Editors' Suggestion

Formation of nonclassical and non-Gaussian states of a strong electromagnetic field due to its interaction with free electrons produced by ionization of a target gas

Evgeny S. Andrianov[1,2] and Oleg I. Tolstikhin[1]
[1]Moscow Institute of Physics and Technology, Dolgoprudny 141700, Russia
[2]Dukhov Research Institute of Automatics (VNIIA), 22 Sushchevskaya, Moscow 127055, Russia

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We consider an exactly solvable model describing the interaction of a quantized strong laser field with free electrons produced in the ionization of a target gas by the field. The interaction is shown to strongly affect the quantum state of the field. It squeezes and displaces the coherent state of a free laser field, which under realistic experimental conditions can result in the formation of nonclassical and non-Gaussian field states with a ring-shaped Wigner function. Our results demonstrate that the effect of the interaction with free electrons on the quantum state of the ionizing laser field should be taken into account simultaneously with that of harmonic generation and above-threshold ionization studied recently.

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I. INTRODUCTION

In studies of the interaction of strong infrared laser fields with matter, which led to the emergence of attosecond physics [1], the field is usually treated classically [2]. This is justified by the high photon numbers characterizing coherent states representing such fields in quantum mechanics [3]. However, recently it has been recognized theoretically and demonstrated experimentally that generation of high-order harmonics (HHG) occurring when a strong laser pulse interacts with a target gas affects the quantum state of the driving field [4]. The very fact that quantum nature of the strong field reveals itself in field observables and can be detected experimentally was a surprise. We mention that the field of emitted harmonics was treated quantum mechanically in previous studies [5–8], but the laser field was generally considered classical, with a few exceptions [9–11], since no quantum effects associated with it were expected. The pioneering paper [4] promoted the development of quantum-optical perspective in strong-field physics. This work has been elaborated [12] and extended theoretically [13] to include the effect of another fundamental strong-field process—above-threshold ionization (ATI)—on the quantum state of the laser field. It should be noted that the observation of nonclassical states of a strong laser field in Refs. [4,12] became possible due to new experimental techniques [14,15] introduced earlier to test the prediction that HHG affects infrared photon counting [16] and implementing so-called conditioning measurements [17]. The quantum-optical paradigm of strong-field processes summarizing these results is presented in reviews [18,19]. Other related recent theoretical developments include studies of photon statistics resulting from the interaction of a strong quantized field with a gas of two-level atoms [20], the effect of correlations between atoms in the gas phase [21] and electrons in the condensed phase [22] on the quantum state of the HHG field, an effective photon-statistics force induced by quantum fluctuations of the laser field [23], and squeezing of the laser mode caused by the HHG process [24].

In this paper we investigate how the interaction with free electrons produced by strong-field ionization of the target gas affects the quantum state of the ionizing laser field. Note that free electrons necessarily appear under conditions where HHG and ATI processes occur, so their effect on the evolution of the laser field should be taken into account simultaneously with that of HHG and ATI studied in Refs. [4,12,13,24]. However, as far as we know, until now this effect has not been investigated. We consider a model in which a quantized electromagnetic field interacts with free electrons. This model admits an exact treatment, which allows us to analyze the effect of the electron-field interaction on the field state.

The paper is organized as follows. In Sec. II we introduce our model. In Sec. III we exactly diagonalize the model Hamiltonian and obtain its eigenvalues and eigenstates which fully incorporate the effect of the electron-field interaction. In Sec. IV we consider the dynamics of the system and show how to average field observables over the momentum distribution of electrons. In Sec. V we discuss two field observables which demonstrate nonclassicality and non-Gaussianity of the field state. Section VI concludes the paper. Some formulas from quantum optics used in the main text are summarized in Appendix A. The gauge transformation of photoelectron momentum distributions and a specific distribution resulting from strong-field ionization are discussed in Appendixes B and C, respectively.

II. MODEL HAMILTONIAN

As a strong laser pulse propagates through a target gas, ionization of gas atoms occurs. This results in the appearance of free electrons. We are interested in the effect that these electrons produce on the quantum state of the laser field. To analyze the effect, we consider a system of N free electrons interacting with a single mode of a quantized electromagnetic field representing the laser field. The Hamiltonian of the system is

$$\hat{H} = \frac{1}{2m}\sum_{i=1}^{N}\left[\hat{\mathbf{p}}_i + \frac{e}{c}\hat{\mathbf{A}}(\mathbf{r}_i)\right]^2 + \hbar\omega(\hat{a}^\dagger\hat{a} + 1/2), \quad (1)$$

where

$$\hat{\mathbf{A}}(\mathbf{r}) = A_0\mathbf{e}_z(\hat{a}e^{i\mathbf{k}\cdot\mathbf{r}} + \hat{a}^\dagger e^{-i\mathbf{k}\cdot\mathbf{r}}) \tag{2}$$

is the vector potential of the field and $A_0 = \sqrt{2\pi\hbar c^2/\omega V}$. Here $e > 0$ is the absolute value of the electron charge, $\omega = ck$ and $\mathbf{k} = k\mathbf{e}_x$ are the frequency and wave vector of the laser mode, $\hat{a}^\dagger$ and $\hat{a}$ are the creation and annihilation operators for a photon in this mode obeying the commutation relation $[\hat{a}, \hat{a}^\dagger] = 1$, and V is the quantization volume. The laser pulse is assumed to propagate along the x axis and be linearly polarized along the z axis, with $\mathbf{e}_x$ and $\mathbf{e}_z$ denoting unit vectors in these directions.

The system described by Eq. (1) is still too complex to allow an analytical treatment. We further simplify it by adopting the dipole approximation which holds for low-frequency laser fields with a typical wavelength of $\lambda \approx 800$ nm used in strong-field physics [2]. In this approximation the exponents in Eq. (2) can be replaced by unity. Such an approach was used in Refs. [4,12,13]. However, while the wavelength is certainly larger than the size of an atom, it is usually smaller than the extent of the region where the laser-gas interaction takes place. To account for this fact, we assign to each electron a phase determined by its location in the interaction region, that is, we substitute

$$\hat{\mathbf{A}}(\mathbf{r}_i) \to A_0\mathbf{e}_z(e^{i\phi_i}\hat{a} + e^{-i\phi_i}\hat{a}^\dagger). \tag{3}$$

In the context of strong-field physics this substitution can be justified as follows. As a result of tunneling ionization an electron is liberated with zero initial momentum. Its subsequent motion is driven by the laser field and hence proceeds along the polarization z axis, while its position in the transverse to this axis direction remains unchanged. These assumptions are consistent with the three-step model [25]. We have $\mathbf{kr}_i = kx_i$, where x_i is the x component of the coordinate $\mathbf{r}_i$ of the ith electron. Replacing x_i with its value at the moment of ionization, that is, with the x component X_i of the coordinate $\mathbf{R}_i$ of the parent ion, we obtain Eq. (3) with $\phi_i = kX_i$. Substituting Eq. (3) into Eq. (1) gives

$$\hat{H} = \sum_{i=1}^{N}\frac{\hat{\mathbf{p}}_i^2}{2m} + \frac{eA_0}{mc}\sum_{i=1}^{N}(e^{i\phi_i}\hat{a} + e^{-i\phi_i}\hat{a}^\dagger)\hat{p}_{iz} + \frac{\hbar\omega}{2}[(1+\xi)(\hat{a}^\dagger\hat{a} + \hat{a}\hat{a}^\dagger) + \xi(\eta\hat{a}\hat{a} + \eta^*\hat{a}^\dagger\hat{a}^\dagger)]. \tag{4}$$

Here

$$\xi = \frac{Ne^2A_0^2}{mc^2\hbar\omega} = \frac{\omega_p^2}{2\omega^2}, \tag{5}$$

where

$$\omega_p^2 = \frac{4\pi n_e e^2}{m} \tag{6}$$

is the plasma frequency squared for a given density $n_e = N/V$ of electrons, and

$$\eta = \frac{1}{N}\sum_{i=1}^{N} e^{2i\phi_i} = |\eta|e^{i\phi_\eta}. \tag{7}$$

The Hamiltonian (4) defines our model treated in subsequent sections. The model is characterized by one real ξ and one generally complex η dimensionless parameter. Note that the electron-field interaction is represented in Eq. (4) by terms containing the electron charge e, i.e., by the second term in the first line and terms $\propto \xi$ in the second line.

III. DIAGONALIZATION OF THE MODEL HAMILTONIAN

In this section we find eigenvalues and eigenstates of the Hamiltonian (4). The derivation proceeds in two steps which clarify the meaning of the two transformations involved.

Let us introduce shorthand notation

$$|\mathbf{P}\rangle = \prod_{i=1}^{N}|\mathbf{p}_i\rangle, \quad \mathbf{P} = (\mathbf{p}_1, \dots, \mathbf{p}_N). \tag{8}$$

Here $|\mathbf{p}_i\rangle$ are eigenstates of $\hat{\mathbf{p}}_i$ satisfying $\hat{\mathbf{p}}_i|\mathbf{p}_i\rangle = \mathbf{p}_i|\mathbf{p}_i\rangle$ and normalized by $\langle\mathbf{p}_i|\mathbf{p}_i'\rangle = (2\pi\hbar)^3\delta(\mathbf{p}_i - \mathbf{p}_i')$, and $\mathbf{P}$ is a $3N$-dimensional vector composed of $\mathbf{p}_i$. The Hamiltonian (4) can be presented in the form

$$\hat{H} = \int \hat{H}(\mathbf{P})|\mathbf{P}\rangle\langle\mathbf{P}|\frac{d\mathbf{P}}{(2\pi\hbar)^{3N}}, \tag{9}$$

where $\hat{H}(\mathbf{P})$ is the momentum representation of $\hat{H}$ obtained from Eq. (4) by substituting $\hat{\mathbf{p}}_i \to \mathbf{p}_i$. This operator acts only on the field degrees of freedom and depends on $\mathbf{P}$ as a parameter. It is more convenient to deal with when discussing the transformations.

