Interaction-induced thermal conductivity of the unitary Fermi superfluids

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(Received 4 June 2024; accepted 7 August 2024; published 21 August 2024)

In recent experimental investigations of unitary Fermi gases, it has been observed that the thermal conductivity κ approaches the quantum limit under the phase transition temperature T_c. While relevant theoretical calculations have been conducted for this phenomenon, the interactions among particles within the heat current were neglected, violating conservation laws and revealing a significant disparity between computed results and experimental data. This paper addresses this issue by incorporating particle interactions within the heat current in Feynman diagrams, based on the Ward identity. We derive a modified Kubo-based expression for κ that accounts for the interaction. We also consider the anomalous thermal conductivity from current transfer via Cooper pairs for the theory completeness. Our computations reveal that near T_c, the fluctuations in the t-matrix paradigm are characterized by the pseudogap order and bosonic excitations contribute significantly to κ, leading to a conspicuous peak in addition to the critical universal scaling laws. Our results align well with the experiment near T_c.

DOI: 10.1103/PhysRevA.110.023323

I. INTRODUCTION

In recent years, the development of experiments involving ultracold atomic gases has injected new vitality into the fields of atomic and molecular physics, as well as condensed matter physics. Particularly, significant progress has been made in understanding quantum fluid behavior. Local transport measurement experiments conducted under nearly uniform density conditions [1–3] have provided robust support for direct comparisons between theory and experiment. By measuring density responses, researchers have successfully extracted the sound diffusion coefficient in the fluid [1,4,5], revealing contributions from various transport quantities. These experiments have not only deepened our understanding of the fundamental properties of ultracold atomic gases but have also taken an important step in theoretical validation, allowing for more precise verification and refinement of existing physical models.

Using the framework of two-fluid hydrodynamics [6], a description of two sound waves associated with the general transport coefficients has been formulated [7–10]. Despite these remarkable achievements, significant limitations persist in the study of thermal conductivity κ in unitary Fermi gases, particularly within superfluids below the phase transition temperature T_c. Existing research mainly focuses on the kinetic level of the Boltzmann equation. While kinetic theory can accurately calculate κ under certain conditions, such as the high-temperature extrapolation of Boltzmann results [10,11] or the low-temperature calculations by phonons [9], it fails below the pseudogap temperature T^* of strong correlation due to the presence of uncondensed fermion pairs [12,13]. Current research has employed t-matrix formalisms [14–19] and linear response theory [20] in the BCS-BEC crossover regime to calculate κ of ultracold Fermi gases [21]. These studies have not considered the interaction term in the heat current, and such vertex corrections do not guarantee conservation laws [22,23]. As the temperature approaches T_c, the contribution of bosonic pairs due to the interactions to κ will become significant. A simple calculation on low-temperature κ with the mean-field approximation has been reported [24] using the linear response theory and Kubo formulas without considering the atom interactions in the heat current. In particular, the work in Ref. [25] has reported the successful prediction of isothermal compressibility using the t matrix within the thermodynamical approaches, that is, a step discontinuity in the compressibility at T_c for finite damping ratio γ. They take $\gamma/T_F = \alpha' T/T_c$, and a large value of $\alpha' \simeq 1.0$ appears for a large step discontinuity of the compressibility at T_c. However, we calculate γ as a function of $(T/T_c, 1/k_F a_s)$, opening all channels of particle-particle scattering progress based on three-particle hydrodynamics [39], where n_{b_0} is the density of condensed pairs. The numerical results show $\alpha' = 0.48$. The full quantum theoretic frameworks for this issue have been presented [25–27], but the density-density correlation function has been used to only study the compressibility [25], the current-current correlation function is too complex to compute κ below T_c [26], and the relative theoretical work is above T_c [27]. It needs to extend our previous work [21] to general cases.

The diagram calculations under the t-matrix methods consider the fluctuations that dominate the critical phenomenon during the phase transitions, willing to provide more accurate results for judging the system's physics. However, as shown

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by Taylor [28], the critical region of ultracold Fermi gases near T_c is large (of the order unity). In this critical region, the fluctuations of the order parameter are particularly significant, rendering thermodynamic quantities computed by any method intrinsically unreliable. Instead, critical universal scaling laws prevail. In the context of the BCS-BEC crossover, the critical exponents in the vicinity of the phase transition region are the same as those of the three-dimensional (3D) XY model [12]. Based on this model, in the ^{4}He system [29] and the BCS-BEC crossover system [30], the specific heat with a λ-type phase transition exhibits identical critical exponents, $c_V \sim |\tau|^{-\alpha}$, where the reduced temperature $\tau = 1 - T/T_c$, $\alpha \simeq -0.012$, and the critical exponent for the coherence length $\nu = (2-\alpha)/3 \simeq 0.6707$. The thermal conductivity κ of the unitary gas of interest also exhibits a λ-type divergence at T_c [11], specifically $\kappa \sim |\tau|^{-\nu/2} \sim |\tau|^{-1/3}$ [27,31].

Building upon the existing theory (in our previous work [21], the theoretical value of κ is much less than the experimental data [5]), further considering the influence of particle interactions on κ will contribute to a more comprehensive understanding of its behavior in ultracold Fermi gases near and below T_c with a simple expression on κ. In this paper, we first provide a brief overview of microscopic theoretical research on thermal conductivity, and then address the above limitations in previous theories. Since we do not take the Nambu representation in the superfluid phase [26], we consider the anomalous thermal conductivity from current transfer via Cooper pairs [32]. Including the interaction term in the heat current, our computational results show that it is just the substantial contributions to thermal conductivity near T_c, and the critical universal scaling laws guarantee the divergence of κ at T_c.

