25) and (27) for $n = 1$ further reduce to

$$|\langle \hat{a}_1 \hat{a}_2 \hat{a}_3\rangle| > \sqrt{\left\langle \left(\hat{I}_2 + \frac{1}{2}\right)\left(\hat{I}_3 + \frac{1}{2}\right) + \frac{1}{4}\right\rangle \langle \hat{I}_1\rangle} \quad (28)$$

and

$$|\langle \hat{a}_1 \hat{a}_2 \hat{a}_3\rangle| > \sqrt{\left\langle \left(\hat{I}_2 + \frac{1}{2}\right)\left(\hat{I}_3 + \frac{1}{2}\right) + \frac{1}{4}\right\rangle \left\langle \hat{I}_1 + \frac{1}{3}\right\rangle}. \quad (29)$$

We see that the tripartite non-Gaussian entanglement and steering criteria for $n = 1$ just depend on the three-order amplitude correlation $\langle \hat{X}_1 \hat{X}_2 \hat{X}_3\rangle = 2\langle \hat{a}_1 \hat{a}_2 \hat{a}_3\rangle$ for the present system, intensity correlations $\langle \hat{I}_k \hat{I}_{k'}\rangle$, and intensities $\langle \hat{I}_k\rangle$, which can be measured in the recent NTPD experiment [53]. Note that it is shown from Eqs. (20) and (28) that the condition for achieving the full inseparable tripartite steering is more strict than that for the full inseparable tripartite entanglement, but it is not true for the genuine tripartite entanglement and steering, as revealed by Eqs. (21) and (29). This is essentially because these conditions are sufficient for detecting the entanglement and steering. Note that in deriving the inequality (A8), six terms on the right-hand side in the inequality (A6) are discarded, different from the derivation of the condition for genuine tripartite entanglement in Eq. (A17).

IV. NUMERICAL RESULTS

In this section, we investigate in detail the features of intracavity and output non-Gaussian tripartite entanglement and steering in the NTPD. We define the quantities $E_{f(g)}^n = |\langle \hat{a}_1^n \hat{a}_2^n \hat{a}_3^n\rangle - \langle \hat{a}_1^n\rangle\langle \hat{a}_2^n \hat{a}_3^n\rangle| - \mathcal{F}(\mathcal{G})_e^n$ and $S_{f(g)}^n = |\langle \hat{a}_1^n \hat{a}_2^n \hat{a}_3^n\rangle - \langle \hat{a}_1^n\rangle\langle \hat{a}_2^n \hat{a}_3^n\rangle| - \mathcal{F}(\mathcal{G})_s^n$ to characterize the full inseparable (genuine) tripartite entanglement and steering. Their existences are signified by the conditions $S_{f(g)}^n > 0$ and $E_{f(g)}^n > 0$.

In Fig. 3, the time evolution of the tripartite entanglement and steering inside the cavity are plotted for $n = 1$ and 2, with initial vacuum and coherent states of the cavity modes $\hat{b}_k$. It is shown that the non-Gaussian tripartite genuine entanglement and fully inseparable steering can be achieved in the short-time regime. The genuine tripartite steering cannot be found (at least) with the quantity S_g^n. It is obviously shown that for the same value of n, the full inseparable entanglement lasts longer than genuine entanglement since the latter exhibits stronger quantum correlations, and both of them for $n = 1$ last longer than those for $n = 2$, which is contrary to the case of the steering. Compared to the case of initial vacuum states, the maximally achievable tripartite

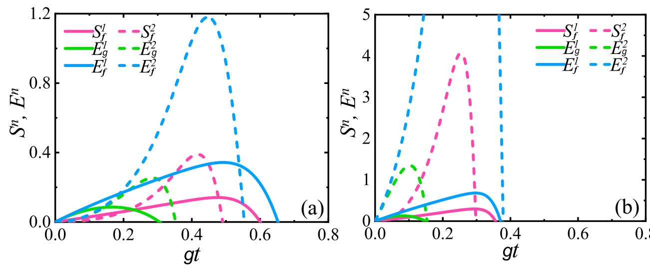

FIG. 3. The time evolution of the tripartite entanglement E^n and steering S^n ($n = 1, 2$) inside the cavity for $\gamma_k = 0$. The initial states of the cavity modes are considered to be vacua in (a) and coherent states in (b) with the amplitude $\beta_k = 1$.

entanglement and steering are obviously enhanced by initial coherent states, as shown in Fig. 3(b). The reason may be that for the NTPD process, the initial coherent seeding of the cavity modes results in nonzero single-mode and two-mode high-order correlations, i.e., $\langle \hat{b}_k^n\rangle \neq 0$ and $\langle \hat{b}_l^n \hat{b}_m^n\rangle \neq 0$, which in turn increase the tripartite non-Gaussian quantum correlations. Figure 4(a) plots the dependence of the maximal tripartite entanglement and steering on the amplitudes $\beta_k = \beta$, and it shows that the increasing of the initial amplitudes β, the maximal tripartite entanglement and steering increase first, then decrease, and finally disappear. This is because in this situation we can express the down-converted cavity mode $\hat{b}_k$ as the sum of average amplitude $\langle \hat{b}_k\rangle$ and corresponding quantum fluctuation $\delta\hat{b}_k$ around the amplitude. Then, the Hamiltonian $\hat{H}_s$ in Eq. (2) can be divided into two parts: the linearized and nonlinear parts which are respectively dependent and independent on the average amplitude $\langle \hat{b}_k\rangle$. As the initial amplitude β increases such that the linearized part dominates over the nonlinear one, the NTPD mainly appears as a Gaussian system, i.e., three concurrent two-mode nondegenerate parametric down-conversion of $\delta\hat{b}_k$ and its non-Gaussian characteristics is suppressed, which leads to that the non-Gaussian entanglement and steering decrease gradually and the NTPD dominantly exhibits Gaussian tripartite entanglement and steering. We therefore see that the initial preparation of the down-converted modes in weak coherent states is helpful to the generation of the tripartite non-Gaussian entanglement and steering. In addition, it shows from Fig. 4(b) that the initial coherent states shorten the existence time of the

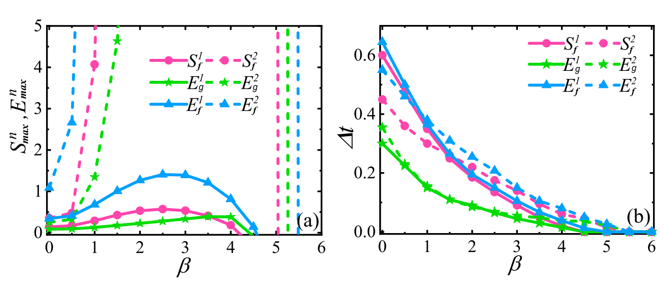

FIG. 4. (a) The dependence of the maximum entanglement $E_{\max}^n$ and steering $S_{\max}^n$ inside the cavity on the amplitude β_k of the initial cavity-field coherent states. (b) The duration Δt of the entanglement and steering versus the coherent amplitude β_k. The parameters are the same as Fig. 3.