A. Squeezing

We first diagonalize the quadratic in $\hat{a}$ and $\hat{a}^\dagger$ part of $\hat{H}(\mathbf{P})$ given by the second line in Eq. (4). This can be done using the Bogoliubov transformation [26]. Let us introduce new operators

$$\hat{b} = u^*\hat{a} + v^*\hat{a}^\dagger, \quad \hat{b}^\dagger = u\hat{a}^\dagger + v\hat{a}. \tag{10}$$

To preserve the commutation relation $[\hat{b}, \hat{b}^\dagger] = 1$, the parameters u and v of this transformation must satisfy

$$|u|^2 - |v|^2 = 1. \tag{11}$$

Under this condition the inverse transformation is given by

$$\hat{a} = u\hat{b} - v^*\hat{b}^\dagger, \quad \hat{a}^\dagger = u^*\hat{b}^\dagger - v\hat{b}. \tag{12}$$

We choose u to be real. To satisfy Eq. (11), we seek u and v in the form

$$u = \cosh\nu, \quad v = e^{i\phi_\eta}\sinh\nu, \tag{13}$$

where ϕ_η is defined by Eq. (7). Substituting Eqs. (12) into Eq. (4) and requiring that the coefficients of $\hat{b}\hat{b}$ and $\hat{b}^\dagger\hat{b}^\dagger$ vanish, we find

$$\nu = \frac{1}{2}\tanh^{-1}\frac{\xi|\eta|}{1+\xi}. \tag{14}$$

The operator $\hat{H}(\mathbf{P})$ expressed in terms of $\hat{b}$ and $\hat{b}^\dagger$ takes the form

$$\hat{H}(\mathbf{P}) = \frac{\mathbf{P}^2}{2m} + \hbar\Omega(\beta^*\hat{b} + \beta\hat{b}^\dagger) + \hbar\Omega(\hat{b}^\dagger\hat{b} + 1/2), \tag{15}$$

where

$$\beta = \frac{eA_0}{mc\hbar\Omega}\sum_{i=1}^{N}(e^{-i\phi_i}u - e^{i\phi_i}\upsilon)p_{iz} \tag{16}$$

is a dimensionless quantity and

$$\Omega = \omega\sqrt{1 + 2\xi + \xi^2(1 - |\eta|^2)} \tag{17}$$

is a modified frequency of the laser mode which incorporates the effect of the electron-field interaction. Note that β depends on $\mathbf{P}$, but we omit this dependence in the notation.

The Bogoliubov transformation (10) with the parameters given by Eqs. (13) and (14) can be presented in the form

$$\hat{b} = \hat{S}(\zeta_0)\hat{a}\hat{S}^\dagger(\zeta_0), \quad \hat{b}^\dagger = \hat{S}(\zeta_0)\hat{a}^\dagger\hat{S}^\dagger(\zeta_0), \tag{18}$$

where

$$\hat{S}(\zeta) = \exp\left(-\frac{\zeta}{2}\hat{a}^\dagger\hat{a}^\dagger + \frac{\zeta^*}{2}\hat{a}\hat{a}\right) \tag{19}$$

is a unitary squeezing operator [27] and

$$\zeta_0 = e^{-i\phi_\eta}\nu. \tag{20}$$

Let $|n\rangle$, $n = 0, 1, \dots,$ denote orthonormal n-photon Fock states of the free field in the laser mode satisfying

$$\hat{a}^\dagger\hat{a}|n\rangle = n|n\rangle. \tag{21}$$

The corresponding squeezed states are defined by

$$|n\rangle_b = \hat{S}(\zeta_0)|n\rangle. \tag{22}$$

It can be seen that

$$\hat{b}^\dagger\hat{b}|n\rangle_b = n|n\rangle_b. \tag{23}$$

Thus squeezing does not change the number of photons in a Fock state, but changes the photon frequency.

B. Displacement

Next, we eliminate the terms in Eq. (15) that are linear in $\hat{b}$ and $\hat{b}^\dagger$. To this end, we introduce new operators

$$\hat{c} = \hat{D}_b(-\beta)\hat{b}\hat{D}_b^\dagger(-\beta) = \hat{b} + \beta, \tag{24a}$$

$$\hat{c}^\dagger = \hat{D}_b(-\beta)\hat{b}^\dagger\hat{D}_b^\dagger(-\beta) = \hat{b}^\dagger + \beta^*, \tag{24b}$$

where

$$\hat{D}_b(\alpha) = \exp(\alpha\hat{b}^\dagger - \alpha^*\hat{b}) \tag{25a}$$

$$= \hat{S}(\zeta_0)\hat{D}(\alpha)\hat{S}^\dagger(\zeta_0) \tag{25b}$$

and

$$\hat{D}(\alpha) = \exp(\alpha\hat{a}^\dagger - \alpha^*\hat{a}) \tag{26}$$

are unitary displacement operators [27]. Substituting Eqs. (24) into Eq. (15) gives

$$\hat{H}(\mathbf{P}) = \frac{\mathbf{P}^2}{2m} - \hbar\Omega|\beta|^2 + \hbar\Omega(\hat{c}^\dagger\hat{c} + 1/2). \tag{27}$$

The displaced states are defined by

$$|n\rangle_c = \hat{D}_b(-\beta)|n\rangle_b. \tag{28}$$

Using Eq. (23), we obtain

$$\hat{c}^\dagger\hat{c}|n\rangle_c = n|n\rangle_c. \tag{29}$$

Thus the displacement changes neither the number of photons in a Fock state nor the photon frequency.

C. Eigenvalues and eigenstates

Let us return to the Hamiltonian (4). It can be seen now that its eigenstates and the corresponding eigenvalues are given by

$$|\mathbf{P}, n\rangle = |\mathbf{P}\rangle|n\rangle_c, \tag{30a}$$

$$E(\mathbf{P}, n) = \frac{\mathbf{P}^2}{2m} - \hbar\Omega|\beta|^2 + \hbar\Omega(n + 1/2). \tag{30b}$$

The eigenstates factorize into a state of electrons $|\mathbf{P}\rangle$ and a state of the field $|n\rangle_c$. Note, however, that $|n\rangle_c$ depends on $\mathbf{P}$, because β defined by Eq. (16) depends on $\mathbf{P}$. The field states can be expressed in terms of the original operators $\hat{a}$ and $\hat{a}^\dagger$ acting on $|n\rangle$. By combining the transformations (22) and (28) and using Eq. (A2), we obtain

$$|n\rangle_c = \hat{D}(\alpha_0)\hat{S}(\zeta_0)|n\rangle, \tag{31}$$

where

$$\alpha_0 = \gamma(-\beta, \zeta_0) = -\beta\cosh\nu + e^{-i\phi_\eta}\beta^*\sinh\nu \tag{32}$$

and ζ_0 is defined by Eq. (20). The composition of the squeezing and displacement operators in Eq. (31) turns an n-photon state of the free field with photon energy $\hbar\omega$ into a state containing n dressed photons with energy $\hbar\Omega$ created and annihilated by the operators $\hat{c}^\dagger$ and $\hat{c}$, respectively. The dressing accounts for the electron-field interaction and results in decoupling of the electron and field subsystems. Equations (30) generalize a similar result for a one-electron model considered in Ref. [9].

IV. DYNAMICS OF THE SYSTEM

In this section we discuss the dynamics of the system with the Hamiltonian (4) in the Schrödinger (Sec. IV A) and Heisenberg (Sec. IV B) pictures. The discussion requires the formulation of initial conditions for the evolution equations, so let us specify our model in this respect. We assume that free electrons are produced by ionization of the target gas due to its interaction with the laser field at times $t < 0$. Starting from $t = 0$ ionization is neglected, because either it is saturated or leads to only a negligible increase of the number of free electrons during the time interval in which the electron-field interaction is considered. Since free electrons are produced by strong-field ionization, they have a specific momentum distribution discussed in Appendix C. We also show how to perform averaging over this distribution, which is needed for calculating field observables (Sec. IV C).

A. Evolution of an initial coherent state of the field

In quantum mechanics, the laser field is represented by a coherent state [27]

$$|\alpha\rangle = \hat{D}(\alpha)|0\rangle. \tag{33}$$

In the absence of electrons, the evolution of this state in time would be given by

$$e^{-i\omega t\hat{a}^\dagger\hat{a}}|\alpha\rangle = |e^{-i\omega t}\alpha\rangle. \tag{34}$$

We are interested in the effect of the electron-field interaction on the dynamics of the field.

