Criteria for stochastic self-focusing

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Analysis of extreme-value statistics of stochastic laser pulses suggests a closed-form, quantitative criterion of self-focusing avoidance. We present an analytical solution for the excess kurtosis of the statistics of nonlinear-optical processes, which is shown to be a rapidly growing function of the nonlinearity order, thus indicating a physically significant redistribution of statistical weight within the probability distribution of the respective nonlinear readouts from its central part to its tails. Unlike deterministic self-focusing, whose criterion is expressed in terms of a well-defined self-focusing threshold $P_{\rm cr}$, its stochastic counterpart is a probabilistic process whose combined probability for a sample of N laser pulses builds up as a function of N, leading to N-dependent self-focusing avoidance criteria. Specifically, for $N \gg 1$ laser pulses with a signal-to-noise ratio a, the criterion of self-focusing avoidance is shown to shift as $a^2 P_{\rm cr}/(2\ln N)$. Instead of dealing with a question as to how to completely avoid self-focusing, stochastic analysis has to deal with a question of how to effectively manage the self-focusing probability over a finite sample of laser shots. The occurrence of self-focusing in stochastic nonlinear optics is thus not a question of *if*, but a question of *when*.

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I. INTRODUCTION

Since its discovery at the dawn of the laser era [1–3], self-focusing has been central to optical science [4–6]. In laser technologies, self-focusing provides a universal mode-locking mechanism [7,8], enabling a robust generation of ultrashort light pulses. In a broad variety of laser-matter interaction scenarios, self-focusing tends to couple to other nonlinear processes [2,4,9], launching complex spatiotemporal transformations of ultrashort field waveforms [10–12] and giving rise to a vast range of intriguing physical phenomena and unusual regimes of nonlinear dynamics, including laser filamentation [10,11,13,14], supercontinuum generation [15,16], high-power pulse compression [17–19], and cross-range wavelength conversion [20]. In high-power lasers, self-focusing is manifested as a primary cause of damage to optical components [4,6,17,21,22], which sets a limit on the peak power of the laser output, setting a stage for the celebrated chirped-pulse amplification technology [23–25].

In the standard framework of nonlinear optics, self-focusing is viewed as a process that plays out for laser beams whose peak power P exceeds a well-defined threshold, which the canonical theory of self-focusing [4,26] sets at

$$P_{\rm cr} \approx (4\pi n_0 n_2)^{-1}\lambda^2, \tag{1}$$

where n_0 is the field-free refractive index, n_2 is the nonlinear refractive index, and λ is the laser wavelength. This view of self-focusing is fully adequate for a vast class of nonlinear-optical phenomena, providing solid grounds for the design of mode-locked laser sources and high-power laser beamlines and helping understand some of the most intriguing and most complex scenarios in ultrafast optical science [4–6,27].

This picture of self-focusing, however, reaches its limits when stochasticity enters the scene. Examples of pertinent physical settings include, but are in no way limited to, noise-seeded stimulated Raman scattering [28–30], parametric gain [31,32] and beam instabilities [31,33], multimode lasing [34,35], and waveform synthesis [36–38] with fluctuating modes, impurity-state- and noise-assisted laser-induced ionization [39–41], as well as random refractive-index variations in crystals and turbulent gases [42–44].

Because the field intensity and the peak power of a laser beam in such settings are no longer sharply defined constants, but can only be described in terms of statistical distributions, well-resolved, constant thresholds are no longer sufficient as self-focusing avoidance criteria. Here, in search of a more adequate framework, we resort to the analysis of extreme-value statistics of self-focusing field waveforms. Such analysis suggests, as one of the central results of this study, that, in stochastic nonlinear optics, self-focusing avoidance criteria can no longer be expressed via constant thresholds, such as the critical power of self-focusing $P_{\rm cr}$ [Eq. (1)], but need to be defined as functions of the number of trials N, i.e., the number of laser shots in a sample. As one of the most significant manifestations of its stochastic nature, the combined probability of self-focusing in a sample of N laser pulses builds up as a function of N, leading to N-dependent self-focusing avoidance criteria. Specifically, for $N \gg 1$ stochastic laser pulses with a signal-to-noise ratio a, the criterion of self-focusing avoidance is shown to shift as $a^2 P_{\rm cr}/(2\ln N)$.

II. STOCHASTIC LASER PULSES AND THEIR SELF-FOCUSING STATISTICS

We consider a nonlinear process in which a laser driver with a field envelope $\rho(\mathbf{r},\ t)$ gives rise to a nonlinear readout $\psi(\rho) = \alpha\rho^n$. Such polynomial nonlinear response is typical of many of the key processes in perturbative nonlinear optics [4,26], including nth-order harmonic generation, wave mixing, and self-phase modulation. In a specific case of

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self-focusing, a laser beam propagating in a medium with a nonlinear refractive index n_2 induces an intensity-dependent change in the refractive index, thus giving rise to a nonlinear lens. The nonlinear readout ψ in this setting can be identified with an on-axis nonlinear phase shift $\psi = \varphi_{nl} = (2\pi/\lambda)n_2 I l_d$ that the laser beam with a field intensity I and beam radius r acquires within the diffraction length $l_d \approx (2\pi/\lambda)n_0 r^2$. Defining the laser peak power as $P = \pi r^2 I = SI$, where S is the beam area, and setting

$$\psi = (4\pi/\lambda^2)n_0 n_2 P = 1 \tag{2}$$

as a criterion of a Kerr lens that is strong enough to compensate the diffraction beam divergence, we find that the peak power P needed to induce such a phase shift and found from Eq. (2) is precisely the critical power of self-focusing P_{cr} as defined by Eq. (1). The α multiplier for the self-focusing nonlinearity is thus

$$\alpha = \alpha_{sf} = S/(2P_{\mathrm{cr}}). \tag{3}$$

Given the probability density function for the driver envelope, $w_\rho(\rho)$, the probability density function of the nonlinear readout ψ is

$$w_\psi(\psi) = |\partial\rho/\partial\psi| w_\rho(\rho = \rho(\psi)), \tag{4}$$

where $\rho = \rho(\psi)$ means that ρ is expressed in terms of ψ via the inverse of $\psi(\rho)$. For the self-focusing nonlinearity, $\psi(\rho) = \alpha\rho^2$, inverting $\rho(\psi)$ yields $\rho(\psi) = (\psi/\alpha)^{1/2}$.