The plan of the paper is as follows. In Sec. II, we briefly review the study of thermal conductivity in ultracold Fermi gases using the Kubo formula based on a current-current correlation. Then we introduced the basic physical quantities of the BCS-BEC crossover region and the calculation of the currents and vertex functions. We employ the t-matrix approximation in the pseudogap model under the $(GG_0)G_0$ scheme. We provide the Kubo expression for κ and correct it in Sec. III by incorporating heat current interactions. We show the calculation results for the complete thermal conductivity, compare them with previous calculations and recent experiments, and reveal the physical pictures of enhancing κ due to the pseudogap order near T_c. Our main findings are summarized in the concluding Sec. IV.

II. THEORY FORMULATION

A. Kubo formula

We first review the Kubo formula for thermal conductivity based on the linear response theory. The Fermi system consists of a particle current and heat current. In physics, each type of current can be defined as the gradient of a scalar field. Here, the particle current is related to the concentration, while the heat current is related to the temperature field. The generation of gradients can be thought of as the external forces applied to the system. Under the assumption of linear response, the system's response is proportional to the external driving forces, given by $\mathbf{J}_i = \sum_{j=1}^{2} L_{ij}\mathbf{X}_j$, where L_{ij} are the second-order tensor elements referred to as the response coefficients. For systems that exhibit time-reversal symmetry, the response coefficients of the normal particle currents have the Onsager relation, $L_{12} = L_{21}$. Since the experimental measurements of thermal conductivity are typically conducted under conditions where $\mathbf{J_1} = \mathbf{0}$, κ is usually defined as $\mathbf{J_2} = -\kappa\nabla T$. For the external forces, we choose the concentration gradient $\mathbf{X_1} = -\nabla\frac{\mu}{T}$ and the temperature gradient $\mathbf{X_2} = -\nabla\frac{1}{T}$, which leads to the Kubo formula

$$\kappa = \frac{1}{T^2}\left(L_{22} - \frac{L_{12}L_{21}}{L_{11}}\right). \tag{1}$$

According to the linear response theory, the response coefficients are expressed as the current-current correlation function. During the heat transfer process, the change in entropy can be viewed as a linear perturbation. We denote the energy associated with the system's response as H', such that $\frac{\partial S}{\partial t} = \sum_i \mathbf{J}_i \cdot \mathbf{X}_i$. The quantity $\hat{H}'$ can be obtained through integration as $\hat{H}'(t) = \frac{\mathrm{i}T}{\Omega}\sum_i \int \mathrm{d}\mathbf{r}\mathbf{J}_i(r)\cdot\mathbf{X}_i(r)$, with $r = (\mathbf{r}, t)$, $\mathbf{X}_i(r) = \int d\Omega\mathbf{X}(\mathbf{r}, \Omega)\mathrm{e}^{-\mathrm{i}\Omega t}\mathrm{e}^{0^+t}$, and excited spectrum Ω. The measurable current $\mathbf{J}_i(\mathbf{r}, t)$ is obtained by calculating the expectation value $\mathbf{J}_i(r) = \frac{T}{\Omega}\sum_j \int \mathrm{d}r'\Theta(t-t')\langle[\hat{\mathbf{j}}_i(r), \hat{\mathbf{j}}_j(r')]\rangle \cdot \mathbf{X}_j(r')$. The step function $\Theta(t-t')$ here ensures causality, and $\overleftrightarrow{L}_{ij}(\mathbf{r}, \mathbf{r}', \tau) = \mathrm{i}\langle\hat{T}_\tau\hat{\mathbf{j}}_i(\mathbf{r}, \tau)\hat{\mathbf{j}}_j(\mathbf{r}', 0)\rangle$ is precisely the retarded current-current correlation function at the equilibrium state for $\tau = t - t'$ with time order operator $\hat{T}_\tau$. In the Matsubara frequency space,

$$\mathbf{J}_i(\mathbf{r}, \mathrm{i}\Omega_m) = \frac{T}{\mathrm{i}\Omega_m}\sum_j \int \mathrm{d}\mathbf{q}'\langle[\hat{\mathbf{j}}_i(\mathbf{r}, \mathrm{i}\Omega_m), \hat{\mathbf{j}}_j(-\mathbf{q}', 0)]\rangle \cdot \mathbf{X}_j(\mathbf{q}', \mathrm{i}\Omega_m). \tag{2}$$

This is the general expression on the measurable current $\mathbf{J}_i(\mathbf{r}, \mathrm{i}\Omega_m)$ within the generalized linear response theory [33]. When the external field is a single-mode excitation, we take $\mathbf{X}_j(\mathbf{q}', \mathrm{i}\Omega_m) = \mathbf{X}_j(\mathbf{q}, \mathrm{i}\Omega_m)\delta_{\mathbf{q}',\mathbf{q}}$, and then have

$$\overleftrightarrow{L}_{ij}(\mathbf{q}, \mathrm{i}\Omega_m) = \langle[\hat{\mathbf{j}}_i(\mathbf{q}, \mathrm{i}\Omega_m), \hat{\mathbf{j}}_j(-\mathbf{q}, 0)]\rangle. \tag{3}$$

Here, we have employed the Matsubara formalism in momentum space for convenience in calculation. Thus the static response coefficients can be obtained by taking the limit after the analytical continuation $\mathrm{i}\Omega_m \to \Omega + \mathrm{i}0^+$:

$$L_{ij} = \lim_{\Omega\to 0}\frac{T}{\Omega}\lim_{\mathbf{q}\to\mathbf{0}}\mathrm{Im}\left(\frac{\mathbf{q}\cdot\overleftrightarrow{L}_{ij}(\mathbf{q}, \Omega)\cdot\mathbf{q}}{q^2}\right). \tag{4}$$

Substituting the response coefficients into the definition of thermal conductivity equation (1), we obtain the Kubo expression for κ in a generalized form.