To investigate the effect, we have to consider the evolution of the whole system described by the time-dependent Schrödinger equation (TDSE)

$$i\hbar\frac{\partial|\Psi(t)\rangle}{\partial t}=\hat{H}|\Psi(t)\rangle. \tag{35}$$

We first discuss a solution $|\Psi(t;\mathbf{P})\rangle$ satisfying the initial condition

$$|\Psi(0;\mathbf{P})\rangle=|\mathbf{P}\rangle|\alpha\rangle, \tag{36}$$

which assumes that all electrons have definite momenta. This solution is given by

$$|\Psi(t;\mathbf{P})\rangle=e^{-i\hat{H}t/\hbar}|\Psi(0;\mathbf{P})\rangle=e^{-iE(\mathbf{P},0)t/\hbar}|\mathbf{P}\rangle|\psi(t;\mathbf{P})\rangle, \tag{37}$$

where

$$|\psi(t;\mathbf{P})\rangle=e^{-i\Omega t\hat{c}^\dagger\hat{c}}|\alpha\rangle. \tag{38}$$

Note that the state (37) is separable, that is, represented by a product of states of the electronic and field subsystems. However, the operators $\hat{c}^\dagger$ and $\hat{c}$ depend on $\mathbf{P}$, and hence so does the field state (38), therefore the subsystems are not actually independent. The field state can be expressed in terms of a coherent squeezed state defined by [27]

$$|\alpha,\zeta\rangle=\hat{D}(\alpha)\hat{S}(\zeta)|0\rangle. \tag{39}$$

Indeed, from Eqs. (18) and (24) we have

$$e^{-i\Omega t\hat{c}^\dagger\hat{c}}=\hat{D}(\alpha_0)\hat{S}(\zeta_0)e^{-i\Omega t\hat{a}^\dagger\hat{a}}\hat{S}^\dagger(\zeta_0)\hat{D}^\dagger(\alpha_0). \tag{40}$$

Substituting this and $|\alpha\rangle=\hat{D}(\alpha)e^{i\Omega t\hat{a}^\dagger\hat{a}}|0\rangle$ into Eq. (38) and using Eqs. (A7) and (A8), we obtain

$$\begin{aligned}|\psi(t;\mathbf{P})\rangle=&\hat{D}(\alpha_0)\hat{S}(\zeta_0)\hat{S}^\dagger(e^{-2i\Omega t}\zeta_0)\hat{D}^\dagger(e^{-i\Omega t}\alpha_0)\\&\times\hat{D}(e^{-i\Omega t}\alpha)|0\rangle.\end{aligned} \tag{41}$$

Using Eqs. (A2), (A4), and (A7), this can be transformed to the form

$$|\psi(t;\mathbf{P})\rangle=e^{i\varphi(t)}|\alpha(t),\zeta(t)\rangle, \tag{42}$$

where

$$\zeta(t)=\Gamma(\zeta_0,-e^{-2i\Omega t}\zeta_0), \tag{43a}$$

$$\begin{aligned}\alpha(t)=&\alpha_0+\gamma(\exp[i\Phi(\zeta_0,-e^{-2i\Omega t}\zeta_0)-i\Omega t]\\&\times(\alpha-\alpha_0),\zeta(t)),\end{aligned} \tag{43b}$$

$$\varphi(t)=\tfrac{1}{2}\Phi(\zeta_0,-e^{-2i\Omega t}\zeta_0)+\mathrm{Im}[\alpha_0(\alpha(t)-\alpha)^*]. \tag{43c}$$

Equation (42) presents one of the main results of this paper. It fully incorporates the effect of the electron-field interaction on the state of the field in the present model. Note that $\zeta(0)=0$, $\alpha(0)=\alpha$, and $\varphi(0)=0$, so $|\psi(0;\mathbf{P})\rangle=|\alpha\rangle$, in agreement with Eq. (38). Also note that in the absence of the electron-field interaction $\zeta_0=\alpha_0=0$, thus $|\psi(t;\mathbf{P})\rangle=|e^{-i\omega t}\alpha\rangle$, in agreement with Eq. (34). We will see shortly (see Sec. V A) that under typical experimental conditions the model parameters defining the state (42) satisfy $\xi\ll1$, $|\eta|\lesssim1$, $|\beta|\approx1$, and $|\alpha|\gg1$. In this case the following approximations hold:

$$\zeta(t)\approx\tfrac{1}{2}\xi\eta^*(1-e^{-2i\omega t}), \tag{44a}$$

$$\alpha(t)\approx\alpha e^{-i\omega t}-i\alpha^*\xi\eta^*\sin(\omega t)-\beta(1-e^{-i\omega t}), \tag{44b}$$

where we have neglected the difference between Ω and ω in the exponents. The last two terms in Eq. (44b) define the displacement of $\alpha(t)$ with respect to the corresponding parameter $\alpha e^{-i\omega t}$ for a free coherent state (34). Note that these terms are quadratic ($\xi\propto e^2$) and linear ($\beta\propto e$) in the electron-field interaction, respectively. The squeezing parameter (44a) is small but nonzero; as shown below this is essential for nonclassicality of the state (42). We mention that another mechanism associated with harmonic generation also resulting in displacement [4,12] and squeezing [24] of the laser mode has been recently discussed.

To account for the fact that free electrons do not have definite momenta but are characterized by a certain momentum distribution, we need the solution of Eq. (35) with the initial condition given by a superposition of states (36):

$$|\Psi(0)\rangle=\int C(\mathbf{P})|\mathbf{P}\rangle|\alpha\rangle\frac{d\mathbf{P}}{(2\pi\hbar)^{3N}}. \tag{45}$$

The coefficients in this superposition determine the momentum distribution of electrons $|C(\mathbf{P})|^2$. This distribution is normalized by

$$\langle\Psi(0)|\Psi(0)\rangle=\int|C(\mathbf{P})|^2\frac{d\mathbf{P}}{(2\pi\hbar)^{3N}}=1. \tag{46}$$

Using Eq. (37), we obtain

$$|\Psi(t)\rangle=\int C(\mathbf{P})e^{-iE(\mathbf{P},0)t/\hbar}|\mathbf{P}\rangle|\psi(t;\mathbf{P})\rangle\frac{d\mathbf{P}}{(2\pi\hbar)^{3N}}. \tag{47}$$

In contrast to Eq. (37), in this state the electronic and field subsystems are entangled.

B. Evolution of the field operators

An alternative way to describe the dynamics of the field is to use the Heisenberg annihilation $\hat{a}(t)$ and creation $\hat{a}^\dagger(t)$ operators defined by

$$\hat{a}(t)=e^{i\hat{H}t/\hbar}\hat{a}e^{-i\hat{H}t/\hbar}=e^{i\Omega t\hat{c}^\dagger\hat{c}}\hat{a}e^{-i\Omega t\hat{c}^\dagger\hat{c}} \tag{48}$$

and similarly for $\hat{a}^\dagger(t)$. Note that this implies the initial conditions $\hat{a}(0)=\hat{a}$ and $\hat{a}^\dagger(0)=\hat{a}^\dagger$ consistent with Eq. (36). From Eqs. (12) and (24) we obtain

$$\hat{a}=u\hat{c}-v^*\hat{c}^\dagger-u\beta+v^*\beta^*, \tag{49a}$$

$$\hat{a}^\dagger=u\hat{c}^\dagger-v\hat{c}-u\beta^*+v\beta. \tag{49b}$$

Thus

$$\hat{a}(t)=u\hat{c}e^{-i\Omega t}-v^*\hat{c}^\dagger e^{i\Omega t}-u\beta+v^*\beta^*, \tag{50a}$$

$$\hat{a}^\dagger(t)=u\hat{c}^\dagger e^{i\Omega t}-v\hat{c}e^{-i\Omega t}-u\beta^*+v\beta. \tag{50b}$$

Using Eqs. (10) and (24), the operators $\hat{c}$ and $\hat{c}^\dagger$ here can be expressed in terms of $\hat{a}$ and $\hat{a}^\dagger$. Note that the Heisenberg operators (50) depend on $\mathbf{P}$.

C. Averaging field observables over the electron momenta

For any field observable $O(\hat{a}, \hat{a}^\dagger)$, its expectation value in the state (47) can be calculated using either of the following expressions:

$$\langle\Psi(t)|O(\hat{a}, \hat{a}^\dagger)|\Psi(t)\rangle = \int \langle\psi(t;\mathbf{P})|O(\hat{a}, \hat{a}^\dagger)|\psi(t;\mathbf{P})\rangle|C(\mathbf{P})|^2 \times \frac{d\mathbf{P}}{(2\pi\hbar)^{3N}} \tag{51a}$$

$$= \int \langle\alpha|O(\hat{a}(t), \hat{a}^\dagger(t))|\alpha\rangle|C(\mathbf{P})|^2 \times \frac{d\mathbf{P}}{(2\pi\hbar)^{3N}}. \tag{51b}$$