We now examine a generic stochastic field waveform represented as a superposition

$$\eta(t) = s(t) + \xi(t) = \rho(t)\cos[\omega_0 t + \theta(t)] \tag{5}$$

of a deterministic waveform $s(t) = \rho_s(t)\cos[\omega_0 t + \varphi_s(t)]$ and a narrow-band noise $\xi(t) = \rho_\xi(t)\cos[\omega_0 t + \varphi_\xi(t)]$ with a slowly varying envelope $\rho(t) = \{[\rho_1(t)]^2 + [\rho_2(t)]^2\}^{1/2}$ and phase $\theta(t) = \mathrm{atan}[\rho_2(t)/\rho_1(t)]$, $\rho_1(t) = \rho_s(t)\cos[\varphi_s(t)] + \rho_+(t)$, $\rho_2(t) = \rho_s(t)\sin[\varphi_s(t)] + \rho_-(t)$, $\rho_+(t) = \rho_\xi(t)\cos[\varphi_\xi(t)]$, and $\rho_-(t) = \rho_\xi(t)\sin[\varphi_\xi(t)]$.

The normalized envelope $v = \rho/\sigma$ of the waveform (5) has a Rice distribution [45,46],

$$w_v(v) = \mathcal{Q}(v, a) = v\exp[-(a^2 + v^2)/2]I_0(av), \tag{6}$$

where $a = \rho_s/\sigma$ is the signal-to-noise ratio and $I_0(x)$ is the modified Bessel function of the first kind.

When $a = 0$, i.e., the deterministic part of the field is zero, $\rho_s = 0$, Eq. (6) recovers the Rayleigh distribution,

$$w_\rho(\rho) = (\rho/\sigma^2)\exp[-\rho^2/(2\sigma^2)]. \tag{7}$$

With $\psi(\rho) = \alpha\rho^n$, $\rho(\psi) = (\psi/\alpha)^{1/n}$ and Eqs. (4) and (7) give

$$w_\psi(\psi) = (\alpha n)^{-1}\sigma^{-2}(\psi/\alpha)^{(2/n)-1}\exp[-(\psi/\alpha)^{2/n}/(2\sigma^2)], \tag{8}$$

or, for $x = \psi/\psi_0$ and $\psi_0 = 2^{n/2}\alpha\sigma^n$,

$$w_x(x) = (2/n)x^{(2/n)-1}\exp(-x^{2/n}). \tag{9}$$

For the Kerr-effect self-focusing nonlinearity, $\psi(\rho) = \alpha\rho^2$,

$$w_\psi(\psi) = (2\alpha)^{-1}\sigma^{-2}\exp[-\psi/(2\alpha\sigma^2)], \tag{10}$$

$$w_x(x) = \exp(-x), \tag{11}$$

with $x = \psi/\psi_0$ and $\psi_0 = 2\alpha\sigma^2$.

In a special case of $\alpha = 1/2$, we find $\psi = \rho^2/2$ and $\psi_0 = \sigma^2$. Equations (10) and (11) then lead to the distribution of the field intensity $I = \rho^2/2$, $w_I(I) = \langle I\rangle^{-1}\exp(-I/\langle I\rangle)$, with $\langle I\rangle = \langle\rho^2\rangle/2 = \sigma^2 = \psi_0$.

The model of a stochastic laser driver as described by Eq. (7) is, of course, an idealization. Yet, this model is well established, fully justifiable as a model of a multimode laser output, and is grounded in a broad class of existing laser sources. Specifically, as one of its properties, this distribution assigns very low, yet finite probabilities to envelopes that are many orders of magnitude larger than its mean, $\langle\rho\rangle = (\pi/2)^{1/2}\sigma$, or median, $\rho_d = (2\ln 2)^{1/2}\sigma$. This property of Eq. (7), however, does not pose any practical or conceptual difficulty. In practical terms, although the pump energy in laser sources of stochastic pulses is limited, for high-energy laser systems, such as electron-beam- or electric-discharge-pumped excimer lasers [35,47–51] and flashlamp- or diode-pumped high-power Nd: glass lasers [52–57], the pump energy is also many orders of magnitude higher than the energy of laser pulses. Therefore, pulses with ρ orders of magnitude larger than $\langle\rho\rangle$ or ρ_d are not prohibited.

The model of Eqs. (5) and (7) is not only grounded in a broad class of laser sources, such as electron-beam- or electric-discharge-pumped excimer lasers [35,47–51] and flashlamp- or diode-pumped high-power Nd: glass lasers [52–57], but is also fully justifiable as a model of statistics of a generic multimode laser source whose output is a superposition of a large number $\mathcal{N}$ of modes with random, uniformly distributed phases [34]. In the Appendix, we provide such a justification, showing that, with a sufficiently large $\mathcal{N}$, the envelope distribution of such a multimode laser output will be arbitrarily close to the distribution of Eq. (7). As a meaningful practical estimate, for a high-power excimer laser [35,47–51], a typical correlation time $\tau_c \approx 1$ ps and pulse width $\tau_p \approx 1$ ns give $\mathcal{N} \approx \tau_p/\tau_c \approx 1000$. For such N, $\langle y^4\rangle$ for a multimode laser deviates from $\langle y^4\rangle$ for a Gaussian statistics by less than $(2N)^{-1} = 0.0005$ (see the details in the Appendix). Energy conservation is never a question in this setting, as envelope fluctuations at the output of such a laser do not involve any extra pump energy, but are totally due to the phase fluctuations of the laser modes, leading to random—sometimes extreme—variations in the envelope of the laser output.

III. EXCESS KURTOSIS AND THE TAIL PROPERTIES OF NONLINEAR READOUT STATISTICS

As pointed out in extensive earlier literature (see Ref. [34] for a review), a polynomial transform of the statistics as described by Eq. (4) with $\psi(\rho) = \alpha\rho^n$, $n > 1$, gives rise to a probability distribution $w_\psi(\psi)$ in Eq. (4), whose rate of convergence to zero in the $\psi \to \infty$ limit is lower than the $w_\rho(\rho) \to 0$ convergence rate of the original distribution $w_\rho(\rho)$. In this section, we seek to quantify this property of nonlinear statistics transform as described by Eq. (4) by examining the excess kurtosis of $w_\psi(\psi)$,

$$\mathcal{K} = \mu_4/\mu_2^2 - 3, \tag{12}$$

where μ_i is the ith central moment of $w_\psi(\psi)$.

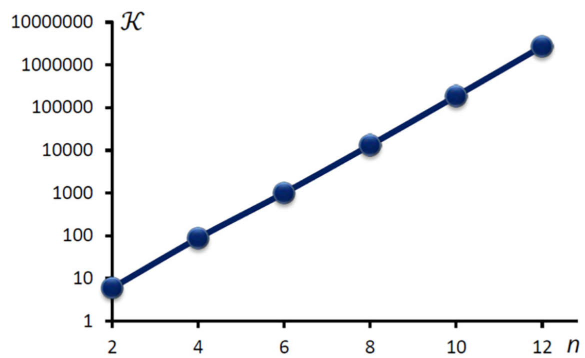

FIG. 1. The excess kurtosis $\mathcal{K}$ of the distribution $w_\psi(\psi)$ as a function of the nonlinearity order n with $\psi(\rho) = \alpha\rho^n$, $n = 2k$, k is an integer, and with $w_\rho(\rho)$ as defined by Eq. (7).

