Hierarchical certification of nonclassical network correlations

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With the increased availability of quantum technological devices, it becomes more important to have tools to guarantee their correct nonclassical behavior. This is especially important for quantum networks, which constitute the platforms where multipartite cryptographic protocols will be implemented, and where guarantees of nonclassicality translate into security proofs. We derive linear and nonlinear Bell-like inequalities for networks, whose violation certifies the absence of a minimum number of classical sources in them. We do so, first, without assuming that nature is ultimately governed by quantum mechanics, providing a hierarchy interpolating between network nonlocality and full network nonlocality. Second we insert this assumption, which leads to results more amenable to certification in experiments.

I. INTRODUCTION

Bell's theorem asserts that models based on classical shared randomness cannot replicate the probabilistic predictions of quantum theory in all circumstances [1]. In particular, certain correlations generated from local measurements on entangled quantum systems [2] cannot be simulated by local hidden variable models. Bell nonlocality, which can be generated from any bipartite entangled system regardless of its dimension [3], represents a type of quantumness that is significant both in the fundamental theory and in various applications [4–8]. The methods for detecting nonlocality can be readily extended to multipartite scenarios by separating the parties along bipartitions [9–11]. Traditionally, the parties within a bipartition are allowed to communicate with each other, and genuine multipartite nonlocality is then defined as the impossibility to write a multipartite probability distribution with a local model along any possible bipartition of the parties.

A recent trend consists of investigating the correlations that can be generated in network structures where, instead of considering a global joint system being shared between the parties, several independent sources distribute systems to different collections of them. For example, quantum networks use sources of entangled systems, as depicted in Fig. 1, providing a scalable framework for constructing the quantum internet and implementing diverse quantum applications [12–16]. Quantum multipartite nonlocality has been investigated in these scenarios [17–23], leading to new fundamental insights on quantum theory [16,24]. Nevertheless, exploring nonlocal correlations in networks implies considerable challenges since the relevant sets of correlations are not convex, and thus objects such as linear Bell inequalities cannot characterize them completely.

Nonlinear Bell inequalities have been used to study network nonlocality (NN), especially in the tripartite network underlying entanglement swapping depicted in Fig. 1(a) [18,25–27]. These inequalities have been extended to more general networks with more parties and sources of physical systems ([28–32], see also the review [23]). However, in analogy with the need to define genuine multipartite nonlocality
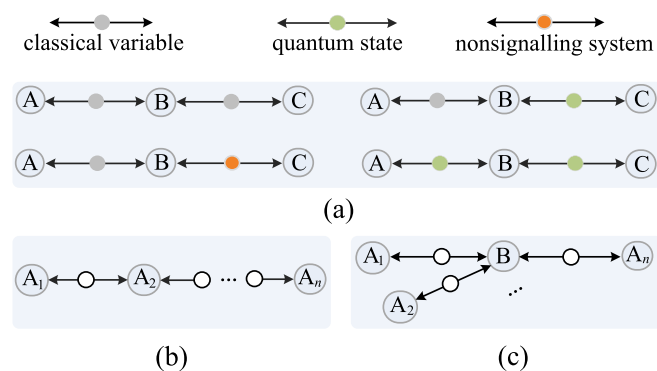

FIG. 1. The main networks discussed in this work. (a) The network underlying simple entanglement swapping experiments. Correlations generated in the network are considered NN if they cannot be generated as in the top-left setup, FNN if they cannot be generated as in the bottom-left setup, and FQNN if they cannot be generated as in the top-right setup. In the appendices we show that realizations in the bottom-right setup produce all these types of correlations. (b) The generalization of the previous scenario to chains, and (c) its generalization to stars.

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as a means to guarantee global nonclassical phenomena, the observation of network nonlocality only guarantees nonclassical behavior in some part of the network. It is possible to guarantee a complete absence of classical components via observing genuine multipartite nonlocality, but it is known that accounting for the network structure allows for milder requirements in experimentally relevant parameters, such as detection efficiency [33]. Currently, there exist two complementary definitions of genuinely nonlocal behaviors in networks: genuine network nonlocality [34] and full network nonlocality (FNN) [35]. The latter uses minimal assumptions, and has been recently observed in experimental implementations of different network structures [36–39].