Thus the corresponding expectation value in the field state (42) should be additionally averaged over the electron momentum distribution $|C(\mathbf{P})|^2$. In our model electrons are independent, therefore

$$|C(\mathbf{P})|^2 = \prod_{i=1}^{N} f(\mathbf{p}_i), \tag{52}$$

where $f(\mathbf{p})$ is a one-electron momentum distribution function normalized by

$$\int f(\mathbf{p})\frac{d\mathbf{p}}{(2\pi\hbar)^3} = 1. \tag{53}$$

This function describing free electrons produced by strong-field ionization is presented in Appendix C. We note that the matrix elements in Eqs. (51) depend on $\mathbf{P}$ only through the quantity β defined by Eq. (16). This quantity is given by a sum of components of electron momenta along the polarization axis, p_{iz}, multiplied by constant (i.e., independent of $\mathbf{P}$) coefficients. According to the central limit theorem [28], for sufficiently large number of electrons N the sum has a normal distribution independently of the particular form of the one-electron distribution $f(\mathbf{p})$. Since $\beta = \beta_\mathrm{r} + i\beta_\mathrm{i}$ is a complex quantity, the distributions in its real β_r and imaginary β_i parts should be considered separately. Taking into account Eqs. (C21) and (C22), for any function $g(\beta)$ we thus obtain

$$\int g(\beta)|C(\mathbf{P})|^2\frac{d\mathbf{P}}{(2\pi\hbar)^{3N}} = \langle g(\beta)\rangle_\beta, \tag{54}$$

where

$$\langle g(\beta)\rangle_\beta \equiv \int g(\beta_\mathrm{r} + i\beta_\mathrm{i})\exp\left(-\frac{\beta_\mathrm{r}^2}{\Delta\beta_\mathrm{r}^2} - \frac{\beta_\mathrm{i}^2}{\Delta\beta_\mathrm{i}^2}\right)\frac{d\beta_\mathrm{r}d\beta_\mathrm{i}}{\pi\Delta\beta_\mathrm{r}\Delta\beta_\mathrm{i}} \tag{55}$$

and the variances are given by

$$\Delta\beta_\mathrm{r}^2 = 2\left(\frac{eA_0}{mc\hbar\Omega}\right)^2\Delta p_z^2\sum_{i=1}^{N}[u\cos\phi_i - |v|\cos(\phi_i+\phi_\eta)]^2, \tag{56a}$$

$$\Delta\beta_\mathrm{i}^2 = 2\left(\frac{eA_0}{mc\hbar\Omega}\right)^2\Delta p_z^2\sum_{i=1}^{N}[u\sin\phi_i + |v|\sin(\phi_i+\phi_\eta)]^2. \tag{56b}$$

Note that $\langle 1\rangle_\beta = 1$. The sums over i in Eqs. (56) can be calculated using Eq. (7). We present results in the form

$$\Delta\beta_\mathrm{r}^2 + \Delta\beta_\mathrm{i}^2 = \frac{2\xi\omega^2}{\Omega^2}(u^2 - 2u|v||\eta|\cos 2\phi_\eta + |v|^2)\chi, \tag{57a}$$

$$\Delta\beta_\mathrm{r}^2 - \Delta\beta_\mathrm{i}^2 = \frac{2\xi\omega^2}{\Omega^2}[u(u|\eta| - 2|v|)\cos\phi_\eta + |v|^2|\eta|\cos 3\phi_\eta]\chi, \tag{57b}$$

from which $\Delta\beta_\mathrm{r}^2$ and $\Delta\beta_\mathrm{i}^2$ can be easily found. Here the factor ω^2/Ω^2 is determined by ξ and $|\eta|$ [see Eq. (17)], and

$$\chi = \frac{\Delta p_z^2}{m\hbar\omega} \tag{58}$$

is an additional independent real dimensionless parameter of our model characterizing the momentum distribution $f(\mathbf{p})$ of free electrons. These equations enable one to average the matrix elements in Eqs. (51) over this distribution. We emphasize that according to the central limit theorem this requires one to know only the variance Δp_z^2 of p_z defined by Eq. (C22), that is, the results are not sensitive to the detailed structure of the function $f(\mathbf{p})$. Accounting for this distribution is another important result of this paper.

V. FIELD OBSERVABLES

We assume that the field eventually leaves the region occupied by electrons and its quantum state can be detected experimentally. In this section we discuss two field observables. The variance of the photoelectron counting distribution discussed in Sec. V B unambiguously indicates the nonclassicality of the field state. The Wigner function discussed in Sec. V C shows that this state is non-Gaussian. Both these modifications of the field state compared to the coherent state of the free laser field are caused by the electron-field interaction.

A. Estimation of model parameters

Before turning to the discussion and illustrative calculation of the observables, we need to estimate parameters of our model corresponding to a typical experimental situation. We begin with parameters characterizing the field. In typical strong-field experiments laser pulses with frequency $\omega \approx 0.057$ a.u. (corresponding to the wavelength $\lambda \approx 800$ nm) and field amplitude $F_0 \approx 0.1$ a.u. (corresponding to the intensity $I \approx 3.5 \times 10^{14}$ W/cm^2) are used. The coherent state (34) corresponds to a classical field $\mathbf{F}(t) = F_0\sin(\omega t - \phi_\alpha)\mathbf{e}_z$ with the amplitude $F_0 = 2|\alpha|\omega A_0/c$ and phase $\phi_\alpha = \arg\alpha$. To estimate the value of $|\alpha|$ we need to know the quantization volume V. For our purposes it is sufficient to note that in any case $V \gtrsim \lambda^3$, which gives $|\alpha| \gtrsim 3 \times 10^5$. The phase ϕ_α can be arbitrary.

We now turn to the parameters ξ and η defining the model Hamiltonian (4). The target gas is assumed to be under normal conditions, which corresponds to the gas density 2.5×10^{19} cm^{-3}. Strong-field ionization produces at most one free electron per atom, which corresponds to $\omega_p = 0.007$ a.u. We thus obtain from Eq. (5) that $0 < \xi \lesssim 0.01$. It is seen from Eq. (7) that $0 \leqslant |\eta| \leqslant 1$. This parameter can be estimated more accurately as follows. Let L be the length of the interaction region occupied by electrons in the direction of propagation of the laser pulse, that is, along the x axis in our geometry, and let S be the area of its section in the transverse direction, so $LS = V$. Let the target gas and hence free electrons be uniformly distributed in this volume. Then, substituting $\phi_i = kX_i$ into Eq. (7), in the continuum limit we obtain

$$\eta = \int_0^L e^{2ikX} \frac{dX}{L} = e^{ikL} \frac{\sin kL}{kL}. \tag{59}$$

This shows that the phase ϕ_η of η can be arbitrary. It can be controlled by varying the length of the interaction region or the frequency of the laser field.

Next we estimate the electron momentum distribution parameter χ defined by Eq. (58). Using Eq. (C22) we find $\Delta p_z^2 \sim e^2F_0^2/\omega^2$, therefore $\chi \approx 80$.

There remain several dependent parameters whose values are determined by the parameters specified above. From Eqs. (13), (14), and (20) we obtain $u \approx 1$ and $|v| \approx \nu = |\zeta_0| \approx \xi|\eta|/2 \lesssim 0.005$, where we have taken into account that $\xi \ll 1$. From Eqs. (57) we find $\Delta\beta_{\rm r}^2 \lesssim 0.8$ and $\Delta\beta_{\rm i}^2 \lesssim 0.8$. It can be seen from Eq. (54) that $|\beta_{\rm r}| \sim \Delta\beta_{\rm r}$ and $|\beta_{\rm i}| \sim \Delta\beta_{\rm i}$. Finally, from Eq. (32) we have $|\alpha_0| \approx 0.9$.

B. Nonclassicality of the field state

When photons emitted from the interaction region hit a photodetector they are converted into photoelectrons. The photon statistics is usually analyzed by measuring the photoelectron counting distribution [29–31]. A sufficient condition for nonclassicality of the field state is that the variance of this distribution (counted from that for the Poisson distribution)

$$\delta = \overline{\Delta n^2} - \overline{n} \tag{60}$$

is negative, where $\Delta n = n - \overline{n}$ and the bar denotes averaging over the distribution. The negativity of δ means that photoelectron counts have a sub-Poissonian statistics, which is impossible for a classical field [29–31]. Let us check whether the quantum state of the field obtained in the preceding section satisfies this criterion of nonclassicality.

The variance of the photoelectron counting distribution (60) measured in a time interval $(t, t+T)$ by a detector with efficiency ϵ is given by [31]

$$\delta = \epsilon^2 \left[\int_t^{t+T} dt_1 \int_{t_1}^{t+T} dt_2 \langle \hat{a}^\dagger(t_1)\hat{a}^\dagger(t_2)\hat{a}(t_2)\hat{a}(t_1)\rangle + \int_t^{t+T} dt_1 \int_t^{t_1} dt_2 \langle \hat{a}^\dagger(t_2)\hat{a}^\dagger(t_1)\hat{a}(t_1)\hat{a}(t_2)\rangle - \left(\int_t^{t+T} dt \langle \hat{a}^\dagger(t)\hat{a}(t)\rangle \right)^2 \right], \tag{61}$$

where $\hat{a}(t)$ and $\hat{a}^\dagger(t)$ are the Heisenberg operators (50) and

$$\langle \dots \rangle \equiv \langle\langle \alpha | \dots | \alpha \rangle\rangle_\beta \tag{62}$$

denotes averaging over the state of the whole system, according to Eqs. (51b) and (54). The normal and time ordering required in Eq. (61) is explicit. Due to the integration over time, we should retain in the integrand only terms that do not depend on time, whose contribution to δ grows with T as T^2. The contribution from oscillating terms proportional to $\exp(\pm i\Omega t)$ and $\exp(\pm 2i\Omega t)$ can be neglected for counting intervals satisfying $T \gg T_\Omega$, where $T_\Omega = 2\pi/\Omega$. This leads to

$$\begin{aligned} \delta = {}& \epsilon^2 T^2 (u^2 + |v|^2)[(u^2 + |v|^2)(\langle \hat{c}^\dagger \hat{c}^\dagger \hat{c}\hat{c}\rangle - \langle \hat{c}^\dagger \hat{c}\rangle^2) \\ & + |v|^2 (2\langle \hat{c}^\dagger \hat{c}\rangle + 1)]. \end{aligned} \tag{63}$$