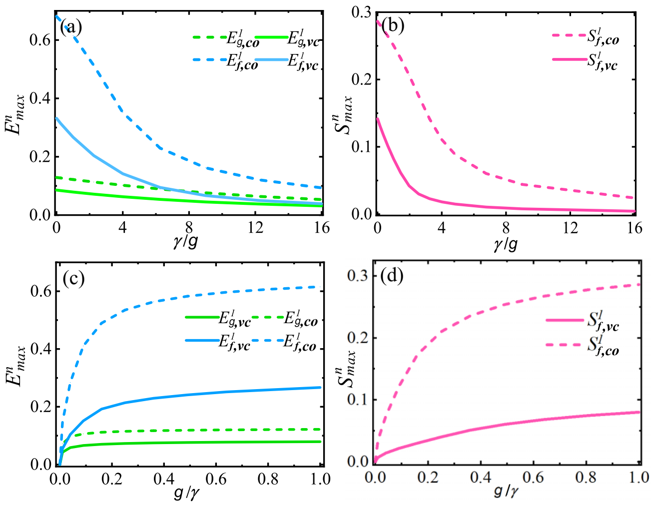

FIG. 5. The maximum entanglement $E^n_{\max}$ and steering $S^n_{\max}$ ($n = 1$) inside the cavity as the functions of the cavity dissipation rates $\gamma_k = \gamma$ [(a) and (b)] and the coupling g [(c) and (d)] for the initial cavity-field coherent amplitude $\beta_k = 1$.

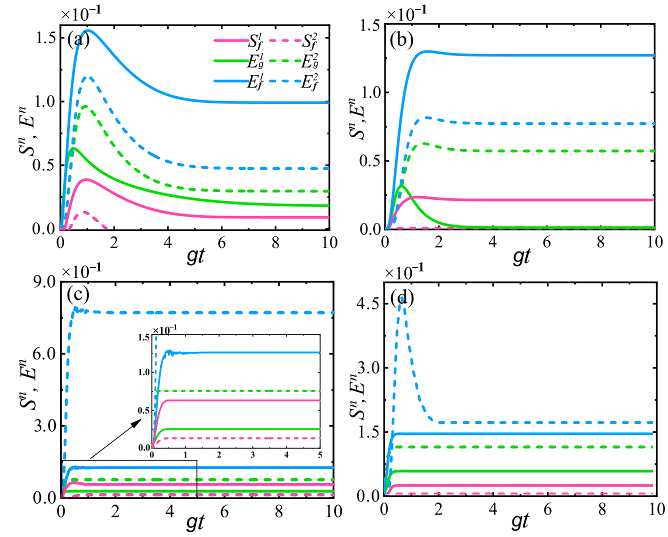

FIG. 6. The time evolution of the entanglement E^n and steering S^n ($n = 1, 2$) of the cavity output field for initial cavity-field vacua [(a) and (b)] and coherent states [(c) and (d)] with the amplitude $\beta_k = 1$, with the time-dependent coupling $g_{\mu_k}(t)$ [(a) and (c)] and constant coupling $g_{\mu_k}(t) = 1.5\sqrt{g}$ [(b)] and (d)]. The cavity dissipation rates $\gamma_k = 9g$.

tripartite entanglement and steering. Figure 5 illustrates the dependence of the maximal tripartite entanglement and steering on the cavity dissipation rates γ_k and interaction strength g. As expected, the tripartite entanglement and steering decrease with the increasing of the dissipation rates, and they eventually disappear when the dissipation rates $\gamma_k \gg g$, irrespective of initial vacuum or coherent states, as plotted in Figs . 5(a) and 5(b), since the intracavity field escapes rapidly from the cavity for the large cavity dissipation rates. In addition, it can be seen from Figs. 5(c) and 5(d) that the maximal tripartite entanglement and steering increase as the interaction strength g increases and the growth rates decrease gradually when the strength further arises.

We next investigate the properties of the tripartite non-Gaussian entanglement and steering in the output field by investigating the virtual cavity modes $\hat{b}_{\mu_k}$. In Fig. 6, the time evolution of the entanglement and steering is plotted for the time-dependent and constant couplings g_{μ_k}, with initial vacuum and coherent states of the intracavity modes. It shows that steady-state non-Gaussian genuine tripartite entanglement and fully inseparable tripartite steering can be achieved, although they are just present in the short-time regime inside the cavity. This can be understood as that the variances in Eqs. (15) and (23) of the intracavity field can be considered as the sum of those of all output modes μ_k and therefore the steady-state tripartite entanglement and steering in output field may be generated, although they only exist in a finite time. Further, we see from Figs. 6(a) and 6(c) that the steady-state tripartite entanglement and steering for $n = 1$ and 2 can also be enhanced by the initial coherent states of the down-converted cavity modes. The entanglement and steering for the initial vacua in Fig. 6(a) drop and slowly stabilize after reaching the maximal values, while they stabilize as the maxima are reached for the case of initial coherent states in Fig. 6(c), as the coupling g_{μ_k} [see Fig. 2(a)] approaches the steady states much faster in the latter case. The entanglement and steering for the constant coupling g_{μ_k} are less improved with the coherent states in Fig. 6(d), but much faster reach the steady states compared to the case of initial vacua in Fig. 6(b). In Figs. 6(c) and 6(d), the entanglements E_g^1 and E_f^2 go down after reaching the highest points because the ratio of the cavity dissipation rates $\sqrt{\gamma_k}$ to the constant coupling g_{μ_k} is not optimized for them and here the same ratio is settled simply. The purity of the output states, defined by $P = \mathrm{Tr}[(\hat{\rho}_{b_{\mu1}b_{\mu2}b_{\mu3}})^2]$, is plotted in Fig. 7. It is shown that the purity for the initial coherent states is decreased, although the entanglement and steering are enhanced by them, compared to the case of the initial vacuum states. In addition, the purity for the constant coupling in Figs. 6(b) and 6(d) is obviously higher than that in Figs. 6(a) and 6(c) because of the larger coupling g_{μ_k}.

We finally consider the application of the steady-state output tripartite non-Gaussian steering to remotely generating negative Wigner-function conditional states via homodyne detection. Specifically, we investigate the conditional states of the output mode $\hat{b}_{\mu_3}$ by homodyning the quadratures $\hat{X}_{b_{\mu_{1(2)}}} =$

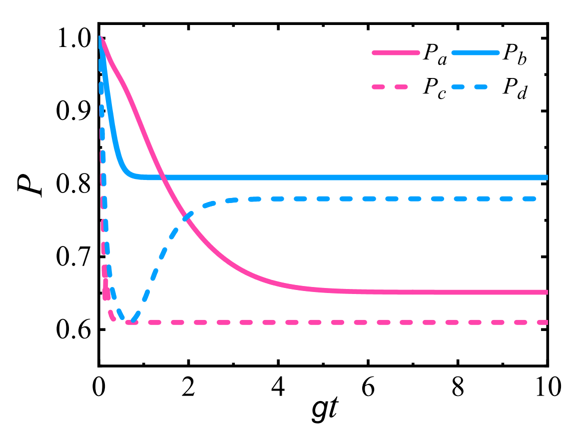

FIG. 7. The purity $P_a - P_d$ of the cavity output states $\hat{\rho}_{b_{\mu1}b_{\mu2}b_{\mu3}}$ corresponding to the states in Figs. 6(a)–6(d), respectively.