The excess kurtosis $\mathcal{K}$ as defined by Eq. (12) provides a meaningful measure of how heavy the tail of the distribution $w_\psi(\psi)$ is relative to the tail of the normal distribution, whose kurtosis $\mathcal{K}_0 = 3$ sets a baseline for $\mathcal{K}$.

With $w_\rho(\rho)$ as defined by Eq. (7) taken as the original distribution in Eq. (4), Eq. (12) allows an instructive closed-form analytical solution for the excess kurtosis $\mathcal{K}$ of $w_\psi(\psi)$. Indeed, with $w_\rho(\rho)$ as defined by Eq. (7) and with $\psi(\rho) = \alpha\rho^{2k}$, where k is an integer, we find $\langle\psi^m\rangle = 2^{km}\alpha^{2km}\Gamma(1+km)\sigma^{2km} = 2^{km}\alpha^{2km}(km)!\sigma^{2km}$, with $m = 1, 2, 3, 4$, as needed for the calculation of the moments μ_2 and μ_4 in Eq. (12). Expressing μ_2 and μ_4 via $\langle\psi^m\rangle$, with $m = 1, 2, 3, 4$, and plugging the result into Eq. (12), we arrive at

$$\mathcal{K} = [(4k)! - 4k!(3k)! + 6(2k)!(k!)^2 - 3(k!)^4]/[(2k)! - (k!)^2]^2 - 3. \tag{13}$$

For the Kerr-effect self-focusing nonlinearity, $\psi(\rho) = \alpha\rho^2$, $k = 1$, $n = 2$, Eq. (13) gives $\mathcal{K} = 6$, thus recovering the excess kurtosis of an exponential distribution. This result is in full agreement with expectations since, for ρ distributed in accordance with Eq. (6), ρ^2 is distributed exponentially. As can be seen from Eq. (13), the excess kurtosis of the statistics of nonlinear readouts rapidly grows with the nonlinearity order (Fig. 1), thus indicating a physically significant redistribution of statistical weight within the distribution $w_\psi(\psi)$ from its central part to its tails.

IV. EXTREME-VALUE STATISTICS: THE GENERAL FRAMEWORK

Because the peak power of a stochastic laser field is a random variable, which can only be described in terms of its statistical distribution, the standard self-focusing avoidance criterion $P < P_{\rm cr}$ is no longer productive. Instead of dealing with a question as to how to completely avoid self-focusing, stochastic analysis has to deal with a question of how to effectively manage the self-focusing probability. Moreover, because self-focusing is the prime cause of a damage of optical components in high-power short-pulse laser beamlines, this self-focusing management needs to apply not to individual pulses, but to laser pulses *en masse*. Since each new stochastic laser pulse adds to the combined probability of self-focusing, field stability against self-focusing over a sample of N laser shots, as another striking distinction from deterministic self-focusing, should be viewed as a function of N.

Central to our study is the search for an adequate framework for the analysis of such problems by resorting to the pertinent extreme-value statistics. To understand the extreme-value statistics of the nonlinear process ψ, we consider a random sample of N independent nonlinear-process readouts, $\{\psi_1, \psi_2, \ldots, \psi_N\}$, and define its maximum, $M_N = \max\psi_1, \psi_2, \ldots, \psi_N\}$. The probability that $M_N \leqslant \xi$ is $Q_N(\xi) = \int_{-\infty}^{\xi} d\psi_1 \int_{-\infty}^{\xi} d\psi_2 \ldots \int_{-\infty}^{\xi} d\psi_N p_\psi(y\psi_1, \psi_2, \ldots, \psi_N)$, where $p_\psi(\psi_1, \psi_2, \ldots, \psi_N)$ is the joint distribution of $\psi_1, \psi_2, \ldots, \psi_N$.

Because $\psi_1, \psi_2, \ldots, \psi_N$ are independent and identically distributed, the joint cumulative distribution $Q_N(\xi)$ is found as

$$Q_N(\xi) = [W(\xi)]^N = [1 - P(\xi)]^N, \tag{14}$$

with $W(\xi) = \int_{-\infty}^{\xi} w_\psi(\psi) d\psi$ and

$$P(\xi) = \int_{\xi}^{\infty} w_\psi(\psi) d\psi. \tag{15}$$

With $w_\psi(\psi)$ and $w_\rho(\rho)$ as defined by Eqs. (4) and (7), integration in Eq. (15) yields

$$P(\xi) = \exp\{-[\rho(\xi)]^2/(2\sigma^2)\}. \tag{16}$$

In accordance with the Fisher-Tippett-Gnedenko (FTG) theorem [58–62], the limit $F(\xi) = \lim_{x\to\infty,\, N\to\infty} Q_N(\xi)$ can only exist, upon linear renormalization $z = (\xi - a_N)/b_N$ with constants $a_N > 0$ and b_N, in one of three classes of functions, $F_1(z)$, $F_2(z)$, or $F_3(z)$, identified as the Gumbel [63], Fréchet [58], and Weibull [64] distributions. The FTG theorem guarantees that, if an extreme-value distribution exists, in a sense of the $\lim_{x\to\infty,\, N\to\infty} Q_N(\xi) = F[(\xi - a_N)/b_N]$ limit for a parent statistics $w_\psi(\psi)$, it is always found in one of three classes of cumulative probability distributions as described by $F_1(z)$, $F_2(z)$, and $F_3(z)$. This theorem, however, does not guarantee the existence of the $\lim_{x\to\infty,\, N\to\infty} Q_N(\xi) = F[(\xi - a_N)/b_N]$ limit for any $w_\psi(\psi)$. The general recipe of finding the a_N and b_N renormalization parameters is not known either.

V. LOW SIGNAL-TO-NOISE LASER DRIVER

To see whether or not the extreme-value theory applies to stochastic self-focusing, we search for $\lim_{x\to\infty,\, N\to\infty}[1 - P(x)]^N$ with

$$P(x) = \exp(-x^{2/n}), \tag{17}$$

as dictated by Eqs. (9) and (16). To this end, we represent $Q_N(x)$ as

$$Q_N(x) = [1 - \exp(-x^\varepsilon)]^N = \exp\{N \ln[1 - \exp(-x^\varepsilon)]\}, \tag{18}$$

with $\varepsilon = 2/n$.