Using Eqs. (10) and (24) and performing the averaging (62), we obtain

$$\delta = \epsilon^2 T^2 \left[A + B\left(\Delta\beta_{\rm r}^2 - \Delta\beta_{\rm i}^2\right) + C\left(\Delta\beta_{\rm r}^2 + \Delta\beta_{\rm i}^2\right) \right], \tag{64}$$

where

$$\begin{aligned} A = {}& \tfrac{1}{8}\sinh(4\nu)[(4|\alpha|^2 + 1)\sinh(4\nu) + 4|\alpha|^2\cosh(4\nu) \\ & \times \cos(\phi_\eta + 2\phi_\alpha)], \end{aligned} \tag{65a}$$

$$B = \tfrac{1}{2}\sinh(2\nu)\cosh^2(2\nu)\cos\phi_\eta, \tag{65b}$$

$$C = \tfrac{1}{2}\sinh^2(2\nu)\cosh(2\nu). \tag{65c}$$

This is an exact result for the variance δ within the present model. As shown in the previous subsection, under typical experimental conditions we have $|\alpha| \gg 1$ and $\xi \ll 1$. In this case Eq. (64) can be simplified:

$$\delta \approx \epsilon^2 T^2 |\alpha|^2 \xi |\eta| [2\xi|\eta| + \cos(\phi_\eta + 2\phi_\alpha)]. \tag{66}$$

It can be seen now that δ becomes negative for certain combinations of the parameters ξ, η, and ϕ_α. Let us set $\phi_\alpha = 0$. Then δ is negative in the interval $\phi_\eta^{(-)} < \phi_\eta < \phi_\eta^{(+)}$, where

$$\phi_\eta^{(\pm)} \approx \pi \pm (\pi/2 - 2\xi|\eta|). \tag{67}$$

The negativity of the variance δ corresponds to a sub-Poissonian photoelectron counting distribution and therefore a nonclassical photon statistics [29–31]. Note that the value of δ in Eq. (66) is proportional to the product $\xi|\eta|$. We assume that there are free electrons, hence $\xi > 0$ [see Eq. (5)]. In this case the product can only become zero if $\eta = 0$. The same product appears in the squeezing parameter (44a). This means that nonzero values of η are essential for both squeezing and formation of a nonclassical state of the field. The presence of the phase ϕ_α in Eq. (66) indicates that the interval of ϕ_η, where the field is nonclassical, can be controlled by varying the phase of the laser field.

Let us illustrate the dependence of the variance δ on the model parameters. We set $|\eta| = 1$, $\chi = 80$, and $\alpha = 3 \times 10^5$ and consider δ as a function of ξ and ϕ_η. Figure 1 shows δ divided by $\epsilon^2 T^2 |\alpha|^2$ calculated using Eqs. (64) and (65). The variance turns to zero along the white lines and is negative in the area between them. In this region of the parameters the field state is nonclassical. The white lines are almost straight, which indicates that the approximation in Eq. (66) works well.

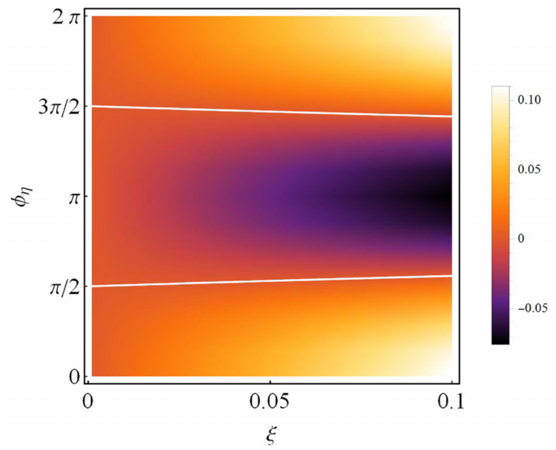

FIG. 1. The variance of the photoelectron counting distribution δ [see Eqs. (60) and (64)] divided by $\epsilon^2 T^2 |\alpha|^2$ as a function of ξ and $\phi_\eta = \arg \eta$. The other parameters are $|\eta| = 1$, $\chi = 80$, and $\alpha = 3 \times 10^5$. The variance turns to zero along the white lines and is negative between them.

As can be seen from Eq. (7), the parameter η accounts for the violation of the dipole approximation for the target gas cloud considered as a whole. This happens if the cloud size L in the direction of the laser pulse propagation (in the x direction in the present geometry) becomes comparable to or larger than the laser wavelength λ. For subwavelength clouds, $L \ll \lambda$, all the phases $\phi_i = kX_i$ in Eq. (7) are small. In the dipole approximation they are neglected, which gives $\eta = 1$. The same follows from Eq. (59) for $kL \ll 1$. In this case $\phi_\eta = 0$, which lies beyond the region between the two white lines in Fig. 2. Thus the dipole approximation used in Refs. [4,12,13] does not account for the formation of nonclassical field states characterized by negative values of the variance δ in the present model. As the cloud size L grows, the parameter η becomes complex and varies according to Eq. (59). Its phase ϕ_η as a function of kL can take any value from 0 to 2π and the criterion of nonclassicality of the field state $\delta < 0$ can be satisfied.

C. Non-Gaussian Wigner function of the field

Another observable which helps to visualize the quantum state of a field is the Wigner function [32]. Here we show that the Wigner function of the field in the present model has a non-Gaussian shape, which is of interest for applications in quantum-information theory [33]. If the whole system is in the state (47), the density matrix of the field subsystem is given by

$$\hat{\rho}(t) = \int \langle \mathbf{P}|\Psi(t)\rangle\langle\Psi(t)|\mathbf{P}\rangle \frac{d\mathbf{P}}{(2\pi\hbar)^{3N}}. \tag{68}$$

Using Eqs. (42) and (54), we obtain

$$\hat{\rho}(t) = \langle |\alpha(t), \zeta(t)\rangle\langle\alpha(t), \zeta(t)|\rangle_\beta. \tag{69}$$

The corresponding Wigner function defined with respect to a local oscillator with frequency ω_0 (in a rotating frame) is [27]

$$W(z,t) = \frac{2}{\pi^2} \int e^{z^* z' - z z'^*} \langle e^{-i\omega_0 t}(z+z')|\hat{\rho}(t) \times |e^{-i\omega_0 t}(z-z')\rangle d^2 z', \tag{70}$$

where both z and z' are complex and $d^2 z' = d(\mathrm{Re} z')d(\mathrm{Im} z')$. Substituting here Eq. (69) gives

$$W(z,t) = \langle W(z,t';\beta)\rangle_\beta \tag{71}$$

where $W(z,t;\beta)$ is the Wigner function for the state (42) given by [34,35]

$$W(z,t;\beta) = \frac{2}{\pi} \exp\{-2|[e^{-i\omega_0 t} z - \alpha(t)] \cosh|\zeta(t)| + e^{i \arg \zeta(t)}[e^{-i\omega_0 t} z - \alpha(t)]^* \sinh|\zeta(t)||^2\}. \tag{72}$$

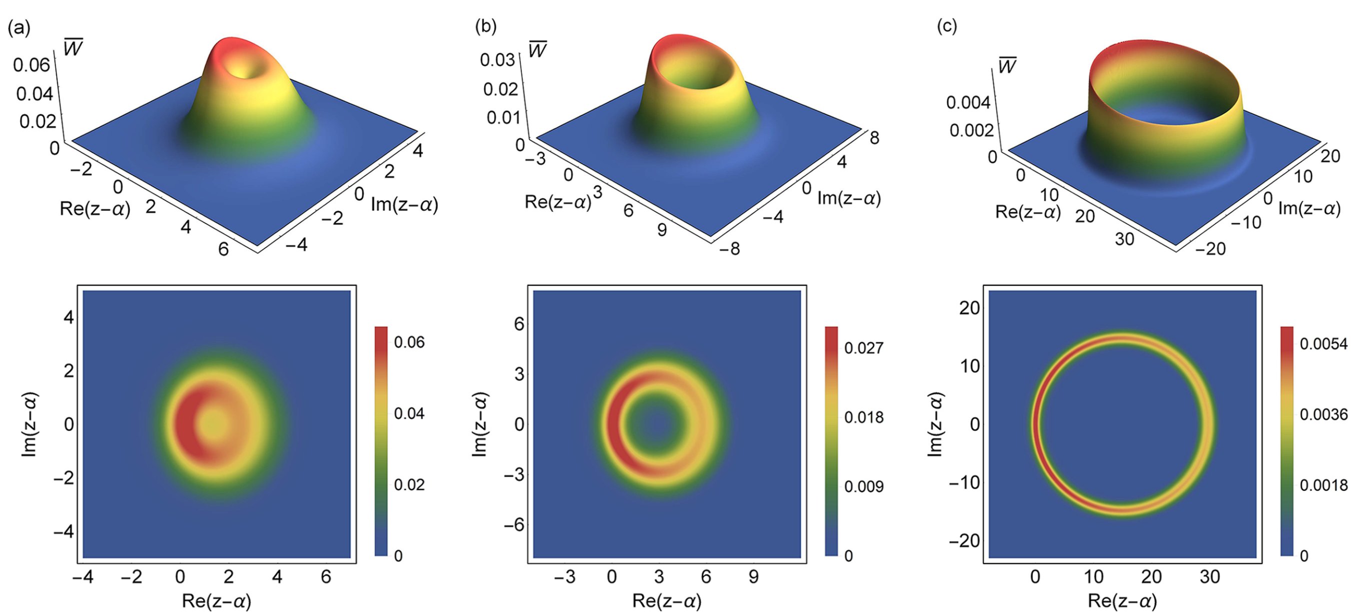

FIG. 2. Three-dimensional (top row) and two-dimensional (bottom row) plots of the Wigner function (73) for (a) $\eta = 10^{-3}$, (b) $\eta = 2 \times 10^{-3}$, and (c) $\eta = 10^{-2}$. The other parameters are $\xi = 0.01$, $\chi = 80$, and $\alpha = 3 \times 10^5$.