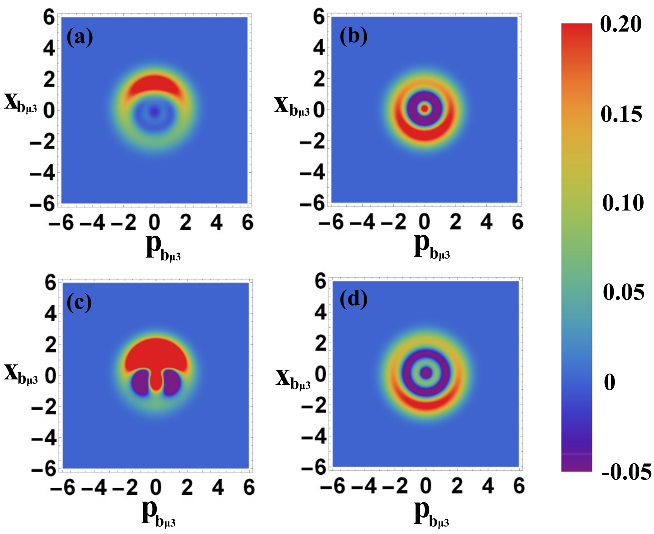

FIG. 8. The density plots of the Wigner functions $W_{b_{\mu_3}}(x_{b_{\mu_3}}, p_{b_{\mu_3}})$ of the conditioned final state $\hat{\rho}_{b_{\mu_3}}(x_{b_{\mu_1}} = x_{b_{\mu_2}} = 5)$ corresponding to the states in Figs. 6(a)–6(d), respectively.

$(\hat{b}_{\mu_{1(2)}} + \hat{b}^{\dagger}_{\mu_{1(2)}})$ of the output modes $\hat{b}_{\mu_1}$ and $\hat{b}_{\mu_2}$. Conditioned on the homodyne detection outcomes $x_{b_{\mu_1}}$ and $x_{b_{\mu_2}}$, the density operator of the mode $\hat{b}_{\mu_3}$ [67]

$$\hat{\rho}_{b_{\mu_3}}(x_{b_{\mu_1}}, x_{b_{\mu_2}}) = \frac{\hat{\tilde{\rho}}_{b_{\mu_3}}(x_{b_{\mu_1}}, x_{b_{\mu_2}})}{\mathrm{Tr}_{b_{\mu_3}}\left[\hat{\tilde{\rho}}_b(x_{b_{\mu_1}}, x_{b_{\mu_2}})\right]}, \tag{30}$$

where the unnormalzied operator $\hat{\tilde{\rho}}_{b_{\mu_3}}(x_{b_{\mu_1}}, x_{b_{\mu_2}}) = \mathrm{Tr}_{b_{\mu_1}b_{\mu_2}}[(\hat{\mathcal{M}}_{b_{\mu_1}b_{\mu_2}} \otimes \hat{I}_{b_{\mu_3}})\hat{\rho}_{b_{\mu_1}b_{\mu_2}b_{\mu_3}}(\hat{I}_{b_{\mu_3}} \otimes \hat{\mathcal{M}}_{b_{\mu_1}b_{\mu_2}})]$ and the projection operator $\hat{\mathcal{M}}_{b_{\mu_1}b_{\mu_2}} = |b_{\mu_1}, b_{\mu_2}\rangle\langle b_{\mu_1}, b_{\mu_2}|$, which can be calculated in the Fock space with $\langle x_o \mid n\rangle = \frac{1}{\pi^{1/4}}\frac{1}{\sqrt{2^{n_o} n_o!}} e^{-x_o^2/2} H_{n_o}(x_o)$, with H_{n_o} being the Hermite polynomial of order n_o and $o = \{b_{\mu_1}, b_{\mu_2}\}$.

In Figs. 8(a)–8(d), the density plots of the Wigner functions $W_{b_{\mu_3}}(x_{b_{\mu_3}}, p_{b_{\mu_3}})$, obtained by performing Fourier transform on the characteristic function defined via $\chi_{b_{\mu_3}}(\xi) = \mathrm{Tr}[e^{\xi\hat{b}^{\dagger}_{\mu_3} - \xi^*\hat{b}_{\mu_3}}\hat{\rho}_{b_{\mu_3}}(x_{b_{\mu_1}}, x_{b_{\mu_2}})]$, are presented for the tripartite non-Gaussian steerable states Figs. 6(a)–6(d), respectively, with the homodyne detection outcomes $x_{b_{\mu_1}} = x_{b_{\mu_2}} = 5$. It shows that the Wigner function exhibits negativity

$$\mathcal{N} = \int [|W(\alpha, \alpha^*)| - W(\alpha, \alpha^*)] d^2\alpha, \tag{31}$$

with phase-space variable $\alpha = x_{b_{\mu_3}} + ip_{b_{\mu_3}}$, indicating genuine non-Gaussian nonclassicality. Essentially, the capability for this remote generation of negative Wigner states is endowed with the non-Gaussian steerable nonlocality generated in the NTPD process. In Fig. 9, we shown the effects of the cavity dissipation rates γ_k on the steady-state tripartite entanglement, steering, and Wigner negativity for the case of the constant coupling g_{μ_k} in Figs. 6(b) and 6(d). In Fig. 9(a), the fully inseparable tripartite entanglement increases rapidly first and then decreases with the dissipation rates, but the genuine tripartite entanglement just increases only after the dissipation rate reaches a certain value, due to the fact that the latter exhibits stronger correlations than the former. As the dissipation rate increases, the dissipative cascaded coupling increases

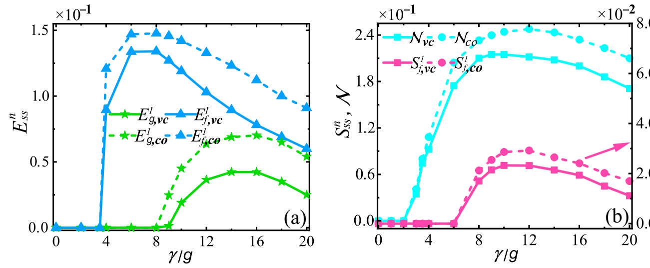

FIG. 9. (a) The steady-state tripartite entanglement E^n_{ss} for the constant coupling $g_{\mu_k} = 1.5\sqrt{g}$ (initial vacua and coherent states with the amplitude $\beta_k = 1$) as the function of the cavity dissipation rate γ_k. (b) The same for the steady-state tripartite steering S^n_{ss} and the Wigner negativity $\mathcal{N}$ of the conditioned final state $\hat{\rho}_{b_{\mu_3}}(x_{b_{\mu_1}} = x_{b_{\mu_2}} = 5)$.

and the entanglement thus increases accordingly, and as the dissipation rate continues to increase, the entanglement and steering are decreased by the dissipation. In Fig. 9(b), the Wigner negativity and the fully inseparable tripartite steering increases rapidly and then decreases slowly over the range of the dissipation rate. It is shown clearly that the negativity has the similar dependence on the dissipation rate to that of the tripartite steering and the improved steering for initial coherent states gives larger negativity, which therefore reflects the intrinsic capability of quantum steering for manipulating local quantum states via remote detection.