B. Current operators and vertex functions

In the BCS-BEC crossover region, the ultracold Fermi gases can be described by a Hamiltonian with zero-range interactions,

$$\hat{H} = \sum_{\mathbf{k}\sigma}\xi_{\mathbf{k}}\hat{c}^\dagger_{\mathbf{k}\sigma}\hat{c}_{\mathbf{k}\sigma} + \frac{g}{2}\sum_{\mathbf{k}\mathbf{k}'\mathbf{q}\sigma\sigma'}\hat{c}^\dagger_{\mathbf{k}+\mathbf{q}\sigma}\hat{c}^\dagger_{\mathbf{k}'-\mathbf{q}\sigma'}\hat{c}_{\mathbf{k}'\sigma'}\hat{c}_{\mathbf{k}\sigma}, \tag{5}$$

where $\xi_{\mathbf{k}} = \mathbf{k}^2/2m - \mu$ is the energy calculated from the chemical potential μ, m is the atomic mass, $\hat{c}^\dagger_{\mathbf{k}\sigma}$ ($\hat{c}_{\mathbf{k}\sigma}$) are the creation (annihilation) operators for fermions, and g is the bare s-wave interaction strength related to the adjustable s-wave scattering length a_s through the regulation relation $\frac{1}{g} = \frac{m}{4\pi a_s} - \sum_{\mathbf{k}} \frac{m}{k^2}$. The dimensionless interaction strength $y = 1/k_F a_s$ vanishes in the unitary limit. The imaginary-time Green's function is defined as $G(\mathbf{k}, \tau) = -\langle \hat{T}_\tau \hat{c}_{\mathbf{k}\sigma}(\tau)\hat{c}^\dagger_{\mathbf{k}\sigma}(0)\rangle$. The Fermi current operators can be expressed as [6,27,34]

$$\mathbf{j}_1(\mathbf{q}, t) = \frac{1}{2m}\sum_{\mathbf{k}\sigma}(2\mathbf{k} + \mathbf{q})\hat{c}^\dagger_{\mathbf{k}\sigma}(t)\hat{c}_{\mathbf{k}+\mathbf{q}\sigma}(t), \tag{6}$$

$$\mathbf{j}_2(\mathbf{q}, t) = \frac{1}{2m}\sum_{\mathbf{k}\sigma}[\mathbf{k}\xi_{\mathbf{k}+\mathbf{q}} + (\mathbf{k}+\mathbf{q})\xi_{\mathbf{k}}]\hat{c}^\dagger_{\mathbf{k}\sigma}(t)\hat{c}_{\mathbf{k}+\mathbf{q}\sigma}(t) + g\sum_{\mathbf{k}\mathbf{k}'\mathbf{q}'}\frac{\mathbf{q}'}{m}\hat{c}^\dagger_{\mathbf{k}+\mathbf{q}+\mathbf{q}'\uparrow}(t)\hat{c}^\dagger_{\mathbf{k}'-\mathbf{q}'\downarrow}(t)\hat{c}_{\mathbf{k}'\downarrow}(t)\hat{c}_{\mathbf{k}\uparrow}(t). \tag{7}$$

Equation (3) involves a total of four correlation functions, each of which can be expressed as a combination of two single-particle propagator lines GG forming a bubble diagram and a full vertex function Γ_j, which can be represented as [26,35]

$$\overleftrightarrow{L}_{ij}(q) = -\sum_k \Gamma^0_i(k, k+q)G(k)G(k+q)\Gamma_j(k+q, k). \tag{8}$$

Here, the bare vertex factors are denoted as $\Gamma^0_1(k, k+q) = (2\mathbf{k} + \mathbf{q})/2m$, and $\Gamma^0_2(k, k+q) = [\mathbf{k}\xi_{\mathbf{k}+\mathbf{q}} + (\mathbf{k}+\mathbf{q})\xi_{\mathbf{k}}]/2m$, while the dressed vertex function can be expressed through an integral equation as [26,35,36]

$$\Gamma_i(k+q, k) = \Gamma^0_i(k+q, k) + \sum_p \Lambda(k, q, p) \times G(p-k)G(p-k-q) \times \Gamma_i(p-k-q, p-k). \tag{9}$$

$\Lambda(k, q, p)$ represents all irreducible representations of the interaction between two Green's functions. For the short notations, $q = (\mathbf{q}, \mathrm{i}\Omega_n)$ and $p = (\mathbf{p}, \mathrm{i}\nu_n)$ are Bose indices, while $k = (\mathbf{k}, \mathrm{i}\omega_n)$ is a Fermi index. The summation includes both frequency summation and integration over three-dimensional momenta, i.e., $\sum_p = T\sum_{\mathrm{i}\nu_n}\int\frac{\mathrm{d}\mathbf{p}}{(2\pi)^3}$, and so on.