The Wigner function measurable in an experiment, e.g., using the homodyne detection scheme [32], is obtained by averaging Eq. (71) over a time interval $(t, t+T)$ of photodetector counting, similarly to how it was done in Eq. (61):

$$\overline{W}(z) = \frac{1}{T}\int_t^{t+T} W(z,t')dt' = \frac{1}{T}\int_t^{t+T} \langle W(z,t';\beta)\rangle_\beta dt'. \tag{73}$$

The density matrix (69) in the present model is periodic in time, $\hat{\rho}(t+T_\Omega) = \hat{\rho}(t)$. We set $\omega_0 = \Omega$, then the function (72) is also periodic with the same period T_Ω. Substituting $T = T_\Omega$ makes the function (73) strictly independent of t. The same independent of t result follows from Eq. (73) by averaging over any counting interval satisfying $T \gg T_\Omega$.

We have calculated the Wigner function (73) for $\xi = 0.01$, $\chi = 80$, $\alpha = 3\times10^5$, and three values of $\eta = 10^{-3}$, 2×10^{-3}, and 10^{-2}. The results are shown in Fig. 2. To explain these results we recall for reference that the Wigner function for the coherent free field state (34) is given by [27]

$$w(z) = \frac{2}{\pi}\exp(-2|z-\alpha|^2). \tag{74}$$

This is a stationary Gaussian whose center is located at $z = \alpha$. The function (72) is a squeezed Gaussian whose center moves along the trajectory $z = e^{i\Omega t}\alpha(t)$. In Eq. (73) this moving Gaussian is averaged, first over the electron momentum distribution for a given t [see Eq. (55)] and then over t. For the present parameters $\alpha(t)$ is approximately given by Eq. (44b), so the center of the moving Gaussian is located at $z = \alpha - \frac{1}{2}\alpha^*\xi\eta^*(e^{2i\omega t}-1) - \beta(e^{i\omega t}-1)$, where $|\beta| \approx 1$. Let us first consider the situation shown in Fig. 2(c). In this case $\frac{1}{2}|\alpha\xi\eta| = 15$, so $\frac{1}{2}|\alpha\xi\eta| \gg |\beta|$. Then the center moves along a circle centered at $z - \alpha = \frac{1}{2}\alpha^*\xi\eta^* = 15$ of radius $\frac{1}{2}|\alpha\xi\eta| = 15$. The width of the moving Gaussian is ≈ 1, so averaging over time yields a ring-shaped Wigner function localized near the circle $|z-(\alpha+15)| = 15$. For the parameters in Fig. 2(b) we have $\frac{1}{2}|\alpha\xi\eta| = 3$. This is still larger than $|\beta|$, so the same argumentation applies, and we again see a ring-shaped Wigner function localized near the circle $|z-(\alpha+3)| = 3$. The parameters in Fig. 2(a) are such that $\frac{1}{2}|\alpha\xi\eta| = 1.5$. In this case the ring shape is not so pronounced because the width of the moving Gaussian is comparable to the radius of the ring. For $\eta = 0$ the Wigner function has a Gaussian shape similar to Eq. (74), slightly broadened because of averaging over the electron momenta. We conclude that nonzero values of η are crucial for obtaining a ring-shaped Wigner function in the present model.

It is also instructive to consider the dependence of the Wigner function on ξ. This parameter is proportional to the electron density n_e [see Eqs. (5) and (6)], which in turn is proportional to the density of the target gas, and hence it can be controlled by varying the gas pressure. As explained above, the ring-shaped structure reveals itself under the condition $\frac{1}{2}|\alpha\xi\eta| > |\beta|$, with $\frac{1}{2}|\alpha\xi\eta|$ giving the radius of the ring. Therefore there exists a lower boundary of the electron density for which the ring-shaped structure can be observed. We have calculated the Wigner function (73) for $\chi = 80$ and $\alpha = 3\times10^5$ as in Fig. 2, $\eta = 0.01$, and four values of $\xi = 10^{-4}$, 10^{-3}, 5×10^{-3}, and 10^{-2} which for $\lambda =$ 800 nm corresponds to the electron density $n_e = 3.5\times10^{17}$, 3.5×10^{18}, 1.7×10^{19}, and 3.5×10^{19} cm^{-3}, respectively. Figure 3 shows one-dimensional cuts of the Wigner function along the line $\mathrm{Im}(z-\alpha) = 0$ multiplied by ξ to bring the results for different ξ to a common scale. For sufficiently low electron densities ($\xi = 10^{-4}$) the Wigner function has a simple Gaussian shape similar to Eq. (74). As the density increases ($\xi = 10^{-3}$), a ring-shaped structure appears, which is reflected in the appearance of the second maximum in the cut shown in Fig. 3. The radius of the Wigner function ring, and consequently the distance between the two maxima in its cut, grow proportionally to the density as it increases further ($\xi = 5\times10^{-3}$ and 10^{-2}). These results demonstrate that field states with a non-Gaussian ring-shaped Wigner function can be formed for realistic electron densities. We mention that strong-field experiments with multiatmosphere target gas pressures (atomic density $\approx 10^{21}$ cm^{-3}) were reported [36].

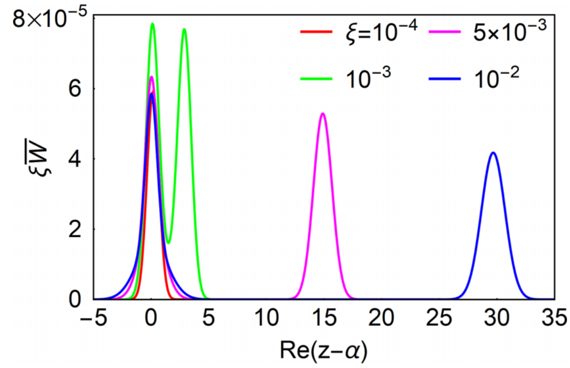

FIG. 3. One-dimensional cuts of the Wigner function (73) at $\mathrm{Im}(z-\alpha) = 0$ multiplied by ξ for $\chi = 80$ and $\alpha = 3\times10^5$, as in Fig. 2, $\eta = 0.01$, and four values of ξ indicated in the figure. The results for $\xi = 10^{-2}$ (the blue line) show the cut of the Wigner function presented in Fig. 2(c).

VI. CONCLUSION

A recent paper [4] has initiated the study of quantum properties of strong laser fields used in attosecond physics. There are three major processes whose effect on the quantum state of the driving laser field can be expected: strong-field ionization, harmonic generation, and the interaction with free electrons produced by strong-field ionization. The first two of them have been considered in Refs. [4,12,13,18,19,24]. In this paper we have analyzed the third mechanism. We have shown that the electron-field interaction can strongly affect the quantum state of the laser field under realistic conditions of strong-field experiments. Namely, it squeezes and displaces the coherent state of a free laser field, which can result in the formation of nonclassical and non-Gaussian field states. This shows that the full theory of quantum strong-field effects should take the electron-field interaction into account.

We mention that the present results may also have implications for quantum information processing. Squeezed states of a strong laser field and non-Gaussian states with a ring-shaped Wigner function measurable using homodyne detection may find applications in quantum computations and communication with continuous variables [37,38]. Indeed, squeezed states are important in continuum-variable quantum computations as a part of a universal quantum gate to perform quantum floating point computations [37]. While non-Gaussian field states are a necessary building block to achieve quantum computational advantages with continuum variables [33,39].

ACKNOWLEDGMENTS

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APPENDIX A: FORMULAS FROM QUANTUM OPTICS

For completeness of the presentation, here we summarize some formulas from quantum optics used in the main text. We adopt the notation (19) and (26) from Ref. [27]. A product of two displacement operators is given by [27]

$$\hat{D}(\alpha_1)\hat{D}(\alpha_2) = \exp[i\mathrm{Im}(\alpha_1\alpha_2^*)]\hat{D}(\alpha_1+\alpha_2). \tag{A1}$$

The squeezing and displacement operators can be permuted using [27]

$$\hat{S}(\zeta)\hat{D}(\alpha) = \hat{D}[\gamma(\alpha,\zeta)]\hat{S}(\zeta), \tag{A2}$$

where

$$\gamma(\alpha,\zeta) = \alpha\cosh|\zeta| - e^{i\phi_\zeta}\alpha^*\sinh|\zeta| \tag{A3}$$

and $\phi_\zeta = \arg\zeta$. A product of two squeezing operators can be presented in the form [35] (the argument of the squeezing operator in Ref. [35] has the opposite sign compared to that in Ref. [27])

$$\hat{S}(\zeta_1)\hat{S}(\zeta_2) = \hat{S}[\Gamma(\zeta_1,\zeta_2)]\exp[i\Phi(\zeta_1,\zeta_2)(\hat{a}^\dagger\hat{a}+1/2)], \tag{A4}$$

where

$$\Phi(\zeta_1,\zeta_2) = \frac{1}{2i}\ln\left(\frac{1+z_1z_2^*}{1+z_1^*z_2}\right), \tag{A5a}$$

$$|\Gamma(\zeta_1,\zeta_2)| = \tanh^{-1}\left|\frac{z_1+z_2}{1+z_1^*z_2}\right|, \tag{A5b}$$

$$\arg\Gamma(\zeta_1,\zeta_2) = \arg\left(\frac{z_1+z_2}{1+z_1^*z_2}\right), \tag{A5c}$$

and

$$z_i = e^{i\phi_i}\tanh|\zeta_i|, \quad \phi_i = \arg\zeta_i. \tag{A6}$$

Note that $\Phi(\zeta_1,\zeta_2)$ is real, while $\Gamma(\zeta_1,\zeta_2)$ is generally complex. In the case $\phi_1-\phi_2 = n\pi$, $n = 0, \pm1, \ldots,$ we obtain $\Phi(\zeta_1,\zeta_2) = 0$ and $\Gamma(\zeta_1,\zeta_2) = \zeta_1+\zeta_2$. Finally, for any real ϕ we have

$$e^{i\phi\hat{a}^\dagger\hat{a}}\hat{D}(\alpha)e^{-i\phi\hat{a}^\dagger\hat{a}} = \hat{D}(e^{i\phi}\alpha) \tag{A7}$$

and

$$e^{i\phi\hat{a}^\dagger\hat{a}}\hat{S}(\zeta)e^{-i\phi\hat{a}^\dagger\hat{a}} = \hat{S}(e^{2i\phi}\zeta). \tag{A8}$$