In the limit of large N,

$$Q_N(x) \approx \exp[-\exp(-x^\varepsilon + \ln N)]. \tag{19}$$

With $x = a_N + b_N z$ and with x^ε expanded in the large-N limit as $x^\varepsilon \approx a_N^\varepsilon + \varepsilon a_N^{\varepsilon-1} b_N z$,

$$Q_N(x) \approx \exp\left[-\exp\left(-a_N^\varepsilon + \ln N - \varepsilon a_N^{\varepsilon-1} b_N z\right)\right], \tag{20}$$

leading to

$$F(x) = \lim_{x,\, N\to\infty}[1 - \exp(-x^\varepsilon)]^N = F_G[(x - a_N)/b_N], \tag{21}$$

where

$$F_G(z) = \exp[-\exp(-z)], \tag{22}$$

$$a_N = (\ln N)^{1/\varepsilon} = (\ln N)^{n/2}, \tag{23}$$

and

$$b_N = (\ln N)^{1/\varepsilon - 1}/\varepsilon = n(\ln N)^{n/2-1}/2. \tag{24}$$

The respective probability distribution function,

$$f_G(z) = F'_G(z) = \exp[-z - \exp(-z)], \tag{25}$$

is recognized as the Gumbel distribution [61–63].

Because the Gumbel distribution is one of the universal extreme-value distributions predicted by the Fisher-Tippett-Gnedenko theorem, Eqs. (21)–(25) prove that the FTG does apply to the extreme-value statistics of a nonlinear signal $\psi(\rho) = \alpha\rho^n$. Moreover, Eqs. (23) and (24) provide explicit analytic solutions for the location and scale parameters of the extreme-value distribution of such a nonlinear signal.

We see from Eqs. (21)–(25) that the class of extreme-value distribution for a polynomial nonlinearity is independent of the nonlinearity order n. However, the location and scale parameters a_N and b_N of the respective extreme-value distribution are n-dependent. As an important property of the extreme-value distribution defined by Eqs. (22)–(25), the location of its maximum, $x_m = a_N$, i.e., the mode of the distribution, shifts as $(\ln N)^{n/2}$ toward larger x with the growth in N,

$$\psi_m = a_N \psi_0 = 2^{n/2}\alpha\sigma^n(\ln N)^{n/2}. \tag{26}$$

The extreme-value distribution for the Kerr-effect self-focusing nonlinearity is found from Eqs. (21)–(25) with $n = 2$, leading to the Gumbel distribution with location and scale parameters as dictated $a_N = \ln N$ and $b_N = 1$. The mode of this distribution, found from Eq. (26), is

$$\psi_m = 2\alpha\sigma^2 \ln N = 2\alpha\langle I\rangle \ln N. \tag{27}$$

While the maximum of this extreme-value distribution function shifts with N as $\ln N$, its width remains constant. These tendencies are clearly seen in Fig. 2, where N is set at 10^2, 10^3, and 10^4, leading to a noticeable shift of the extreme-value distribution.

With α as defined by Eq. (3) for the self-focusing nonlinearity, Eq. (27) becomes

$$\psi_m = \bar{P}\ln N/P_{\rm cr} = \langle I\rangle S \ln N/P_{\rm cr}. \tag{28}$$

Searching for the critical peak power from the threshold condition of Eq. (2) at the mode [Eq. (27)] of the extreme-value distribution defined by Eqs. (22)–(26), we find $\bar{P}_{\rm cr} = P_{\rm cr}/\ln N$.

At the peak of the Gumbel distribution function, as defined by Eqs. (26)–(28), its cumulative distribution function [Eq. (19)] takes a value of $1/e \approx .037$ regardless of its scale parameter. Keeping $\bar{P}$ below $P_{\rm cr}/\ln N$, i.e.,

$$\bar{P} < P_{\rm cr}/\ln N, \tag{29}$$

thus provides $\mathcal{P} \geqslant 37\%$ probability of self-focusing avoidance for all N pulses in a sample.

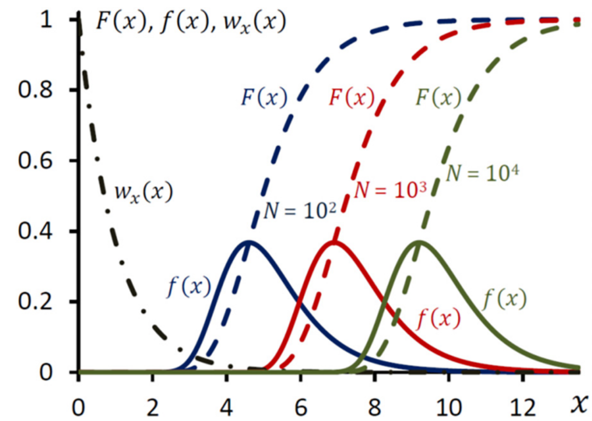

FIG. 2. Extreme-value probability density functions (solid lines) and their respective cumulative distribution functions (dashed lines) for the self-focusing nonlinearity $\psi = \alpha\rho^2$, driven by a laser field with an envelope distribution (7). The abscissa is $x = \psi/\psi_0$, with $\psi_0 = 2\alpha\sigma^2$. The number of nonlinear-signal readouts is $N = 10^2$, 10^3, and 10^4, as shown in the plot. The probability density function of ψ/ψ_0 for $\psi = \alpha\rho^2$ is shown by the dashed-dotted line.

VI. HIGH SIGNAL-TO-NOISE LASER DRIVER

To appreciate the significance of the criterion of Eq. (29), we resort to a more general model of the envelope distribution as described by Eq. (6) and examine the case of high signal-to-noise ratios, $a \gg 1$. In this limit, the modified Bessel function in Eq. (6) is expanded as $I_0(x) \approx (2\pi x)^{-1/2}\exp(-x)$ to yield

$$w_\rho(\rho) = (2\pi)^{-1/2}\sigma^{-1}\exp[-(\rho - \rho_s)^2/(2\sigma^2)]. \tag{30}$$

The probability density function $w_\rho(\rho)$ as described by Eq. (30) is a narrow Gaussian distribution that is centered at $\rho = \rho_s$ with a distribution width of σ.

The extreme-value distribution of a standard Gaussian probability distribution $w(x) = (2\pi)^{-1/2}\sigma^{-1}\exp[-x^2/(2\sigma^2)]$ is found in the class of the Gumbel distribution functions [61–63], as defined by Eqs. (22)–(25), with location and scale parameters

$$a_N = \sigma\{(2\ln N)^{1/2} - \ln(\ln N)/[2(2\ln N)^{1/2}]\}, \tag{31}$$

and

$$b_N = \sigma\{(2\ln N)^{1/2} - \ln(\ln N)/[2(2\ln N)^{1/2}]\}^{-1}. \tag{32}$$

For $N \gg 1$, $a_N \approx \sigma(2\ln N)^{1/2}$ and $b_N \approx \sigma(2\ln N)^{-1/2}$.