C. t-matrix approximation

Past work in the literature has addressed the main features of the t-matrix approximation in the broken-symmetry phase. The $(GG_0)G_0$ scheme [19] is one of the best schemes in the calculation of the fundamental physical quantities in the BCS-BEC crossover region of the ultracold Fermi superfluids. In this scheme, there exists an approximate relationship [16] given by $\Lambda(k, q, p) \approx t(p)$, where the t matrix is defined as $\frac{1}{t(q)} = \frac{1}{g} + \sum_k G(k)G_0(q-k)$, and the self-energy is $\Sigma(k) = \sum_q t(q)G_0(q-k)$. The bare propagator is given as $G_0(k) = 1/(\xi_{\mathbf{k}} - \mathrm{i}\omega_n)$. The full Green's function satisfies the Dyson equation $G^{-1}(k) = G_0^{-1}(k) - \Sigma(k)$. This is a closed form of the fundamental equations.

For the modified vertex Γ_i in Eqs. (8) and (9), it exhibits five dominant contributions within the $(GG_0)G_0$ scheme, which can be seen in Fig. 1 (these Feynman diagrams are visually depicted in Fig. 1 of Ref. [22]).

These contributions are the vertex function expressed by [22,23]

$$\Gamma_i = \Gamma^0_i + \Gamma^{\mathrm{MT}}_i + \Gamma^{\mathrm{AL1}}_i + \Gamma^{\mathrm{AL2}}_i + \Gamma^{\mathrm{Int}}_i. \tag{10}$$

Here the so-called Maki-Thompson (MT) contribution and two versions of the Aslamazov-Larkin (AL) contributions have been presented as the specific expressions,

$$\Gamma^{\mathrm{MT}}_i(k+q, k) = \sum_p t(p)G_0(p-k)G_0(p-k-q) \times \Gamma_i(p-k, p-k-q), \tag{11}$$

$$\Gamma^{\mathrm{AL1}}_i(k+q, k) = -\sum_{p,l} G_0(l+q)G(p-l)t(p+l) \times G_0(p-k)t(p)G_0(l)\Gamma^0_i(l+q, l), \tag{12}$$

$$\Gamma^{\mathrm{AL2}}_i(k+q, k) = -\sum_{p,l} G(l+q)G_0(l)t(p-l) \times G_0(p-k)t(p)G(l)\Gamma_i(l+q, l). \tag{13}$$

The interaction-induced Γ^{Int}_i is related to (25) below. Based on the t-matrix approximation, $\Gamma_i = \Gamma_{i(\mathrm{sc})} + \Gamma_{i(\mathrm{pg})}$, decomposing as superfluid and pseudogap components. The many-body vertex Γ^{AL}_i calculation is very hard. Fortunately, at the $q \to 0$ limit (within the linear response theory, we address in this case), one has [23]

$$\Gamma_i = \Gamma^0_i - \Gamma^{\mathrm{MT}}_i + \Gamma^{\mathrm{Int}}_i. \tag{14}$$

For convenience, we note the single-particle energies of bosons and fermions $\varepsilon_b = \frac{\mathbf{p}^2}{2M^*} - \mu_{\mathrm{pair}}$, $\varepsilon_f = \frac{(\mathbf{p}-\mathbf{k})^2}{2m} - \mu$, and the difference $\delta = \varepsilon_b - \varepsilon_f$. $n_{b(f)}$ is the Bose (Fermi) distribution function. In the pseudogap model [37], the energy gap Δ can be also divided into two parts, $\Delta^2 = \Delta^2_{\mathrm{sc}} + \Delta^2_{\mathrm{pg}}$. Then $t(q) = -\Delta^2_{\mathrm{sc}}\delta(q)/T + t_{\mathrm{pg}}(q)$ with $t^{-1}_{\mathrm{pg}}(q) = Z(\Omega - \mathbf{q}^2/2M^* + \mu_{\mathrm{pair}})$ in the small q approximation. The physical quantities appearing here, namely, μ_{pair}, Z, and M^*, can all be obtained in Ref. [37].

D. κ expression

In Ref. [21], it is mentioned that by considering only the contributions from the first four terms of Eq. (10), an expression for the static coefficient $L^{(0)}_{ij}$ in the absence of interactions in $q \to 0$ can be obtained as

$$L^{(0)}_{ij} = \frac{T}{3\pi^2 m^2}\int_0^\infty \mathrm{d}k\mathbf{k}^4\xi^{i+j-2}_{\mathbf{k}}\int_{-\infty}^\infty \frac{\mathrm{d}\varepsilon}{4\pi}\left(-\frac{\partial n_f(\varepsilon)}{\partial\varepsilon}\right) \times \left[A^2(\mathbf{k}, \varepsilon) + B^2_{\mathrm{sc}}(\mathbf{k}, \varepsilon) - B^2_{\mathrm{pg}}(\mathbf{k}, \varepsilon)\right]. \tag{15}$$

Here, the spectral functions $A(\mathbf{k}, \varepsilon) = -2\,\mathrm{Im}\,G(\mathbf{k}, \varepsilon)$ and $B_{\mathrm{sc(pg)}}(\mathbf{k}, \varepsilon) = -2\,\mathrm{Im}\,F_{\mathrm{sc(pg)}}(\mathbf{k}, \varepsilon)$ are respectively for the normal and anomalous Green's functions. The former is

$$G(\mathbf{k}, \omega) = \left(\omega - \xi_{\mathbf{k}} + \mathrm{i}\gamma - \frac{\Delta^2_{\mathrm{pg}}}{\omega + \xi_{\mathbf{k}} + \mathrm{i}\gamma} - \frac{\Delta^2_{\mathrm{sc}}}{\omega + \xi_{\mathbf{k}}}\right)^{-1}, \tag{16}$$