APPENDIX B: GAUGE TRANSFORMATION OF MOMENTUM DISTRIBUTIONS

Consider a quantum electron interacting with a classical homogeneous time-dependent electric field. Let $\hat{\mathbf{p}}$ be the usual canonical momentum of the electron; in the coordinate representation $\hat{\mathbf{p}} = -i\hbar\nabla$. Its eigenstates are defined by $\hat{\mathbf{p}}|\mathbf{p}\rangle = \mathbf{p}|\mathbf{p}\rangle$ and $\langle\mathbf{p}|\mathbf{p}'\rangle = (2\pi\hbar)^3\delta(\mathbf{p}-\mathbf{p}')$. Let $\hat{\mathbf{P}} = m\hat{\mathbf{v}}$ be the kinetic momentum, where

$$\hat{\mathbf{v}} = \widehat{\frac{d\mathbf{r}}{dt}} = \frac{i}{\hbar}[\hat{H},\hat{\mathbf{r}}] \tag{B1}$$

is the electron velocity and $\hat{H}$ is its Hamiltonian. In the case of a homogeneous field the operators $\hat{\mathbf{p}}$ and $\hat{\mathbf{P}}$ commute, and therefore have common eigenstates $|\mathbf{p}\rangle$. Let $\hat{\mathbf{P}}|\mathbf{p}\rangle = \mathbf{P}(\mathbf{p})|\mathbf{p}\rangle$, where $\mathbf{P}(\mathbf{p})$ denotes the corresponding eigenvalue. Then $\hat{\mathbf{P}}|\mathbf{p}(\mathbf{P})\rangle = \mathbf{P}|\mathbf{p}(\mathbf{P})\rangle$, where $\mathbf{p}(\mathbf{P})$ is the inverse function of $\mathbf{P}(\mathbf{p})$. Let the electron be in the state $|\psi(t)\rangle$. The canonical momentum distribution in this state is defined by

$$w(\mathbf{p}) = |\langle\mathbf{p}|\psi(t)\rangle|^2. \tag{B2}$$

Similarly, the kinetic momentum distribution is defined by

$$W(\mathbf{P}) = |\langle\mathbf{p}(\mathbf{P})|\psi(t)\rangle|^2. \tag{B3}$$

It is well known that the function $\mathbf{p}(\mathbf{P})$ and the state $|\psi(t)\rangle$ depend on the gauge. Let us discuss the gauge dependence of the distributions (B2) and (B3).

For definiteness, we consider two gauges commonly used in strong-field physics. In the velocity gauge the field is described by a four-potential $(\varphi_v, \mathbf{A}_v) = (0, \mathbf{A}(t))$. The electron Hamiltonian is

$$\hat{H}_v = \frac{1}{2m}\left[\hat{\mathbf{p}} + \frac{e}{c}\mathbf{A}(t)\right]^2. \tag{B4}$$

The kinetic momentum is given by

$$\hat{\mathbf{P}}_v = \hat{\mathbf{p}} + \frac{e}{c}\mathbf{A}(t) \rightarrow \mathbf{p}(\mathbf{P}) = \mathbf{P} - \frac{e}{c}\mathbf{A}(t). \tag{B5}$$

The general solution of the TDSE with the Hamiltonian (B4) can be presented in the form

$$|\psi_v(t)\rangle = \int c(\mathbf{p})\exp\left(-\frac{i}{2m\hbar}\int_0^t\left[\mathbf{p}+\frac{e}{c}\mathbf{A}(t')\right]^2dt'\right)|\mathbf{p}\rangle \times \frac{d\mathbf{p}}{(2\pi\hbar)^3}, \tag{B6}$$

where $c(\mathbf{p})$ is an arbitrary function. Thus

$$w_v(\mathbf{p}) = |\langle\mathbf{p}|\psi_v(t)\rangle|^2 = |c(\mathbf{p})|^2 \tag{B7}$$

and

$$W_v(\mathbf{P}) = \left|\left\langle\mathbf{P}-\frac{e}{c}\mathbf{A}(t)\middle|\psi_v(t)\right\rangle\right|^2 = \left|c\left[\mathbf{P}-\frac{e}{c}\mathbf{A}(t)\right]\right|^2. \tag{B8}$$

In the length gauge the field is described by a four-potential $(\varphi_l, \mathbf{A}_l) = (-\mathbf{F}(t)\mathbf{r}, \mathbf{0})$, where

$$\mathbf{F}(t) = -\frac{1}{c}\frac{d\mathbf{A}(t)}{dt}. \tag{B9}$$

The Hamiltonian is

$$\hat{H}_l = \frac{\hat{\mathbf{p}}^2}{2m} + e\mathbf{F}(t)\hat{\mathbf{r}}. \tag{B10}$$

The kinetic momentum is given by

$$\hat{\mathbf{P}}_l = \hat{\mathbf{p}} \rightarrow \mathbf{p}(\mathbf{P}) = \mathbf{P}. \tag{B11}$$

The state (B6) is represented by

$$|\psi_l(t)\rangle = \exp\left[i\frac{e}{c\hbar}\mathbf{A}(t)\hat{\mathbf{r}}\right]|\psi_v(t)\rangle. \tag{B12}$$

Thus

$$w_l(\mathbf{p}) = |\langle\mathbf{p}|\psi_l(t)\rangle|^2 = \left|c\left[\mathbf{p} - \frac{e}{c}\mathbf{A}(t)\right]\right|^2 \tag{B13}$$

and

$$W_l(\mathbf{P}) = |\langle\mathbf{P}|\psi_l(t)\rangle|^2 = \left|c\left[\mathbf{P} - \frac{e}{c}\mathbf{A}(t)\right]\right|^2. \tag{B14}$$

Summarizing, we arrive at the following conclusions. First, the canonical momentum distribution depends on the gauge and

$$w_l(\mathbf{p}) = w_v\left[\mathbf{p} - \frac{e}{c}\mathbf{A}(t)\right], \tag{B15}$$

while the kinetic momentum distribution is gauge invariant,

$$W_v(\mathbf{P}) = W_l(\mathbf{P}). \tag{B16}$$

This means, in particular, that only the latter is observable, and the former is not. Second, in the length gauge the two distributions coincide:

$$w_l(\mathbf{p}) = W_l(\mathbf{p}). \tag{B17}$$

APPENDIX C: MOMENTUM DISTRIBUTION OF ELECTRONS PRODUCED BY STRONG-FIELD IONIZATION

Free electrons discussed in the main text are produced in the ionization of the target gas by the laser field. Here we derive the one-electron momentum distribution $f(\mathbf{p})$ introduced in Sec. IV C. The Hamiltonian (1) corresponds to the velocity gauge, and hence so does the Hamiltonian (4) obtained by substituting Eq. (3). The momenta $\mathbf{p}_i$ defining the argument of $f(\mathbf{p})$ are introduced in Eq. (8) as eigenvalues of canonical momentum operators $\hat{\mathbf{p}}_i$. Thus $f(\mathbf{p})$ is the canonical momentum distribution in the velocity gauge, using the terminology of Appendix B.

To derive this distribution, let us consider the interaction of the ith atom with the ionizing laser field. The atom is treated in the single-active-electron approximation and described by a spherically symmetric potential $V(r)$, where $\mathbf{r} = \mathbf{r}_i - \mathbf{R}_i$ is the electron coordinate measured from the atomic nucleus. The field is assumed to be classical and corresponding to the coherent state $|\alpha\rangle$ of the quantum field appearing in the initial condition (36). It is described by

$$\mathbf{A}(t) = \langle e^{-i\omega t}\alpha|\hat{\mathbf{A}}(\mathbf{r}_i)|e^{-i\omega t}\alpha\rangle = A(t)\mathbf{e}_z \tag{C1}$$

and

$$\mathbf{F}(t) = -\frac{1}{c}\frac{d}{dt}\mathbf{A}(t) = F(t)\mathbf{e}_z, \tag{C2}$$

where $\hat{\mathbf{A}}(\mathbf{r}_i)$ is given by Eq. (3). We obtain

$$A(t) = \frac{cF_0}{\omega}\cos(\omega t - \phi_i - \phi_\alpha), \tag{C3a}$$

$$F(t) = F_0\sin(\omega t - \phi_i - \phi_\alpha), \tag{C3b}$$

where $F_0 = 2|\alpha|\omega A_0/c$ and $\phi_\alpha = \arg\alpha$. The TDSE describing the active electron in the dipole approximation and length gauge reads

$$i\hbar\frac{\partial\psi(\mathbf{r},t)}{\partial t} = \left[-\frac{\hbar^2}{2m}\Delta + V(r) + eF(t)z\right]\psi(\mathbf{r},t). \tag{C4}$$