V. CONCLUSION

In summary, we study in this paper the properties of tripartite non-Gaussian entanglement and steering in an intracavity NTPD process. We derive the criteria for full inseparability and genuine tripartite non-Gaussian entanglement and steering with high-order field quadratures for the present system. With the criteria and visualizing the specific modes in the output continuous field as virtual cavities coupled to the NTPD cavity in a cascade way, the tripartite non-Gaussian entanglement and steering inside and outside the cavity are studied in detail. It is found that the tripartite non-Gaussian entanglement and steering inside the cavity only exist in the short-time regime but they can be generated in the steady-state regime in the output field of the cavity. Moreover, it is revealed that the initial coherent cavity-field states can effectively enhance the output steady-state and intracavity transient entanglement and steering. It is also shown that the output tripartite non-Gaussian steering can be utilized to remotely generate non-Gaussian states with negative Wigner functions by homodyne detection. Our findings unravel the non-Gaussian nonclassical characteristics in the nonlinear NTPD process. Further work may include the investigation on the application of the output triple photons in genuine tripartite non-Gaussian entangled states to various quantum tasks, such as quantum parameter estimation and quantum illumination.

ACKNOWLEDGMENTS

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APPENDIX: DERIVATION OF THE NON-GAUSSIAN TRIPARTITE ENTANGLEMENT AND STEERING

In this paper, the high-order quadrature operators are defined as $\hat{X}_k^n = (\hat{a}_k^{\dagger n} + \hat{a}_k^n)$ and $\hat{Y}_k^n = i(\hat{a}_k^{\dagger n} - \hat{a}_k^n)$, where $\hat{a}_k$ and $\hat{a}_k^\dagger$ are the annihilation and creation operators with $[\hat{a}_k, \hat{a}_k^\dagger] = 1$. $\hat{X}_k^n$ and $\hat{Y}_k^n$ satisfy the commutation relation $[\hat{X}_k^n, \hat{Y}_k^n] = i\hat{C}_k^n$. $\hat{X}_{lm}^n$ and $\hat{Y}_{lm}^n$ satisfy the commutation relation $[\hat{X}_{lm}^n, \hat{Y}_{lm}^n] = i\hat{C}_{lm}^n$.

According to the biseparable state $\rho_{k,lm} = \sum_i \eta_i \rho_k^i \rho_{lm}^i$, the total variance of the pair of operators $\hat{U}_{k,lm}^n$ and $\hat{V}_{k,lm}^n$ satisfies the inequality

$$\begin{aligned}
\langle(\Delta\hat{U}_{k,lm}^n)^2\rangle + \langle(\Delta\hat{V}_{k,lm}^n)^2\rangle &= \langle(\hat{U}_{k,lm}^n)^2\rangle + \langle(\hat{V}_{k,lm}^n)^2\rangle - \langle\hat{U}_{k,lm}^n\rangle^2 - \langle\hat{V}_{k,lm}^n\rangle^2 \\
&= \sum_i \eta_i\{\langle(\hat{X}_k^n)^2 + (g_{k,n}\hat{X}_{lm}^n)^2 + (\hat{Y}_k^n)^2 + (h_{k,n}\hat{Y}_{lm}^n)^2\rangle_i + 2g_{k,n}\langle\hat{X}_k^n\hat{X}_{lm}^n\rangle_i + 2h_{k,n}\langle\hat{Y}_k^n\hat{Y}_{lm}^n\rangle_i\} \\
&\quad - \langle\hat{U}_{k,lm}^n\rangle^2 - \langle\hat{V}_{k,lm}^n\rangle^2 \\
&= \sum_i \eta_i\{\langle(\Delta\hat{X}_k^n)^2 + (\Delta\hat{Y}_k^n)^2 + (g_{k,n}\Delta\hat{X}_{lm}^n)^2 + (h_{k,n}\Delta\hat{Y}_{lm}^n)^2\rangle_i + 2g_{k,n}(\langle\hat{X}_k^n\hat{X}_{lm}^n\rangle_i - \langle\hat{X}_k^n\rangle_i\langle\hat{X}_{lm}^n\rangle_i) \\
&\quad + 2h_{k,n}(\langle\hat{Y}_k^n\hat{Y}_{lm}^n\rangle_i - \langle\hat{Y}_k^n\rangle_i\langle\hat{Y}_{lm}^n\rangle_i)\} + \sum_i \eta_i\langle\hat{U}_k^n\rangle_i^2 - \left(\sum_i \eta_i\langle\hat{U}_k^n\rangle_i\right)^2 + \sum_i \eta_i\langle\hat{V}_k^n\rangle_i^2 - \left(\sum_i \eta_i\langle\hat{V}_k^n\rangle_i\right)^2 \\
&\geqslant \sum_i \eta_i\{\langle(\Delta\hat{X}_k^n)^2 + (\Delta\hat{Y}_k^n)^2 + (g_{k,n}\Delta\hat{X}_{lm}^n)^2 + (h_{k,n}\Delta\hat{Y}_{lm}^n)^2\rangle_i\} \\
&\geqslant C_k^n + |g_{k,n}h_{k,n}|C_{lm}^n. \qquad \text{(A1)}
\end{aligned}$$

In the derivation process, we utilized the Cauchy-Schwartz inequality $\sum_i \eta_i\langle\hat{U}^n\rangle_i^2 \geqslant (\sum_i \eta_i\langle\hat{U}^n\rangle_i)^2$, the sum uncertainty relation $\langle(\Delta\hat{X}^n)^2\rangle + \langle(\Delta\hat{Y}^n)^2\rangle \geqslant |\langle[\hat{X}^n, \hat{Y}^n]\rangle|$, and $\langle\hat{C}^n\rangle \equiv C^n$.