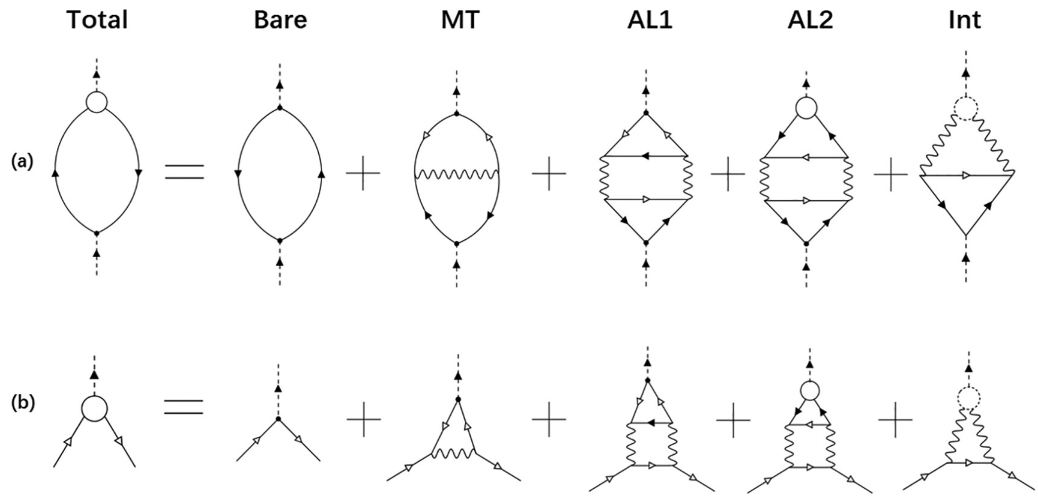

FIG. 1. (a) The polarization bubble, and (b) the schematic diagram of the vertex function used to calculate the response function. The wavy line represents the t matrix, the open (solid) arrow represents the bare (dressed) Green's function, the dashed line represents the external stress tensor field and the small black dot represents the zeroth-order bare vertex Γ^0, the solid open circle represents the full vertex Γ and the dashed open circle represents the first-order bare vertex Γ^{Int}.

and the latter are

$$F_{\text{sc}}(\mathbf{k},\omega)=\frac{-\Delta_{\text{sc}}}{\omega+\xi_{\mathbf{k}}+\mathrm{i}0^+}\frac{1}{\omega-\xi_{\mathbf{k}}-\frac{\Delta^2}{\omega+\xi_{\mathbf{k}}}+\mathrm{i}0^+},\tag{17}$$

$$F_{\text{pg}}(\mathbf{k},\omega)=\frac{-\Delta_{\text{pg}}}{\omega+\xi_{\mathbf{k}}+\mathrm{i}\gamma}G(\mathbf{k},\omega),\tag{18}$$

the same as those of the pseudogap model and some high T_c literature [38,39]. It includes $\gamma=1/2\tau$ the damping ratio and τ the relaxation time. Opening all channels of particle-particle collisions [i.e., (n_f,n_f), (n_f,n_b), (n_f,n_{b_0}), and (n_b,n_{b_0}) scattering progress], we have calculated τ as a function of $(T/T_c,1/k_Fa_s)$ based on three-particle hydrodynamics [40], where n_{b_0} is the density of condensed pairs. The numerical results show that $\gamma/T_F=\alpha' T/T_c$ is correct near T_c for unitary Fermi gases but with $\alpha'\simeq 0.48$ (see Fig. 3 in Ref. [40]). The fermion-boson scattering channel dominates in all channels, resulting in the departure from Fermi-liquid theory and a similar temperature dependence with the Planck time τ_P. Across a wide range of strongly correlated systems, the phenomenological dimensionless constant is $0.45<\alpha'<1.1$ in the anomalous transport regimes [41], which shows the dramatic universality of the unitary Fermi gases due to the pseudogap effects.

The blue line in Fig. 1 shows κ vs T/T_c without interactions in heat current operators [21]. The numerical results show that $\kappa^{(0)}\simeq L_{22}^{(0)}/T^2$, and that the normal spectral function $A(\mathbf{k},\varepsilon)=-2\,\text{Im}\,G(\mathbf{k},\varepsilon)$ is the main contribution to $\kappa^{(0)}\propto[-2\,\text{Im}\,G(\mathbf{k},\varepsilon)]^2$ of the integrable function. Since the pseudogap order (fluctuations) has modified $G(\mathbf{k},\varepsilon)$ both by γ and by Δ_{pg}, we need to check which one is the core contribution to $\kappa^{(0)}$. In any case, the blue line of $\kappa^{(0)}$ in Fig. 1 is much less than the experimental data [5] (red circles) for fixed γ. We need to modify Δ_{pg} and consider the anomalous thermal conductivity.