We assume that the field is smoothly turned on and at times $t > 0$ is given by Eqs. (C3). The initial condition for Eq. (C4) is specified by

$$\psi(\mathbf{r}, t\to-\infty) = e^{-iE_0t/\hbar}\phi_0(r), \tag{C5}$$

where E_0 and $\phi_0(r)$ are the energy and wave function of an initial bound state which for simplicity is assumed to be an s state. For sufficiently low-frequency and high-intensity fields considered in strong-field physics the solution to Eqs. (C4) and (C5) can be found using the adiabatic theory [40]. In this theory, the solution is constructed in the form $\psi(\mathbf{r},t) = \psi_a(\mathbf{r},t) + \psi_r(\mathbf{r},t)$, where the adiabatic $\psi_a(\mathbf{r},t)$ and rescattering $\psi_r(\mathbf{r},t)$ parts of the wave function represent bound and continuum parts of the electron state, respectively. The rescattering part $\psi_r(\mathbf{r},t)$ accounts for the interaction of the liberated electron with both the parent ion and the field. The former interaction results in the appearance of a small fraction of high-energy electrons and can be neglected for the present purposes. In this approximation the continuum part of the wave function is denoted by $\psi_r^{(a)}(\mathbf{r},t)$. This function describes a wave packet of liberated electrons which is driven only by the field. The relation between $\psi_r(\mathbf{r},t)$ and $\psi_r^{(a)}(\mathbf{r},t)$ is similar to that between an exact scattering state and the corresponding incident plane wave, respectively; for more details see Ref. [40] where both functions were explicitly obtained. Let us transform $\psi_r^{(a)}(\mathbf{r},t)$ to the momentum representation:

$$\psi_l(\mathbf{p},t) = \int\psi_r^{(a)}(\mathbf{r},t)e^{-i\mathbf{pr}/\hbar}d\mathbf{r}. \tag{C6}$$

Omitting the derivation based on the results of Ref. [40], we obtain

$$\psi_l(\mathbf{p},t) = e^{i\pi/4}(2\pi\hbar)^{1/2}\sum_i\frac{A(\mathbf{p}_\perp;|F(t_i)|)}{|eF(t_i)|^{1/2}}e^{-iS_i/\hbar}, \tag{C7}$$

where

$$S_i = \frac{\mathbf{p}_\perp^2}{2m}(t-t_i) + \frac{1}{2m}\int_{t_i}^t\left(p_z - \frac{e}{c}[A(t)-A(t')]\right)^2dt' + s(t_i) \tag{C8}$$

and

$$s(t) = E_0t + \int_{-\infty}^t[E(|F(t')|) - E_0]dt'. \tag{C9}$$

Here $\mathbf{p}_\perp = (p_x, p_y)$, $E(F) = \mathcal{E}(F) - \frac{i}{2}\Gamma(F)$ is the complex energy and $A(\mathbf{p}_\perp; F)$ is the transverse momentum distribution amplitude characterizing the Siegert state that originates from the initial bound state (C5) in the presence of a static electric field $\mathbf{F} = F\mathbf{e}_z$ [40,41], t_i is the ionization time defined by

$$p_z = \frac{e}{c}[A(t) - A(t')] \;\to\; t' = t_i < t, \tag{C10}$$

and the summation in Eq. (C7) runs over all solutions of Eq. (C10). According to Eq. (B12), the same wave packet in the velocity gauge is represented by

$$\psi_v(\mathbf{p},t) = \psi_l\Big(\mathbf{p}_\perp, p_z + \frac{e}{c}A(t), t\Big). \tag{C11}$$

The norm of this state gives the total probability of ionization by time t. Let us introduce the differential ionization rate,

$$w(\mathbf{p}) = \frac{1}{T}|\psi_v(\mathbf{p},T)|^2\bigg|_{T\to\infty}. \tag{C12}$$

Using Eqs. (C7) and (C11), we obtain

$$w(\mathbf{p}) = \frac{4(\hbar\omega)^2}{eF(p_z)}|A(\mathbf{p}_\perp;F(p_z))|^2\cos^2\frac{\Delta S}{2\hbar}\sum_{n\geqslant n_0} \times\delta\Big(\frac{p^2}{2m}+U_p-\bar{\mathcal{E}}-n\hbar\omega\Big). \tag{C13}$$

Here

$$\bar{\mathcal{E}} - \frac{i}{2}\bar{\Gamma} = \frac{1}{T_\omega}\int_0^{T_\omega} E(|F(t)|)\,dt = \frac{2}{\pi}\int_0^{F_0}\frac{E(F)dF}{\sqrt{F_0^2-F^2}} \tag{C14}$$

denotes the energy of the Siegert state averaged over the laser period $T_\omega = 2\pi/\omega$ and n_0 is the minimum integer satisfying $n\hbar\omega > U_p - \bar{\mathcal{E}}$. The other notation is defined by

$$F(p_z) = F_0\sqrt{1-\frac{p_z^2}{p_0^2}},\quad U_p = \frac{p_0^2}{4m},\quad p_0 = \frac{eF_0}{\omega}, \tag{C15}$$

and

$$\Delta S = \frac{2(\pi-\phi)}{\omega}\Big(\frac{p^2}{2m}+U_p\Big) + \frac{3p_z}{2m\omega}\sqrt{p_0^2-p_z^2} - \frac{2}{\omega}\int_\phi^\pi \mathcal{E}(F_0\sin\phi')\,d\phi', \tag{C16}$$

where $\phi = \arccos(p_z/p_0)$. In arriving at Eq. (C13) we have neglected ionization that occurs during turning on the field at $t < 0$. Furthermore, we have assumed that $T \gg T_\omega$ and $T\bar{\Gamma} \ll \hbar$, which specifies the meaning of the limit in Eq. (C12) and justifies neglecting the depletion. The $\cos^2$ factor in Eq. (C13) describes the intracycle interference of two contributions with ionization times belonging to the same optical cycle, while the sum of δ functions describes the intercycle interference resulting from the summation over many cycles in the interval $0 < t < T$. In the adiabatic regime [40] these factors are rapidly varying functions of the electron momentum $\mathbf{p}$. By averaging them, we obtain the averaged differential ionization rate,

$$\bar{w}(\mathbf{p}) = \frac{2\hbar\omega}{eF(p_z)}|A(\mathbf{p}_\perp;F(p_z))|^2. \tag{C17}$$

Taking into account the relation [41]

$$\int |A(\mathbf{p}_\perp;F)|^2\frac{d\mathbf{p}_\perp}{(2\pi\hbar)^2} = \Gamma(F)/\hbar \tag{C18}$$

it can be seen that

$$\int \bar{w}(\mathbf{p})\frac{d\mathbf{p}}{(2\pi\hbar)^3} = \bar{\Gamma}/\hbar. \tag{C19}$$

Thus the momentum distribution we need is given by

$$f(\mathbf{p}) = \hbar\bar{\Gamma}^{-1}\bar{w}(\mathbf{p}). \tag{C20}$$

It satisfies the normalization condition (53). Note that this distribution is even in p_z, because p_z enters into Eq. (C17) only through the function $F(p_z) = F(-p_z)$ [see Eqs. (C15)], therefore the average of p_z vanishes:

$$\int p_z f(\mathbf{p})\frac{d\mathbf{p}}{(2\pi\hbar)^3} = 0. \tag{C21}$$

Let us introduce the average of p_z^2:

$$\int p_z^2 f(\mathbf{p})\frac{d\mathbf{p}}{(2\pi\hbar)^3} = \Delta p_z^2. \tag{C22}$$

It defines the parameters (57) needed for averaging over electron momenta.

To determine the particular shape of the distribution $f(\mathbf{p})$ we need to know how the functions $\Gamma(F)$ and $A(\mathbf{p}_\perp;F)$ depend on F. At sufficiently weak fields this dependence is known analytically from the weak-field asymptotic theory [41]. We have

$$\Gamma(F) = \frac{e^2\varkappa g_{00}^2}{2}\Big(\frac{4e\varkappa^2}{F}\Big)^{2\kappa-1}\exp\Big(-\frac{2\hbar^2\varkappa^3}{3meF}\Big) \tag{C23}$$

and

$$|A(\mathbf{p}_\perp;F)|^2 = \frac{4\pi\hbar\varkappa\Gamma(F)}{meF}\exp\Big(-\frac{\varkappa p_\perp^2}{meF}\Big), \tag{C24}$$

where $\kappa = me^2/\hbar^2\varkappa$, $\varkappa = \hbar^{-1}\sqrt{2m|E_0|}$, and g_{00} is a dimensionless coefficient appearing in the asymptotic tail of the unperturbed wave function $\phi_0(r)$ [41]. Using Eqs. (C23) and (C24), we obtain

$$f(\mathbf{p}) = \frac{\mathcal{N}}{(p_0^2-p_z^2)^{\kappa+1/2}}\exp\Bigg(-\frac{\varkappa p_0}{3meF_0}\frac{3p_\perp^2+2\hbar^2\varkappa^2}{\sqrt{p_0^2-p_z^2}}\Bigg), \tag{C25}$$

where $\mathcal{N}$ is a constant normalization coefficient. This function is localized in a region limited by the interval $-p_0 < p_z < p_0$ along the p_z axis and having a radius $p_\perp \sim \sqrt{meF_0/\varkappa}$ in the transverse direction. In the adiabatic regime the ratio of the transverse and longitudinal sizes of this region $\omega\sqrt{m/\varkappa eF_0}$ is small, so the distribution has a cigarlike shape extended along the p_z axis.

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