But if the state of subsystems l and m is not assumed to be a quantum state, there is only the assumption of non-negativity for the associated variances. For the biseparable local hidden state model symbolized as $\rho_{lm\to k} = \sum_i \eta_i\rho_{kQ}^i\rho_{lm}^i$,

$$\begin{aligned}
\langle(\Delta\hat{U}_{k,lm}^n)^2\rangle + \langle(\Delta\hat{V}_{k,lm}^n)^2\rangle &= \langle[\Delta(\hat{X}_k^n + g_{k,n}\hat{X}_{lm}^n)]^2\rangle + \langle[\Delta(\hat{Y}_k^n + h_{k,n}\hat{Y}_{lm}^n)]^2\rangle \\
&\geqslant \sum_i \eta_i\{\langle(\Delta\hat{X}_k^n)^2\rangle_i + \langle\Delta(g_{k,n}\hat{X}_{lm}^n)^2\rangle_i + \langle(\Delta\hat{Y}_k^n)^2\rangle_i + \langle\Delta(h_{k,n}\hat{Y}_{lm}^n)^2\rangle_i\} \\
&\geqslant \sum_i \eta_i\{\langle(\Delta\hat{X}_k^n)^2\rangle_i + \langle(\Delta\hat{Y}_k^n)^2\rangle_i\} \geqslant C_k^n. \qquad \text{(A2)}
\end{aligned}$$

In the derivation process, we utilized the non-negativity of variances that for any local hidden variables, i.e., $\langle\Delta(g_{k,n}\hat{X}_{lm}^n)^2\rangle \geqslant 0$ and $\langle\Delta(h_{k,n}\hat{Y}_{lm}^n)^2\rangle \geqslant 0$, the sum uncertainty relation $\langle(\Delta\hat{X}_k^n)^2\rangle + \langle(\Delta\hat{Y}_k^n)^2\rangle \geqslant |\langle[\hat{X}_k^n, \hat{Y}_k^n]\rangle|$ and $\langle\hat{C}_k^n\rangle \equiv C_k^n$.

Then, the three inequalities can be written as

$$S_1^n = \langle(\Delta\hat{U}_{1,23}^n)^2\rangle + \langle(\Delta\hat{V}_{1,23}^n)^2\rangle \geqslant C_1^n, \quad S_2^n = \langle(\Delta\hat{U}_{2,13}^n)^2\rangle + \langle(\Delta\hat{V}_{2,13}^n)^2\rangle \geqslant C_2^n, \quad S_3^n = \langle(\Delta\hat{U}_{3,12}^n)^2\rangle + \langle(\Delta\hat{V}_{3,12}^n)^2\rangle \geqslant C_3^n. \qquad \text{(A3)}$$

Violation of the three inequalities above can confirm fully inseparable tripartite steering.

In our symmetric system, $S_1^n = S_2^n = S_3^n$, $C_1^n = C_2^n = C_3^n$, $g_{1,n} = g_{2,n} = g_{3,n}$, $h_{1,n} = h_{2,n} = h_{3,n}$, i.e., violating any of the above formulas can confirm fully inseparable tripartite steering. Calculating

$$\begin{aligned}
S_1^n = \langle(\Delta\hat{U}_{1,23}^n)^2\rangle + \langle(\Delta\hat{V}_{1,23}^n)^2\rangle &= \langle(\hat{X}_1^n)^2\rangle - \langle\hat{X}_1^n\rangle^2 + 2g_{1,n}(\langle\hat{X}_1^n\hat{X}_{23}^n\rangle - \langle\hat{X}_1^n\rangle\langle\hat{X}_{23}^n\rangle) + g_{1,n}^2(\langle(\hat{X}_{23}^n)^2\rangle - \langle\hat{X}_{23}^n\rangle^2) + \langle(\hat{Y}_1^n)^2\rangle - \langle\hat{Y}_1^n\rangle^2 \\
&\quad + 2h_{1,n}(\langle\hat{Y}_1^n\hat{Y}_{23}^n\rangle - \langle\hat{Y}_1^n\rangle\langle\hat{Y}_{23}^n\rangle) + h_{1,n}^2(\langle(\hat{Y}_{23}^n)^2\rangle - \langle\hat{Y}_{23}^n\rangle^2), \qquad \text{(A4)}
\end{aligned}$$

where the optimal gain parameters $g_{1,n} = -h_{1,n} = \frac{-(\langle\hat{X}_1^n\hat{X}_{23}^n\rangle - \langle\hat{X}_1^n\rangle\langle\hat{X}_{23}^n\rangle) + (\langle\hat{Y}_1^n\hat{Y}_{23}^n\rangle - \langle\hat{Y}_1^n\rangle\langle\hat{Y}_{23}^n\rangle)}{[(\langle(\hat{X}_{23}^n)^2\rangle - \langle\hat{X}_{23}^n\rangle^2) + (\langle(\hat{Y}_{23}^n)^2\rangle - \langle\hat{Y}_{23}^n\rangle^2]}$. Bringing the parameters back to the first formula in Eq. (A3), we can get a simplified inequality

$$|\langle\hat{a}_1^n\hat{a}_2^n\hat{a}_3^n\rangle - \langle\hat{a}_1^n\rangle\langle\hat{a}_2^n\hat{a}_3^n\rangle| \leqslant \frac{1}{2}\sqrt{\langle\hat{a}_2^{\dagger n}\hat{a}_2^n\hat{a}_3^{\dagger n}\hat{a}_3^n\rangle + \langle\hat{a}_2^n\hat{a}_2^{\dagger n}\hat{a}_3^n\hat{a}_3^{\dagger n}\rangle - 2\langle\hat{a}_2^n\hat{a}_3^n\rangle^2}\sqrt{\langle\hat{a}_1^{\dagger n}\hat{a}_1^n\rangle + \langle\hat{a}_1^n\hat{a}_1^{\dagger n}\rangle - \frac{1}{2}C_1^n - 2\langle\hat{a}_1^n\rangle^2}. \qquad \text{(A5)}$$

Violation of the inequality above can confirm fully inseparable tripartite steering.

Furthermore, we consider that the system is described by mixtures of the type $\rho_{\text{mix}} = P_1\sum_{i_1}\eta_{i_1}\rho_{1Q}^{i_1}\rho_{23}^{i_1} + P_2\sum_{i_2}\eta_{i_2}\rho_{2Q}^{i_2}\rho_{13}^{i_2} + P_3\sum_{i_3}\eta_{i_3}\rho_{3Q}^{i_3}\rho_{12}^{i_3}$, where $\sum_i P_i = 1$ and $\sum_i \eta_i = 1$. Then substituting the mixture state into Eq. (A3), we find

$$S_1^n \geqslant P_1S_{1,1}^n + P_2S_{1,2}^n + P_3S_{1,3}^n, \quad S_2^n \geqslant P_1S_{2,1}^n + P_2S_{2,2}^n + P_3S_{2,3}^n, \quad S_3^n \geqslant P_1S_{3,1}^n + P_2S_{3,2}^n + P_3S_{3,3}^n, \qquad \text{(A6)}$$

where $S^n_{j,j'}$ stands for the total variance of operators $\hat{U}_j$ and $\hat{V}_j$ over the density operator ρ_j. Thus, we have

$$\begin{aligned} S^n_{1,1} &= \langle(\Delta\hat{U}^n_{1,23})^2\rangle_1 + \langle(\Delta\hat{V}^n_{1,23})^2\rangle_1 \\ &= \langle[\Delta(\hat{X}^n_1 + g_{1,n}\hat{X}^n_{23})]^2\rangle_1 + \langle[\Delta(\hat{Y}^n_1 + h_{1,n}\hat{Y}^n_{23})]^2\rangle_1 \\ &\geqslant P_1\sum_{i_1}\eta_{i_1}\{\langle(\Delta\hat{X}^n_1)^2\rangle_{i_1} + \langle\Delta(g_{1,n}\hat{X}^n_{23})^2\rangle_{i_1} + \langle(\Delta\hat{Y}^n_1)^2\rangle_{i_1} + \langle\Delta(h_{1,n}\hat{Y}^n_{23})^2\rangle_{i_1}\} \\ &\geqslant P_1\sum_{i_1}\eta_{i_1}\{\langle(\Delta\hat{X}^n_1)^2\rangle_{i_1} + \langle(\Delta\hat{Y}^n_1)^2\rangle_{i_1}\} \geqslant P_1\langle\hat{C}^n_1\rangle. \end{aligned} \tag{A7}$$

Applying the same conditions as the above inequality on $S_{2,2}$ and $S_{3,3}$, we can get

$$S^n_1 + S^n_2 + S^n_3 \geqslant P_1S_{1,1} + P_2S_{2,2} + P_3S_{3,3} \geqslant P_1C^n_1 + P_2C^n_2 + P_3C^n_3 \geqslant \min\{C^n_1, C^n_2, C^n_3\}. \tag{A8}$$

Violation of above inequality with any n is sufficient to confirm the genuine tripartite steering.