For the theory completeness, we now calculate the anomalous thermal conductivity using the Kubo formula under the t-matrix approximation by considering the Cooper pair current operators [32] as $\mathbf{j}'_1(\mathbf{q},t)=\frac{1}{2m}\sum_{\mathbf{k}}[(\mathbf{q}-\mathbf{k})\hat{c}_{\mathbf{k}\downarrow}(t)\hat{c}_{\mathbf{q}-\mathbf{k}\uparrow}(t)-\mathbf{k}\hat{c}^\dagger_{\mathbf{k}\uparrow}(t)\hat{c}^\dagger_{-\mathbf{q}-\mathbf{k}\downarrow}(t)]$, and their heat current operators as $\mathbf{j}'_2(\mathbf{q},t)=\frac{1}{2m}\sum_{\mathbf{k}}[-\mathbf{k}\xi_{\mathbf{q}-\mathbf{k}}\hat{c}_{\mathbf{k}\downarrow}(t)\hat{c}_{\mathbf{q}-\mathbf{k}\uparrow}(t)+(\mathbf{k}+\mathbf{q})\xi_{\mathbf{k}}\hat{c}^\dagger_{\mathbf{k}\uparrow}(t)\hat{c}^\dagger_{-\mathbf{q}-\mathbf{k}\downarrow}(t)]$. Letting $k'=k+q$ for short notations, the corresponding correlation functions are

$$\overleftrightarrow{L}'_{11}(q)=-\frac{1}{4m^2}\sum_k[\mathbf{k}'\mathbf{k}'\bar{G}(k)G(k')+\mathbf{kk}G(k)\bar{G}(k')],\tag{19}$$

$$\overleftrightarrow{L}'_{22}(q)=\frac{1}{4m^2}\sum_k\big[\mathbf{kk}\xi^2_{\mathbf{k}'}\bar{G}(k)G(k')+\mathbf{k}'\mathbf{k}'\xi^2_{\mathbf{k}}G(k)\bar{G}(k')-2\mathbf{kk}'\xi_{\mathbf{k}}\xi_{\mathbf{k}'}F(k)F(k')\big].\tag{20}$$

Especially, in the symmetry-broken superfluid phase, the antisymmetry correlation functions are

$$\begin{aligned}\overleftrightarrow{L}'_{12}(q)&=-\overleftrightarrow{L}'_{21}(q)\\&=\frac{1}{4m^2}\sum_k\{[\mathbf{kk}\xi_{\mathbf{k}'}-\mathbf{k}'\mathbf{k}'\xi_{\mathbf{k}}]F(k)F(k')\\&\quad+\mathbf{kk}'[\xi_{\mathbf{k}'}\bar{G}(k)G(k')-\xi_{\mathbf{k}}G(k)\bar{G}(k')]\}.\end{aligned}\tag{21}$$

At the mean-field approximation for the interested limit of $q\to 0$, we find $\overleftrightarrow{L}'_{11}(0)=-\frac{1}{2m^2}\sum_k\mathbf{kk}G(k)\bar{G}(k)\neq 0$ due to the Cooper pair current, and $\overleftrightarrow{L}'_{12}(q)=-\overleftrightarrow{L}'_{21}(q)=0$ indeed. But $\overleftrightarrow{L}'_{22}(0)=\frac{1}{2m^2}\sum_k\mathbf{kk}\xi^2_{\mathbf{k}}[G(k)\bar{G}(k)-F(k)F(k)]$ does not seem to vanish. In fact, $\text{Im}\sum_n TG(-k)G(k')=\int_{-\infty}^{\infty}\frac{d\varepsilon}{2}[-\frac{\partial n_F(\epsilon)}{\partial\epsilon}]A(-\mathbf{k},-\varepsilon)A(\mathbf{k}',\varepsilon+\Omega)$ leads to $\kappa'_{\text{MF}}=\frac{L'_{22}}{T^2}\propto\int_{-\infty}^{\infty}\frac{d\varepsilon}{2\pi}[B^2(\mathbf{k},\varepsilon)-A(-\mathbf{k},-\varepsilon)A(\mathbf{k},\varepsilon)]=0$ in the case of $q\to 0$. Therefore according to the mean-field calculations, the Cooper pairs do not contribute to the thermal conductivity physically. We need to go beyond the mean-field approximation for the heat current of the Cooper pairs.

Within the t-matrix approximation, $\overleftrightarrow{L}'_{22}(q)$ becomes

$$\begin{aligned}\overleftrightarrow{L}'_{22}(q) &= -\sum_k \Gamma_2^0(k,k')\Gamma_2(k',k)F(k)F(k')\\ &= -\sum_k \Gamma_2^0(k,k')\Gamma_2^0(k',k)[F(k)F(k')\\ &\quad - F_{\mathrm{sc}}(k)F_{\mathrm{sc}}(k') + F_{\mathrm{pg}}(k)F_{\mathrm{pg}}(k')].\end{aligned} \tag{22}$$

Based on the above mean-field approximation calculation, it simplifies to

$$\overleftrightarrow{L}'_{22}(q) = -2\sum_k \Gamma_2^0(k,k')\Gamma_2^0(k',k)F_{\mathrm{pg}}(k)F_{\mathrm{pg}}(k'). \tag{23}$$

Therefore the quantum fluctuation in the thermal conductivity is expressed on

$$\kappa' = \frac{1}{3\pi^2 m^2 T}\int_0^\infty \mathrm{d}k\mathbf{k}^4\xi_{\mathbf{k}}^2 \int_{-\infty}^{\infty}\frac{\mathrm{d}\varepsilon}{4\pi}\left[-\frac{\partial n_F(\epsilon)}{\partial \epsilon}\right]B_{\mathrm{pg}}^2(\mathbf{k},\epsilon). \tag{24}$$

The green line in Fig. 1 shows this result. Although there is a small hump near T_c, the value is less than the one with the blue line. Therefore we need to consider the interactions within the heat current operator.