The extreme-value envelope distribution of an $a \gg 1$ field waveform (5) is thus the Gumbel distribution [Eq. (25)], whose maximum (mode), centered at

$$\rho_{\max} \approx \rho_s + \sigma(2\ln N)^{1/2}, \tag{33}$$

shifts with N as $\sigma(2\ln N)^{1/2}$ (Fig. 2).

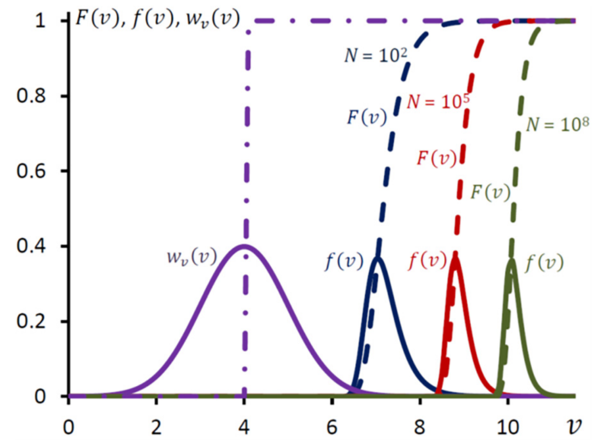

FIG. 3. Extreme-value probability density functions (solid lines) and their respective cumulative distribution functions (dashed lines) for the envelope of the high signal-to-noise field waveform (5) with $a^2 = 16$ and $N = 100$, 10^5, and 10^8, as shown in the plot. The abscissa is $v = \rho/\sigma$. Also shown are (solid purple line) the probability density function for the envelope of the field waveform (5) with $a^2 = 16$ and (dash-dotted purple line) the cumulative probability function for the envelope of the field waveform (5) with $a \to \infty$.

With σ expressed as $= \rho_s/a$, Eq. (33) can be rewritten as

$$\rho_{\max} \approx \rho_s[1 + a^{-1}(2\ln N)^{1/2}]. \quad (34)$$

For $a^{-1}(2\ln N)^{1/2} \gg 1$, or $N \gg \exp(a^2/2)$, Eq. (34) becomes

$$\rho_{\max} \approx (\rho_s/a)(2\ln N)^{1/2} = \sigma(2\ln N)^{1/2}. \quad (35)$$

In this limit, $\rho_{\max}^2 \approx 2\sigma^2 \ln N$, recovering Eq. (27) for the mode of the extreme-value distribution for the field waveform with zero mean.

We now consider an $a \gg 1$ laser field waveform (5) with $\rho_s < \rho_{\rm cr} = (2P_{\rm cr}/S)^{1/2}$. In the $a \to \infty$ limit, the second term in Eqs. (33) and (34) is vanishingly small, indicating that stochastic properties of the laser field are totally suppressed and the peak power is $P = P_s = S\rho_s^2/2$. The cumulative distribution function for the field envelope is then a step function centered at $\rho = \rho_s$ (dash-dotted purple line in Fig. 3). As long as $\rho_s < \rho_{\rm cr}$, self-focusing of such a beam is totally avoided. Thus, as $a \to \infty$, the canonical picture of deterministic self-focusing is recovered in its entirety. Specifically, the beam remains stable, exhibiting no self-focusing, as long as $P_0 < P_{\rm cr}$, in full agreement with the deterministic self-focusing avoidance criterion.

For finite a, however, the second, stochastic term in Eqs. (33) and (34) is nonzero, increasing with N as $a^{-1}(2\ln N)^{1/2}$. Now, as N grows, the whole extreme-value envelope distribution of the field (5) shifts toward larger ρ (Fig. 3). The peak of this distribution, i.e., its mode, is achieved at $\rho = \rho_{\max}$, translating into a peak power

$$P_{\max} \approx P_s[1 + a^{-1}(2\ln N)^{1/2}]^2. \quad (36)$$

It is readily seen from Eq. (36) that, even with the peak power P_s of a laser pulse chosen well below the critical power of self-focusing $P_{\rm cr}$, the mode of its extreme-value distribution for sufficiently large N may fall beyond the $P_{\rm cr}$ borderline, thus making the beam unstable with respect to self-focusing. The requirement $P_{\max} < P_{\rm cr}$ as a criterion of self-focusing avoidance can now be satisfied only with

$$P_s < P_{\rm cr}[1 + a^{-1}(2\ln N)^{1/2}]^{-2}. \quad (37)$$

For $N \gg \exp(a^2/2)$, the criterion of Eq. (37) becomes

$$P_s < a^2 P_{\rm cr}/(2\ln N). \quad (38)$$

Calculating $\langle I\rangle = \langle\rho^2\rangle/2$ via a statistical averaging of ρ^2 with a probability density function as defined by Eq. (6), we find $\langle I\rangle = \sigma^2(1 + a^2/2)$. Using this result to express P_s in the $a \gg 1$ limit as $P_s \approx a^2\sigma^2/2$, we see that Eq. (38) reduces to Eq. (29).

Equations (37) and (38) provide closed-form analytical expressions for the criterion of self-focusing avoidance over a sample of $N \gg 1$ stochastic laser pulses. Thus, when the criterion of Eq. (37) is fulfilled, self-focusing is avoided with a $\mathcal{P} \geqslant 37\%$ probability for all N pulses in the sample.

In Fig. 3, we present the extreme-value probability density functions and their respective cumulative distribution functions for the envelope of a field waveform (5) with $a^2 = 16$. With an increase in N, the mode of the extreme-value distributions is seen to shift toward larger $v = \rho/\sigma$. The magnitude of this shift agrees very well with Eqs. (33) and (34). Specifically, with $N = 100$, $(2\ln N)^{1/2} \approx 3.0$ provides a very accurate estimate for the shift of the mode of the $N = 100$ extreme-value probability density function (blue solid curve in Fig. 3) relative to the maximum of the envelope probability distribution density function (violet solid curve in Fig. 3). With $N = 10^5$, on the other hand, $1 + a^{-1}(2\ln N)^{1/2} \approx 2.2$, so that the maximum of the respective extreme-value probability density function is achieved at $P_{\max} \approx 2.2P_s$. The self-focusing avoidance criterion of Eq. (37) then becomes $P_s < 0.45P_{\rm cr}$. When this condition is satisfied, self-focusing within a sample of $N = 10^5$ laser pulses is avoided with a $\mathcal{P} \geqslant 37\%$ probability, as read out from the respective cumulative distribution function (red dashed curve in Fig. 3).