In our symmetric system, Eq. (A8) can be simplified to $3S^n_1 \geqslant C^n_1$, and

$$\begin{aligned} 3S^n_1 = 3[\langle(\Delta\hat{U}^n_{1,23})^2\rangle + \langle(\Delta\hat{V}^n_{1,23})^2\rangle] &= 3\langle(\hat{X}^n_1)^2\rangle - 3\langle\hat{X}^n_1\rangle^2 + 6g_{1,n}(\langle\hat{X}^n_1\hat{X}^n_{23}\rangle - \langle\hat{X}^n_1\rangle\langle\hat{X}^n_{23}\rangle) + 3g^2_{1,n}[\langle(\hat{X}^n_{23})^2\rangle - \langle\hat{X}^n_{23}\rangle^2] \\ &\quad + 3\langle(\hat{Y}^n_1)^2\rangle - 3\langle\hat{Y}^n_1\rangle^2 + 6h_{1,n}(\langle\hat{Y}^n_1\hat{Y}^n_{23}\rangle - \langle\hat{Y}^n_1\rangle\langle\hat{Y}^n_{23}\rangle) + 3h^2_{1,n}[\langle(\hat{Y}^n_{23})^2\rangle - \langle\hat{Y}^n_{23}\rangle^2], \end{aligned} \tag{A9}$$

where the optimal gain parameters $g_{1,n} = -h_{1,n} = \frac{-(\langle\hat{X}^n_1\hat{X}^n_{23}\rangle - \langle\hat{X}^n_1\rangle\langle\hat{X}^n_{23}\rangle) + (\langle\hat{Y}^n_1\hat{Y}^n_{23}\rangle - \langle\hat{Y}^n_1\rangle\langle\hat{Y}^n_{23}\rangle)}{[\langle(\hat{X}^n_{23})^2\rangle - \langle\hat{X}^n_{23}\rangle^2) + (\langle(\hat{Y}^n_{23})^2\rangle - \langle\hat{Y}^n_{23}\rangle^2]}$. Bringing the parameters back to $3S^n_1 \geqslant C^n_1$, we can get

$$|\langle\hat{a}^n_1\hat{a}^n_2\hat{a}^n_3\rangle - \langle\hat{a}^n_1\rangle\langle\hat{a}^n_2\hat{a}^n_3\rangle| \leqslant \tfrac{1}{2}\sqrt{\langle\hat{a}^{\dagger n}_2\hat{a}^n_2\hat{a}^{\dagger n}_3\hat{a}^n_3\rangle + \langle\hat{a}^n_2\hat{a}^{\dagger n}_2\hat{a}^n_3\hat{a}^{\dagger n}_3\rangle - 2\langle\hat{a}^n_2\hat{a}^n_3\rangle^2}\sqrt{\langle\hat{a}^{\dagger n}_1\hat{a}^n_1\rangle + \langle\hat{a}^n_1\hat{a}^{\dagger n}_1\rangle - \tfrac{1}{6}C^n_1 - 2\langle\hat{a}^n_1\rangle^2}. \tag{A10}$$

Violation of the above inequality can confirm the genuine tripartite steering.

When all the subsystems are constrained to be quantum states, we will get $\rho'_{\mathrm{mix}} = P_1\sum_{i_1}\eta_{i_1}\rho^{i_1}_1\rho^{i_1}_{23} + P_2\sum_{i_2}\eta_{i_2}\rho^{i_2}_2\rho^{i_2}_{13} + P_3\sum_{i_3}\eta_{i_3}\rho^{i_3}_3\rho^{i_3}_{12}$. Referring to the inequalities (A3), we can get

$$\begin{aligned} E^n_1 &= \langle(\Delta\hat{U}^n_{1,23})^2\rangle + \langle(\Delta\hat{V}^n_{1,23})^2\rangle \geqslant C^n_1 + |g_{1,n}h_{1,n}|C^n_{23}, \\ E^n_2 &= \langle(\Delta\hat{U}^n_{2,13})^2\rangle + \langle(\Delta\hat{V}^n_{2,13})^2\rangle \geqslant C^n_2 + |g_{2,n}h_{2,n}|C^n_{13}, \\ E^n_3 &= \langle(\Delta\hat{U}^n_{3,12})^2\rangle + \langle(\Delta\hat{V}^n_{3,12})^2\rangle \geqslant C^n_3 + |g_{3,n}h_{3,n}|C^n_{12}. \end{aligned} \tag{A11}$$

Violation of the three inequalities above can confirm fully inseparable tripartite entanglement. The same as the fully inseparable tripartite steering,

$$\begin{aligned} E^n_1 = \langle(\Delta\hat{U}^n_{1,23})^2\rangle + \langle(\Delta\hat{V}^n_{1,23})^2\rangle &= \langle(\hat{X}^n_1)^2\rangle - \langle\hat{X}^n_1\rangle^2 + 2g_{1,n}(\langle\hat{X}^n_1\hat{X}^n_{23}\rangle - \langle\hat{X}^n_1\rangle\langle\hat{X}^n_{23}\rangle) + g^2_{1,n}[\langle(\hat{X}^n_{23})^2\rangle - \langle\hat{X}^n_{23}\rangle^2] + \langle(\hat{Y}^n_1)^2\rangle \\ &\quad - \langle\hat{Y}^n_1\rangle^2 + 2h_{1,n}(\langle\hat{Y}^n_1\hat{Y}^n_{23}\rangle - \langle\hat{Y}^n_1\rangle\langle\hat{Y}^n_{23}\rangle) + h^2_{1,n}[\langle(\hat{Y}^n_{23})^2\rangle - \langle\hat{Y}^n_{23}\rangle^2] \end{aligned} \tag{A12}$$

in our system, where the $g_{1,n} = -h_{1,n} = \frac{-(\langle\hat{X}^n_1\hat{X}^n_{23}\rangle - \langle\hat{X}^n_1\rangle\langle\hat{X}^n_{23}\rangle) + (\langle\hat{Y}^n_1\hat{Y}^n_{23}\rangle - \langle\hat{Y}^n_1\rangle\langle\hat{Y}^n_{23}\rangle)}{[\langle(\hat{X}^n_{23})^2\rangle - \langle\hat{X}^n_{23}\rangle^2] + [\langle(\hat{Y}^n_{23})^2\rangle - \langle\hat{Y}^n_{23}\rangle^2] - C^n_{23}}$. Bringing the parameters back to the first formula in Eq. (A11), we can get a simplified inequality

$$|\langle\hat{a}^n_1\hat{a}^n_2\hat{a}^n_3\rangle - \langle\hat{a}^n_1\rangle\langle\hat{a}^n_2\hat{a}^n_3\rangle| \leqslant \sqrt{\langle\hat{a}^{\dagger n}_2\hat{a}^n_2\hat{a}^{\dagger n}_3\hat{a}^n_3\rangle - 2\langle\hat{a}^n_2\hat{a}^n_3\rangle^2}\sqrt{\langle\hat{a}^{\dagger n}_1\hat{a}^n_1\rangle - \langle\hat{a}^n_1\rangle^2}. \tag{A13}$$