In this work, we study fully network nonlocal correlations under the assumption that physical systems are ultimately governed by quantum mechanics. We derive Bell-like inequalities, both linear and nonlinear, for star and chain networks [see Figs. 1(b) and 1(c)]. The violation of these inequalities certifies that the corresponding statistics could not have been produced had there been at least a number of classical sources in the network. In many cases, we are able to provide a single inequality that guarantees that no source in the network distributes classical physical systems, in contrast with previous works [35,38] that needed one inequality per source in the network. In other cases, we provide bounds for each number of classical sources in the network. In addition to this, we provide FNN bounds for the inequalities, and demonstrate violations of both types of bounds within quantum mechanics. Finally, we discuss how these results can be exploited to guarantee nonclassicality of the sources of arbitrary networks.

II. MODELS FOR NON-FULL CORRELATIONS

Nonlocality in networks is defined in an analogous manner to the bipartite case [1,5], namely, by opposition to having a local model. This is, network nonlocal correlations do not admit a decomposition in the following form:

$$P(\vec{a}|\vec{x}) = \int_\Omega \prod_{j=1}^m d\mu(\lambda_j) \prod_{i=1}^n P(a_i|x_i, \bar{\lambda}_i), \quad (1)$$

where $\vec{a} = a_1 \ldots a_n$ denotes the parties' outputs, $\vec{x} = x_1 \ldots x_n$ their inputs, λ_j the classical variable distributed by source j, and $\bar{\lambda}_i$ the set of variables that arrive to party i, i.e., the set of classical physical systems that arrive to party i from the sources that connect to it. As an illustration, take the network known as the bilocality scenario, depicted in Fig. 1(a). Network-local correlations in this network (i.e., bilocal correlations) admit a description of the form [18,25]

$$P(a, b, c|x, y, z) = \int d\mu(\lambda_1) d\mu(\lambda_2)$$
$$\times P(a|x, \lambda_1) P(b|y, \lambda_1, \lambda_2) P(c|z, \lambda_2).$$

When $m = 1$, Eq. (1) recovers the standard notion of a local hidden variable model [5].

Many works have focused on detecting when multipartite correlations do and do not admit realizations in the form of Eq. (1), producing many Bell-like inequalities [18,23,25,28–32]. However, in generalizing the standard notion of a Bell inequality, inequalities tailored for detecting network

nonlocality can be violated even when just one of the sources in the network is nonclassical [19]. This motivated the development of new notions beyond network nonlocality whose observation guarantees stronger quantum properties, for instance the absence of sources of classical systems [35] or the presence of joint quantum measurements [34] in the network. Concretely, Ref. [35] defines as "not interesting" correlations [i.e., the analogous of Eq. (1) in the setting of network nonlocality] those that can be generated if and only if there is at least one source in the network that distributes classical physical systems. Correlations that cannot be generated in this way are called FNN.

In this work we consider two refinements of the definition of FNN in Ref. [35]. First, we define a hierarchy of levels between NN and FNN, determined by the number of classical sources that can be used for reproducing the correlation.

Definition 1 (ℓ-level network nonlocality). A multipartite probability distribution is ℓ-level network nonlocal (ℓ-NN) relative to a network if it cannot be generated when at least ℓ sources in the network distribute classical physical systems, and the rest are allowed to distribute physical systems only limited by no-signaling. Otherwise, the distribution is ℓ-level network local (ℓ-NL).

It is clear that the above definition coincides with that of NN when ℓ equals the number of sources in the network [i.e., when $\ell = m$ in Eq. (1)], and with FNN when $\ell = 1$. As an illustration, consider the star network in Fig. 1(c) with three branch parties, $\{A_1, A_2, A_3\}$. ℓ-NL correlations in this network admit distributions of the forms

$$P^{\text{2-NL}}(a_1, a_2, a_3, b|x_1, x_2, x_3, y)$$
$$= \int d\mu(\lambda_1) d\mu(\lambda_2) P(a_1|x_1, \lambda_1) P(a_2|x_2, \lambda_2)$$
$$\times P_{\text{NSI}}(a_3, b|x_3, y, \lambda_1, \lambda_2), \quad (2)$$