Should this level of self-focusing avoidance probability $\mathcal{P}$ become insufficient, higher levels of $\mathcal{P}$ can be achieved by tightening the upper bounds in inequalities (37) and (38) to impose more stringent requirements on the laser peak power. Specifically, with the upper bounds in these inequalities defined from the median rather than the mode of the respective extreme-value distribution, a $\mathcal{P} \geqslant 50\%$ probability of self-focusing avoidance will be provided. As an aid to a practical implementation of this approach, the solution for the median x_d for the Gumbel distribution [Eq. (25)] is known analytically, $x_d = a_N - b_N \ln(\ln 2)$, with a_N and b_N as defined by Eqs. (23) and (24). More generally, unlike the specific probability distribution of a stochastic laser field, its extreme-value distribution is known—it is always found in the class of the Gumbel distribution functions as long as optical nonlinearity is $\psi(\rho) = \alpha\rho^n$. Therefore, for any given ε, a $\mathcal{P} \geqslant \varepsilon$ self-focusing avoidance probability can be achieved by relating the upper bounds in Eqs. (37) and (38) to ψ chosen close enough to the $F_G(\psi) = 1$ asymptotic limit of the respective extreme-value distribution (dashed lines in Figs. 2 and 3).

Equations (29), (37), and (38) provide a closed-form analytical solution for the criterion of self-focusing avoidance within large samples of stochastic laser pulses. This solution establishes conditions under which stochastic laser pulses become prone to self-focusing. While it answers the question of *when* stochastic laser pulses become unstable with respect to self-focusing, this solution is not intended to address the question of *how* self-focusing unfolds. To answer this latter question, nonlinear spatiotemporal field evolution equations need to be solved for a specific stochastic pulse envelope and a specific stochastic beam profile [65–67]. Aimed at understanding conditions and criteria of stochastic self-focusing rather than simulating a specific self-focusing scenario, the present study is a critically important step in the analysis of stochastic self-focusing.

VII. CORRELATION EFFECTS

The analysis above was performed in the approximation of independent laser pulses. While this approximation is fully justified for laser sources operating at relatively low pulse repetition rates, its validity may and should be questioned for advanced high-repetition-rate laser systems, where pulse-to-pulse correlations are possible due to gain depletion or thermal-wake effects in a laser medium. While there is currently no universal framework for the analysis of extreme-value statistics of correlated random processes, some powerful and rather general approaches have been developed for certain practically significant special cases [68–78].

Of particular relevance for pulsed laser sources is the case of weak correlations [68–71], in which the correlation length N_c is much shorter than the length N of the sample of random readouts $\{\psi_1, \psi_2, \ldots, \psi_N\}$. The sample of readouts can then be divided into $M = N/N_c \gg 1$ blocks. While the readouts within each such block are correlated, readouts from different blocks are not. Now, the local maxima χ_j found within each of these M blocks, $j = 1, \ldots, M$ are uncorrelated. Thus, if the probability distribution of χ_i can be found, then the problem is reduced to searching for the maximum of M uncorrelated random variables, i.e., the problem that can be solved in terms of extreme-value statistics [68].

When correlations between ψ_i are so strong that the correlation length N_c is comparable to the sample length, analysis of extreme-value statistics is possible when the lasing process can be described in terms of exactly solvable models. Pertinent to pulsed laser sources are the models of Brownian motion, random walks, and the Ornstein-Uhlenbeck process [72–78]. Within their respective applicability realms, these models will yield analytical solutions for the extreme-value distribution of the driver field, which can be then used to find the extreme value distribution of the nonlinear readouts in accordance with the procedure described in the previous sections.

VIII. CONCLUSION

To summarize, we have shown that analysis of extreme-value statistics of stochastic laser pulses suggests a closed-form, quantitative criterion of self-focusing avoidance. Unlike deterministic self-focusing, whose criterion is expressed in terms of a well-defined self-focusing threshold P_{cr}, its stochastic counterpart is a probabilistic process whose combined probability for a sample of N laser pulses builds up as a function of N, leading to N-dependent self-focusing avoidance criteria. Specifically, for $N \gg 1$ laser pulses with a signal-to-noise ratio a, the criterion of self-focusing avoidance is shown to shift as $a^2 P_{cr}/(2\ln N)$.

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APPENDIX

Stochastic properties of a multimode laser: Rationale behind the model of a stochastic laser driver

Equation (29) provides a closed-form criterion of self-focusing avoidance applicable to a vast range of stochastic field-evolution scenarios in which the envelope of the input field (5) has a Rayleigh distribution, as described by Eq. (7). To justify this model of a stochastic laser driver and to provide a quantitative criterion of applicability of this model, we consider a generic laser source, whose output is a mixture of $\mathcal{N}$ modes,

$$y(t) = \sum_{n=1}^{\mathcal{N}} b_n \cos(\omega_n t + \varphi_n) = \sum_{n=1}^{\mathcal{N}} b_n \cos \Phi_n. \tag{A1}$$

The characteristic function of such a process [34],

$$\theta(u) = \langle \exp(iyu) \rangle, \tag{A2}$$

is

$$\theta(u) = \prod_{n=1}^{eucalN} \theta(ub_n), \tag{A3}$$

where

$$\theta(ub_n) = \langle \exp(iub_n \cos \Phi_n) \rangle$$
$$= (2\pi)^{-1} \int_{-\pi}^{\pi} \exp[iub_n \cos(\omega_n t + \varphi_n)] d\varphi_n = J_0(ub_n)$$

is the characteristic function of the nth mode, and $J_0(\zeta)$ is the zeroth-order Bessel function of ζ.

The distribution function of y is then found as

$$w(y) = (2\pi)^{-1} \int_{-\infty}^{\infty} \theta(u) \exp(-iuy) du, \tag{A4}$$

leading to

$$w(y) = (2A)^{-1} \left[1 + 2 \sum_{k=1}^{\infty} \cos(\pi k y/A) \prod_{n=1}^{\mathcal{N}} J_0(\pi k a_n/A) \right], \tag{A5}$$

where $A = \sum_{n=1}^{\mathcal{N}} a_n$.

For low $\mathcal{N}$, the distribution $w(y)$ is distinctly non-Gaussian. It becomes Gaussian, however, as $\mathcal{N} \to \infty$. To see this, we consider a process $z(t) = \mathcal{N}^{-1/2}y(t)$. The characteristic function of such a process with $b_n = b$ is

$$\theta_z(u) = [J_0(N^{-1/2}bu)]^{\mathcal{N}}. \tag{A6}$$

In the $\mathcal{N} \gg 1$ limit,

$$\ln[\theta_z(u)] \approx -(bu)^2/4 + (bu)^4/(64\mathcal{N}), \tag{A7}$$

leading to [33]

$$w(z) = (\pi)^{-1/2}b^{-1}[1-(64\mathcal{N})^{-1}H_4(z/b)]\exp(-z^2/b^2), \tag{A8}$$

where $H_n(x)$ is the nth-order Hermite polynomial.