Violation of the inequality above can confirm fully inseparable tripartite entanglement.

According to inequalities (A11) and ρ'_{mix}, we have

$$E^n_1 \geqslant P_1E^n_{1,1} + P_2E^n_{1,2} + P_3E^n_{1,3}, \quad E^n_2 \geqslant P_1E^n_{2,1} + P_2E^n_{2,2} + P_3E^n_{2,3}, \quad E^n_3 \geqslant P_1E^n_{3,1} + P_2E^n_{3,2} + P_3E^n_{3,3}, \tag{A14}$$

where

$$E_{1,1}^n \geqslant P_1\big(\langle \hat{C}_1^n\rangle_1 + |g_{1,n}h_{1,n}|\langle \hat{C}_{23}^n\rangle_1\big), \tag{A15}$$

$$\begin{aligned}
E_{1,2}^n &= \langle(\Delta\hat{U}_{1,23}^n)^2\rangle_2 + \langle(\Delta\hat{V}_{1,23}^n)^2\rangle_2\\
&\geqslant P_2\sum_{i_2}\eta_{i_2}\big\{\big\langle(\hat{X}_1^n)^2 + (g_{1,n}\hat{X}_{23}^n)^2 + (\hat{Y}_1^n)^2 + (h_{1,n}\hat{Y}_{23}^n)^2\big\rangle_{i_2} + 2g_{1,n}\langle\hat{X}_1^n\hat{X}_{23}^n\rangle_{i_2} + 2h_{1,n}\langle\hat{Y}_1^n\hat{Y}_{23}^n\rangle_{i_2}\big\} - P_2^2\langle\hat{U}_{1,23}^n\rangle_2^2 - P_2^2\langle\hat{V}_{1,23}^n\rangle_2^2\\
&\geqslant P_2\sum_{i_2}\eta_{i_2}\big\{\big\langle(\Delta\hat{X}_1^n)^2 + (\Delta\hat{Y}_1^n)^2 + (g_{1,n}\Delta\hat{X}_{23}^n)^2 + (h_{1,n}\Delta\hat{Y}_{23}^n)^2\big\rangle_{i_2} + 2g_{1,n}\langle\hat{X}_1^n\hat{X}_{23}^n\rangle_{i_2} + 2h_{1,n}\langle\hat{Y}_1^n\hat{Y}_{23}^n\rangle_{i_2}\big\}\\
&\quad + \left(P_2\sum_{i_2}\eta_{i_2}\langle\hat{X}_1^n\rangle_{i_2}^2 - P_2^2\langle\hat{X}_1^n\rangle_2^2 + P_2\sum_{i_2}\eta_{i_2}\langle\hat{Y}_1^n\rangle_{i_2}^2 - P_2^2\langle\hat{Y}_1^n\rangle_2^2\right)\\
&\quad + \left(P_2\sum_{i_2}\eta_{i_2}\langle\hat{X}_{23}^n\rangle_{i_2}^2 - P_2^2\langle\hat{X}_{23}^n\rangle_2^2 + P_2\sum_{i_2}\eta_{i_2}\langle\hat{Y}_{23}^n\rangle_{i_2}^2 - P_2^2\langle\hat{Y}_{23}^n\rangle_2^2\right) - 2P_2^2g_{1,n}\langle\hat{X}_1^n\rangle_2\langle\hat{X}_{23}^n\rangle_2 - 2P_2^2h_{1,n}\langle\hat{Y}_1^n\rangle_2\langle\hat{Y}_{23}^n\rangle_2\\
&\geqslant P_2\big(\langle\hat{C}_1^n\rangle_2 + |g_{1,n}h_{1,n}|\langle\hat{C}_{23}^n\rangle_2 + 2g_{1,n}\langle\hat{X}_1^n\hat{X}_{23}^n\rangle_2 + 2h_{1,n}\langle\hat{Y}_1^n\hat{Y}_{23}^n\rangle_2\big) - 2P_2^2g_{1,n}\langle\hat{X}_1^n\rangle_2\langle\hat{X}_{23}^n\rangle_2 - 2P_2^2h_{1,n}\langle\hat{Y}_1^n\rangle_2\langle\hat{Y}_{23}^n\rangle_2.
\end{aligned} \tag{A16}$$

Combining those inequalities in Eqs. (A14)–(A16), we find that

$$\begin{aligned}
E_1^n + E_2^n + E_3^n \geqslant{}& C_1^n + |g_{1,n}h_{1,n}|C_{23}^n + C_2^n + |g_{2,n}h_{2,n}|C_{13}^n + C_3^n + |g_{3,n}h_{3,n}|C_{12}^n\\
&+ 2(g_{2,n}+g_{3,n})P_1\langle\hat{X}_1^n\hat{X}_{23}^n\rangle_1 + 2(g_{1,n}+g_{3,n})P_2\langle\hat{X}_2^n\hat{X}_{13}^n\rangle_2 + 2(g_{1,n}+g_{2,n})P_3\langle\hat{X}_3^n\hat{X}_{12}^n\rangle_3\\
&+ 2(h_{2,n}+h_{3,n})P_1\langle\hat{Y}_1^n\hat{Y}_{23}^n\rangle_1 + 2(h_{1,n}+h_{3,n})P_2\langle\hat{Y}_2^n\hat{Y}_{13}^n\rangle_2 + 2(h_{1,n}+h_{2,n})P_3\langle\hat{Y}_3^n\hat{Y}_{12}^n\rangle_3\\
&- 2P_2^2\big(g_{1,n}\langle\hat{X}_1^n\rangle_2\langle\hat{X}_{23}^n\rangle_2 + h_{1,n}\langle\hat{Y}_1^n\rangle_2\langle\hat{Y}_{23}^n\rangle_2\big) - 2P_3^2\big(g_{1,n}\langle\hat{X}_1^n\rangle_3\langle\hat{X}_{23}^n\rangle_3 + h_{1,n}\langle\hat{Y}_1^n\rangle_3\langle\hat{Y}_{23}^n\rangle_3\big)\\
&- 2P_1^2\big(g_{2,n}\langle\hat{X}_2^n\rangle_1\langle\hat{X}_{13}^n\rangle_1 + h_{2,n}\langle\hat{Y}_2^n\rangle_1\langle\hat{Y}_{13}^n\rangle_1\big) - 2P_3^2\big(g_{2,n}\langle\hat{X}_2^n\rangle_3\langle\hat{X}_{13}^n\rangle_3 + h_{2,n}\langle\hat{Y}_2^n\rangle_3\langle\hat{Y}_{13}^n\rangle_3\big)\\
&- 2P_1^2\big(g_{3,n}\langle\hat{X}_3^n\rangle_1\langle\hat{X}_{12}^n\rangle_1 + h_{3,n}\langle\hat{Y}_3^n\rangle_1\langle\hat{Y}_{12}^n\rangle_1\big) - 2P_2^2\big(g_{3,n}\langle\hat{X}_3^n\rangle_2\langle\hat{X}_{12}^n\rangle_2 + h_{3,n}\langle\hat{Y}_3^n\rangle_2\langle\hat{Y}_{12}^n\rangle_2\big).
\end{aligned} \tag{A17}$$