III. THE INTERACTION-INDUCED THERMAL CONDUCTIVITY AND NUMERICAL RESULTS

In the aforementioned calculations, the interaction terms of Fermi particles within the heat current operator, i.e., Γ_j^{Int} in the Feynman diagrams (the superscript Int represents the interaction), were not included. This simplification renders the conservation laws invalid. On the other hand, in the vicinity of the unitary limit, both bosonic and fermionic excitations (corresponding to the vertex function and normal Green's function) are equally important. Therefore, a proper description of thermal conductivity must encompass the effects of the interaction terms. These interaction terms within the heat current include four fermionic operators, and transforming them into two bosonic operators yields contributions from bosonic excitations. Hence, it is appropriate to say that the interaction terms involve contributions from bosons. Regarding Γ_j^{Int} in (10), its second-order tensor form is discussed in Ref. [22], which is given as

$$\Gamma_{\mathrm{Int}}^{ij}(k+q,k) = \frac{\delta^{ij}}{g}\sum_p t_{\mathrm{pg}}(p+q)t_{\mathrm{pg}}(p)G_0(p-k) \tag{25}$$

[see Eq. (49) in Ref. [22]]. For the vertex function in thermal conductivity calculations, in comparison to Eq. (8), we provide the following expression,

$$\begin{aligned}\overleftrightarrow{L}^{\mathrm{Int}}_{ij}(q) = &-\sum_k \Gamma_i^0(k,k+q)\Gamma_j^0(k+q,k)G(k)\\ &\times G(k+q)\frac{\delta_{ij}}{gT}\sum_p t(p)t(p+q)G_0(p-k).\end{aligned} \tag{26}$$

This is the interaction-dependent term within the heat current operator.

To carry out the coefficient calculation, we denote $\mathcal{L} = -T\sum_{\mathrm{i}\omega_n} G(k)G(k+q)T\sum_{\mathrm{i}\nu_n} t(p)t(p+q)G_0(p-k)$. First, we perform two frequency summations on $\mathcal{L}$ and then take the limit of $\mathbf{q}$ approaching 0,

$$\begin{aligned}\lim_{\mathbf{q}\to\mathbf{0}} \mathrm{Im}\,\mathcal{L} = &\frac{n_b(\varepsilon_b)+n_f(\varepsilon_f)}{4\pi Z^2}\int_{-\infty}^{\infty}\mathrm{d}\varepsilon A(\mathbf{k},\varepsilon)\\ &\times\bigg[A(\mathbf{k},\varepsilon+\Omega)\frac{n_f(\varepsilon)-n_f(\varepsilon+\Omega)}{(\varepsilon-\delta)(\varepsilon-\delta+\Omega)}\\ &+A(\mathbf{k},\delta+\Omega)\frac{n_f(\delta+\Omega)-n_f(\delta)}{\Omega(\varepsilon-\delta)}\\ &+A(\mathbf{k},\delta-\Omega)\frac{n_f(\delta-\Omega)-n_f(\delta)}{\Omega(\varepsilon-\delta)}\bigg].\end{aligned} \tag{27}$$

Next, dividing Ω and letting $\Omega\to 0$, one has

$$\mathcal{L}_0 = \lim_{\Omega\to 0}\frac{1}{\Omega}\lim_{\mathbf{q}\to\mathbf{0}}\mathrm{Im}\,\mathcal{L} = \frac{n_b(\varepsilon_b)+n_f(\varepsilon_f)}{4\pi Z^2}(I_1+2I_2+2I_3), \tag{28}$$

with three principal value integrals

$$\begin{aligned}I_1 &= \lim_{\Omega\to 0}\int_{-\infty}^{\infty}\mathrm{d}\varepsilon\frac{A\big(\mathbf{k},\varepsilon-\frac{1}{2}\Omega\big)A\big(\mathbf{k},\varepsilon+\frac{1}{2}\Omega\big)}{\big(\varepsilon-\delta-\frac{1}{2}\Omega\big)\big(\varepsilon-\delta+\frac{1}{2}\Omega\big)}\frac{-\partial n_f(\varepsilon)}{\partial\varepsilon}\\ &= \lim_{\Omega\to 0}\frac{1}{\Omega}\int_{-\infty}^{\infty}\mathrm{d}\varepsilon\left(-\frac{\partial n_f(\varepsilon)}{\partial\varepsilon}\right)A\left(\mathbf{k},\varepsilon-\frac{1}{2}\Omega\right)\\ &\quad\times A\left(\mathbf{k},\varepsilon+\frac{1}{2}\Omega\right)\left(\frac{1}{\varepsilon-\delta-\frac{1}{2}\Omega}-\frac{1}{\varepsilon-\delta+\frac{1}{2}\Omega}\right),\end{aligned} \tag{29}$$

$$I_2 = \int_{-\infty}^{\infty}\mathrm{d}\varepsilon A(\mathbf{k},\delta)\frac{A(\mathbf{k},\varepsilon)}{\varepsilon-\delta}\frac{\partial^2 n_f(\delta)}{\partial\delta^2}, \tag{30}$$

$$I_3 = \int_{-\infty}^{\infty}\mathrm{d}\varepsilon\frac{A(\mathbf{k},\varepsilon)}{\varepsilon-\delta}\frac{\partial A(\mathbf{k},\delta)}{\partial\delta}\frac{\partial n_f(\delta)}{\partial\delta}. \tag{31}$$