As $\mathcal{N} \to \infty$, both $w(z)$ and $w(y)$ become Gaussian, with

$$w(y) = (2\pi)^{-1/2}\sigma^{-1}\exp[-y^2/(2\sigma^2)], \tag{A9}$$

$$2\sigma^2 = \mathcal{N}b^2.$$

We now see that, as the number of lasing modes N increases, the distribution of a multimode lasing process (A1) converges to a Gaussian distribution regardless of the spectrum of the lasing modes $\{\omega_n\}$. The lowest-order moments of y are

$$\langle y^2\rangle = \mathcal{N}b^2/2 = \sigma^2 = \bar{I},$$
$$\langle y^4\rangle = 3[1-(2\mathcal{N})^{-1}]\bar{I}^2,$$
$$\langle y^6\rangle = 15[1-3(2\mathcal{N})^{-1} + 2(3\mathcal{N}^2)^{-1}]\bar{I}^3.$$

With $\mathcal{N} \to \infty$, these moments recover the moments of a Gaussian random process [79]. As a meaningful practical estimate, for a high-power excimer laser [35,47–51], a typical correlation time, $\tau_c \approx 1$ ps, and pulse width, $\tau_p \approx 1$ ns, give $\mathcal{N} \approx \tau_p/\tau_c \approx 1000$. For such $\mathcal{N}$, $\langle y^4\rangle$ for a multimode laser deviates from $\langle y^4\rangle$ for a Gaussian statistics by less than $(2N)^{-1} = 0.0005$.

[1] N. F. Pilipetsky and A. R. Rustamov, JETP. Lett. **2**, 55 (1965).
[2] P. Lallemand and N. Bloembergen, Phys. Rev. Lett. **15**, 1010 (1965).
[3] E. Garmire, R. Y. Chiao, and C. H. Townes, Phys. Rev. Lett. **16**, 347 (1966).
[4] Y. R. Shen, *The Principles of Nonlinear Optics* (Wiley-Interscience, New York, 1984).
[5] S. A. Akhmanov, A. P. Sukhorukov, and R. V. Khokhlov, Sov. Phys. Usp. **10**, 609 (1968).
[6] *Self-Focusing: Past and Present*, edited by R. W. Boyd, S. G. Lukishova, and Y. Shen, Topics in Applied Physics Vol. 114 (Springer, New York, 2009).
[7] D. E. Spence, P. N. Kean, and W. Sibbett, Opt. Lett. **16**, 42 (1991).
[8] H. A. Haus, J. Sel. Top. Quantum Electron. **6**, 1173 (2000).
[9] A. L. Gaeta, Phys. Rev. Lett. **84**, 3582 (2000).
[10] A. Couairon and A. Mysyrowicz, Phys. Rep. **441**, 47 (2007).
[11] L. Bergé, S. Skupin, R. Nuter, J. Kasparian, and J.-P. Wolf, Rep. Prog. Phys. **70**, 1633 (2007).
[12] A. M. Zheltikov, Phys. Usp. **49**, 605 (2006).
[13] R. Y. Chiao, M. A. Johnson, S. Krinsky, H. A. Smith, C. H. Townes, and E. M. Garmire, IEEE J. Quantum Electron. **2**, 467 (1966).
[14] A. M. Zheltikov, J. Phys. B: At. Mol. Opt. Phys. **50**, 092001 (2017).
[15] *The Supercontinuum Laser Source: The Ultimate White Light*, edited by R. Alfano (Springer, New York, 2022).
[16] A. Zheltikov, Appl. Phys. B **77**, 143 (2003).
[17] M. D. Perry and G. Mourou, Science **264**, 917 (1994).
[18] V. Shumakova, P. Malevich, S. Ališauskas, A. Voronin, A. M. Zheltikov, D. Faccio, D. Kartashov, A. Baltuška, and A. Pugžlys, Nat. Commun. **7**, 12877 (2016).
[19] A. A. Voronin, A. M. Zheltikov, T. Ditmire, B. Rus, and G. Korn, Opt. Commun. **291**, 299 (2013).
[20] A. V. Mitrofanov, D. A. Sidorov-Biryukov, M. M. Nazarov, A. A. Voronin, M. V. Rozhko, A. B. Fedotov, and A. M. Zheltikov, Opt. Lett. **46**, 1081 (2021).
[21] N. Bloembergen, IEEE J. Quantum Electron. **10**, 375 (1974).
[22] B. C. Stuart, M. D. Feit, S. Herman, A. M. Rubenchik, B. W. Shore, and M. D. Perry, Phys. Rev. B **53**, 1749 (1996).
[23] D. Strickland and G. Mourou, Opt. Commun. **55**, 447 (1985).
[24] E. A. Khazanov, S. Yu. Mironov, and G. Mourou, Phys. Usp. **62**, 1096 (2019).
[25] P. Maine, D. Strickland, P. Baldo, M. Pessot, and G. Mourou, IEEE J. Quantum Electron. **24**, 398 (1988).
[26] R. W. Boyd, *Nonlinear Optics* (Academic, Boston, MA, 1992).
[27] A. M. Zheltikov, A. L'Huillier, and F. Krausz, in *Handbook of Lasers and Optics*, edited by F. Träger (Springer, New York, 2007), pp. 157–248.
[28] R. L. Carman, F. Shimizu, C. S. Wang, and N. Bloembergen, Phys. Rev. A **2**, 60 (1970).
[29] S. A. Akhmanov, Y. E. D'yakov, and L. I. Pavlov, Sov. Phys. JETP **39**, 249 (1974).
[30] G. S. Agarwal, Opt. Commun. **35**, 267 (1980).
[31] V. I. Bespalov and V. I. Talanov, JETP Lett. **3**, 307 (1966).
[32] A. Zheltikov, Opt. Express **24**, 20716 (2016).
[33] M. M. T. Loy and Y. R. Shen, Phys. Rev. Lett. **22**, 994 (1969).
[34] S. A. Akhmanov, Yu. E. D'yakov, and A. S. Chirkin, *Introduction to Statistical Radiophysics and Optics* (Nauka, Moscow, 1981).
[35] S. Obenschain, R. Lehmberg, D. Kehne, F. Hegeler, M. Wolford, J. Sethian, J. Weaver, and M. Karasik, Appl. Opt. **54**, F103 (2015).
[36] E. Goulielmakis, M. Schultze, M. Hofstetter, V. S. Yakovlev, J. Gagnon, M. Uiberacker, A. L. Aquila, E. M. Gullikson, D. T. Attwood, R. Kienberger, F. Krausz, and U. Kleineberg, Science **320**, 1614 (2008).
[37] M. Th. Hassan, T. T. Luu, A. Moulet, O. Raskazovskaya, P. Zhokhov, M. Garg, N. Karpowicz, A. M. Zheltikov, V. Pervak, F. Krausz, and E. Goulielmakis, Nature (London) **530**, 66 (2016).
[38] A. Zheltikov, J. Opt. Soc. Am. B **36**, A168 (2019).
[39] N. Bloembergen, Appl. Opt. **12**, 661 (1973).
[40] M. Lenzner, J. Krüger, S. Sartania, Z. Cheng, Ch. Spielmann, G. Mourou, W. Kautek, and F. Krausz, Phys. Rev. Lett. **80**, 4076 (1998).