Violation of the above inequality is the condition for genuine tripartite entanglement.

In our system, Eq. (A17) can be further simplified to

$$3E_1^n \geqslant 3C_1^n + 3|g_{1,n}h_{1,n}|C_{23}^n + 4g_{1,n}\langle\hat{X}_1^n\hat{X}_{23}^n\rangle + 4h_{1,n}\langle\hat{Y}_1^n\hat{Y}_{23}^n\rangle - 4g_{1,n}\langle\hat{X}_1^n\rangle\langle\hat{X}_{23}^n\rangle - 4h_{1,n}\langle\hat{Y}_1^n\rangle\langle\hat{Y}_{23}^n\rangle, \tag{A18}$$

where we use the operator properties

$$\langle\hat{X}_1^n\hat{X}_{23}^n\rangle = P_1\langle\hat{X}_1^n\hat{X}_{23}^n\rangle_1 + P_2\langle\hat{X}_2^n\hat{X}_{13}^n\rangle_2 + P_3\langle\hat{X}_3^n\hat{X}_{12}^n\rangle_3,\quad \langle\hat{Y}_1^n\hat{Y}_{23}^n\rangle = P_1\langle\hat{Y}_1^n\hat{Y}_{23}^n\rangle_1 + P_2\langle\hat{Y}_2^n\hat{Y}_{13}^n\rangle_2 + P_3\langle\hat{Y}_3^n\hat{Y}_{12}^n\rangle_3, \tag{A19}$$

and

$$\begin{aligned}
-P_1^2\langle\hat{X}_1^n\rangle_1\langle\hat{X}_{23}^n\rangle_1 - P_2^2\langle\hat{X}_1^n\rangle_2\langle\hat{X}_{23}^n\rangle_2 - P_3^2\langle\hat{X}_1^n\rangle_3\langle\hat{X}_{23}^n\rangle_3 &\geqslant -P_1\langle\hat{X}_1^n\rangle_1\langle\hat{X}_{23}^n\rangle_1 - P_2\langle\hat{X}_1^n\rangle_2\langle\hat{X}_{23}^n\rangle_2 - P_3\langle\hat{X}_1^n\rangle_3\langle\hat{X}_{23}^n\rangle_3\\
&= -\langle\hat{X}_1^n\rangle\langle\hat{X}_{23}^n\rangle,\\
&\quad -P_1^2\langle\hat{Y}_1^n\rangle_1\langle\hat{Y}_{23}^n\rangle_1 - P_2^2\langle\hat{Y}_1^n\rangle_2\langle\hat{Y}_{23}^n\rangle_2 - P_3^2\langle\hat{Y}_1^n\rangle_3\langle\hat{Y}_{23}^n\rangle_3\\
&\geqslant -P_1\langle\hat{Y}_1^n\rangle_1\langle\hat{Y}_{23}^n\rangle_1 - P_2\langle\hat{Y}_1^n\rangle_2\langle\hat{Y}_{23}^n\rangle_2 - P_3\langle\hat{Y}_1^n\rangle_3\langle\hat{Y}_{23}^n\rangle_3\\
&= -\langle\hat{Y}_1^n\rangle\langle\hat{Y}_{23}^n\rangle.
\end{aligned} \tag{A20}$$

Then, the same as the genuine tripartite steering,

$$\begin{aligned}
3E_1^n = 3[\langle(\Delta\hat{U}_{1,23}^n)^2\rangle + \langle(\Delta\hat{V}_{1,23}^n)^2\rangle] ={}& 3\langle(\hat{X}_1^n)^2\rangle - 3\langle\hat{X}_1^n\rangle^2 + 6g_{1,n}(\langle\hat{X}_1^n\hat{X}_{23}^n\rangle - \langle\hat{X}_1^n\rangle\langle\hat{X}_{23}^n\rangle) + 3g_{1,n}^2[\langle(\hat{X}_{23}^n)^2\rangle - \langle\hat{X}_{23}^n\rangle^2]\\
&+ 3\langle(\hat{Y}_1^n)^2\rangle - 3\langle\hat{Y}_1^n\rangle^2 + 6h_{1,n}(\langle\hat{Y}_1^n\hat{Y}_{23}^n\rangle - \langle\hat{Y}_1^n\rangle\langle\hat{Y}_{23}^n\rangle) + 3h_{1,n}^2[\langle(\hat{Y}_{23}^n)^2\rangle - \langle\hat{Y}_{23}^n\rangle^2]
\end{aligned} \tag{A21}$$

in Eq. (A18), where the optimal gain parameters $g_{1,n} = -h_{1,n} = \frac{-(\langle\hat{X}_1^n\hat{X}_{23}^n\rangle - \langle\hat{X}_1^n\rangle\langle\hat{X}_{23}^n\rangle) + (\langle\hat{Y}_1^n\hat{Y}_{23}^n\rangle - \langle\hat{Y}_1^n\rangle\langle\hat{Y}_{23}^n\rangle)}{3[\langle(\hat{X}_{23}^n)^2\rangle - \langle\hat{X}_{]23}^n\rangle^2) + 3[\langle(\hat{Y}_{23}^n)^2\rangle - \langle\hat{Y}_{23}^n\rangle^2] - 3C_{23}^n}$. Bringing the parameters back to Eq. (A18), we can get

$$|\langle\hat{a}_1^n\hat{a}_2^n\hat{a}_3^n\rangle - \langle\hat{a}_1^n\rangle\langle\hat{a}_2^n\hat{a}_3^n\rangle| \leqslant 3\sqrt{\langle\hat{a}_2^{\dagger n}\hat{a}_2^n\hat{a}_3^{\dagger n}\hat{a}_3^n\rangle - 2\langle\hat{a}_2^n\hat{a}_3^n\rangle^2}\sqrt{\langle\hat{a}_1^{\dagger n}\hat{a}_1^n\rangle - \langle\hat{a}_1^n\rangle^2}. \tag{A22}$$

Violation of the above inequality is the condition for genuine tripartite entanglement in our system.

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