We observe that for a small enough but fixed value Ω, the integrals in the above equations are finite through the principal value integrals. As Ω approaches zero, the integrals I_1, I_2, and I_3 converge. Comparing $\mathcal{L}_0$ with Eqs. (4) and (26), we derive the specific expression for the response coefficients as follows,

$$L_{ij}^{\mathrm{Int}} = \delta_{ij}\int\frac{\mathrm{d}\mathbf{p}}{(2\pi)^3}\int\frac{\mathrm{d}\mathbf{k}}{(2\pi)^3}\frac{k^2\cos^2\theta\,\xi_k^{i+j-2}}{gm^2}\mathcal{L}_0. \tag{32}$$

In the above equation, θ represents the angle between $\mathbf{p}$ and $\mathbf{k}$. This expression involves a sixfold integral, and we employ the Monte Carlo algorithm Vegas, developed based on the adaptive stratified sampling method, to perform the computation [42]. Substituting $L_{ij} = L_{ij}^{(0)} + L_{ij}^{\mathrm{Int}}$ into Eq. (1), one finally obtains the complete expression of the thermal conductivity after including the interaction term in the heat current as

$$\kappa = \frac{1}{T^2}\left[L_{22}^{(0)} + L_{22}^{\mathrm{Int}} - \frac{L_{12}^{(0)}L_{21}^{(0)}}{L_{11}^{(0)}+L_{11}^{\mathrm{Int}}}\right]. \tag{33}$$

Using the above expression combined with the critical scaling laws, Fig. 2 displays the thermal conductivity calculated at a strong interaction strength of $y = 0.01$. The red data points correspond to the experimental data for the thermal

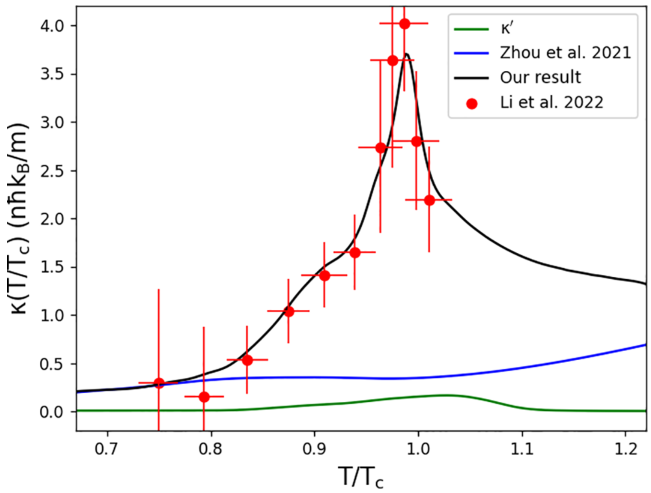

FIG. 2. Thermal conductivity κ in units of $n\hbar k_B/m$ vs dimensionless reduced temperature T/T_c. The solid black line represents our theoretical results, and the red circles denote the experimental data [5]. The blue solid line represents the calculation results without accounting interactions in heat current [21]. The green solid line represents the calculation results contributed from the Cooper pairs.

conductivity at the unitary limit [5]. Our numerical results show that $\kappa \simeq L_{22}/T^2 = [L_{22}^{(0)} + L_{22}^{\rm Int}]/T^2$, and that the normal spectral function $A(\mathbf{k}, \varepsilon) = -2\,{\rm Im}\,G(\mathbf{k}, \varepsilon)$ is the main contribution to κ. Since the pseudogap order (fluctuations) modifies $G(\mathbf{k}, \varepsilon)$, its $\Delta_{\rm pg}$ both in $L_{22}^{(0)}$ and primarily in $L_{22}^{\rm Int}$ is the core contribution to κ for the fixed value of $\alpha' = 0.48$.

IV. REMARKS AND CONCLUSION

It is worthy to remark the following: (1) The physics picture of the fluctuations in the t-matrix paradigm is characterized by the pseudogap order. The interaction-induced pseudogap order combined with the critical scaling laws leads to κ increasing sharply. (2) Our results align well with the recent experiment [5] near T_c with the experimental value of $T_c \approx 0.167T_F$ [5,43]. (3) The contribution of the correction term to the thermal conductivity reaches its maximum at T_c, and as the temperature decreases, the contribution of the correction term continuously diminishes. (4) At T_c, critical fluctuations are expected to lead to a divergence of κ [44], obeying the critical universal scaling laws. (5) Around $0.9T_c$, κ displays behavior approaching the quantum limit value; below approximately $0.8T_c$, the result yields very small values as reported in Ref. [24], above T_c, the contribution of the correction term gradually decreases with increasing temperature, reducing to zero at around $1.4T_c$. (6) Starting from a microscopic theory, the computed thermal conductivity exhibits a rapid decrease below T_c. The significant difference in κ on both sides of the phase transition temperature is also mentioned in Ref. [11]. Below approximately $0.8T_c$, both Ref. [21] without the interaction and this work with the interaction yields very small values for a small enough pseudogap order.

In summary, we derive the Kubo-based expression for the thermal conductivity of a unitary Fermi gas by taking into account the interactions in the heat current. Our results demonstrate that the interaction-induced pseudogap order combined with the critical universal scaling laws lead to the thermal conductivity increasing sharply near the phase transition temperature (although the thermal conductivity is also sensitive to the damping ratio, it is not an adjusted parameter).

ACKNOWLEDGMENT

This work is supported by the State Key Programs of China under Grant No. 2017YFA0304204 and the National Natural Science Foundation of China under Grant No. 12175040.

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