[41] P. A. Zhokhov and A. M. Zheltikov, Phys. Rev. Lett. **113**, 133903 (2014).
[42] S. L. Chin, A. Talebpour, J. Yang, S. Petit, V. P. Kandidov, O. G. Kosareva, and M. P. Tamarov, Appl. Phys. B **74**, 67 (2002).
[43] R. Ackermann, G. Méjean, J. Kasparian, J. Yu, E. Salmon, and J.-P. Wolf, Opt. Lett. **31**, 86 (2006).
[44] A. Houard, M. Franco, B. Prade, A. Durécu, L. Lombard, P. Bourdon, O. Vasseur, B. Fleury, C. Robert, V. Michau, A. Couairon, and A. Mysyrowicz, Phys. Rev. A **78**, 033804 (2008).
[45] S. O. Rice, Bell System Tech. J. **23**, 282 (1944).
[46] S. O. Rice, Bell System Tech. J. **27**, 109 (1948).
[47] *Excimer Lasers*, edited by C. K. Rhodes (Springer, Berlin, 1979).
[48] E. M. Campbell, T. C. Sangster, V. N. Goncharov, J. D. Zuegel, S. F. B. Morse, C. Sorce, G. W. Collins, M. S. Wei, R. Betti, S. P. Regan, D. H. Froula, C. Dorrer, D. R. Harding, V. Gopalaswamy, J. P. Knauer, R. Shah, O. M. Mannion, J. A. Marozas, P. B. Radha, M. J. Rosenberg *et al.*, Philos. Trans. R. Soc. A **379**, 0011 (2020).
[49] J. W. Bates, R. K. Follett, J. G. Shaw, S. P. Obenschain, J. F. Myatt, J. L. Weaver, M. F. Wolford, D. M. Kehne, M. C. Myers, and T. J. Kessler, Phys. Plasmas **30**, 052703 (2023).
[50] S. P. Obenschain, A. J. Schmitt, J. W. Bates, M. F. Wolford, M. C. Myers, M. W. McGeoch, M. Karasik, and J. L. Weaver, Philos. Trans. R. Soc. A **378**, 20200031 (2020).
[51] J. Weaver, R. Lehmberg, S. Obenschain, D. Kehne, and M. Wolford, Appl. Opt. **56**, 8618 (2017).
[52] A. B. Zylstra, O. A. Hurricane, D. A. Callahan *et al.*, Nature (London) **601**, 542 (2022).
[53] A. L. Kritcher, C. V. Young, H. F. Robey *et al.*, Nat. Phys. **18**, 251 (2022).
[54] H. Abu-Shawareb *et al.* (Indirect Drive ICF Collaboration), Phys. Rev. Lett. **129**, 075001 (2022).
[55] T. Döppner, M. Bethkenhagen, D. Kraus *et al.*, Nature (London) **618**, 270 (2023).
[56] D. Turnbull, J. Katz, M. Sherlock, L. Divol, N. R. Shaffer, D. J. Strozzi, A. Colaïtis, D. H. Edgell, R. K. Follett, K. R. McMillen, P. Michel, A. L. Milder, and D. H. Froula, Phys. Rev. Lett. **130**, 145103 (2023).
[57] S. Zhang, A. Chien, L. Gao *et al.*, Nat. Phys. **19**, 909 (2023).
[58] M. Fréchet, Ann. Soc. Polon. Math. **6**, 93 (1927).
[59] R. A. Fisher and L. H. C. Tippett, Proc. Cambridge Philos. Soc. **24**, 180 (1928).
[60] B. V. Gnedenko, Ann. Math. **44**, 423 (1943).
[61] J. Galambos, *The Asymptotic Theory of Extreme Order Statistics* (Krieger, Malabar, FL, 1987).
[62] S. Coles, *An Introduction to Statistical Modeling of Extreme Events* (Springer, Berlin, 2001).
[63] E. J. Gumbel, *Statistics of Extremes* (Columbia University Press, New York, 1958).
[64] W. Weibull, *Fatigue Testing and Analysis of Results* (Pergamon, New York, 1961).
[65] A. A. Voronin and A. M. Zheltikov, Phys. Rev. A **94**, 023824 (2016).
[66] A. A. Voronin and A. M. Zheltikov, Sci. Rep. **7**, 36263 (2017).
[67] A. A. Voronin and A. M. Zheltikov, Phys. Usp. **59**, 869 (2016).
[68] S. N. Majumdar, A. Pal, and G. Schehr, Phys. Rep. **840**, 1 (2020).
[69] T. Antal, M. Droz, G. Gyorgyi, and Z. Racz, Phys. Rev. Lett. **87**, 240601 (2001).
[70] G. Gyorgyi, N. R. Moloney, K. Ozogany, and Z. Racz, Phys. Rev. Lett. **100**, 210601 (2008).
[71] S. N. Majumdar and R. M. Ziff, Phys. Rev. Lett. **101**, 050601 (2008).
[72] J. Bechhoefer and B. Marshall, Phys. Rev. Lett. **98**, 098105 (2007).
[73] G. Schehr, S. N. Majumdar, A. Comtet, and J. Randon-Furling, Phys. Rev. Lett. **101**, 150601 (2008).
[74] J. Randon-Furling, S. N. Majumdar, and A. Comtet, Phys. Rev. Lett. **103**, 140602 (2009).
[75] S. N. Majumdar, A. Rosso, and A. Zoia, Phys. Rev. Lett. **104**, 020602 (2010).
[76] G. Schehr and S. N. Majumdar, Phys. Rev. Lett. **108**, 040601 (2012).
[77] A. Perret, A. Comtet, S. N. Majumdar, and G. Schehr, Phys. Rev. Lett. **111**, 240601 (2013).
[78] T. Sadhu, M. Delorme, and K. J. Wiese, Phys. Rev. Lett. **120**, 040603 (2018).
[79] W. Bryc, *The Normal Distribution: Characterizations with Applications* (Springer, Berlin, 1995).