$$P^{\text{1-NL}}(a_1, a_2, a_3, b|x_1, x_2, x_3, y)$$
$$= \int d\mu(\lambda_1) P(a_1|x_1, \lambda_1)$$
$$\times P_{\text{NSI}}(a_2, a_3, b|x_2, x_3, y, \lambda_1), \quad (3)$$

or of analogs under permutations of the branch parties. P_{NSI} are distributions only constrained by the principles of no-signaling and the independence of the sources [40]. This means that for the distribution in Eq. (2) we have that $\sum_{a_3} P_{\text{NSI}}(a_3, b|x_3, y, \lambda_1, \lambda_2) = P_{\text{NSI}}(b|y, \lambda_1, \lambda_2)$ and $\sum_b P_{\text{NSI}}(a_3, b|x_3, y, \lambda_1, \lambda_2) = P(a_3|x_3)$, and for the case of the distribution in Eq. (3) we require that $\sum_{a_i} P_{\text{NSI}}(a_2, a_3, b|x_2, x_3, y, \lambda_1) = P_{\text{NSI}}(a_{j \neq i}, b|x_{j \neq i}, y, \lambda_1)$ and $\sum_b P_{\text{NSI}}(a_2, a_3, b|x_2, x_3, y, \lambda_1) = P(a_2|x_2) P(a_3|x_3)$.

Definition 1 allows us to make theory-independent statements on the presence of classical sources in networks [38]. However, inserting the additional assumption that the nonclassical sources in the network distribute systems that obey quantum theory is a natural one that, as we show, leads to milder requirements on the experimentally relevant parameters (such as the visibility under white noise) needed to observe nonlocality. Thus, we also study the hierarchy in Definition 1 under this assumption. This leads to an analogous hierarchy, and to the definition of fully quantum network nonlocality (FQNN):

Definition 2 (ℓ-level quantum network nonlocality). A multipartite probability distribution is ℓ-level quantum network nonlocal (ℓ-QNN) relative to a network if it cannot be generated when at least ℓ sources in the network distribute classical physical systems, and the rest are allowed to distribute arbitrary multipartite quantum systems. Otherwise, the distribution is ℓ-level quantum network local (ℓ-QNL). We say that a correlation is fully quantum network nonlocal if it is 1-QNN.

Going back to the example of the star network with three branch parties, the corresponding distributions $P^{2\text{-QNL}}$ and $P^{1\text{-QNL}}$ have the same form as Eqs. (2) and (3), where now the distributions P_{NSI} are instead distributions that follow the corresponding Born's rule, i.e., $\text{Tr}(\Pi_{a_3}^{x_3} \otimes \Pi_b^{y,\lambda_1,\lambda_2} \cdot \rho_{A_3 B})$ for Eq. (2) and $\text{Tr}(\Pi_{a_2}^{x_2} \otimes \Pi_{a_3}^{x_3} \otimes \Pi_b^{y,\lambda_1} \cdot \rho_{A_2 B} \otimes \rho_{A_3 B})$ for Eq. (3).

In the remainder of this paper we analytically prove inequalities, both linear and nonlinear, satisfied by ℓ-NL and ℓ-QNL correlations in large families of networks and for several values of ℓ, including complete hierarchies (i.e., $\ell \in [1, \ldots, m]$) for some scenarios. We must note that, in general, obtaining analytical results for ℓ-NL is hard. This is due to the lack of a formal description of all generalized states and effects that lead to nonsignaling distributions. For a more systematic analysis than the one done in this work one could use inflation methods (see, e.g., Refs. [35,38,41] for examples of its use in the derivation of witnesses of FNN), to create tractable relaxations of the sets of ℓ-NL distributions.

III. NONLINEAR INEQUALITIES FOR CHAIN NETWORKS

The first family of networks that we study is that of chain networks, depicted in Figs. 1(b) and 2. Chain networks are comprised of sources of bipartite states with the parties arranged in the form of a chain, such that with the exception of the two, extremal parties, all the remaining ones receive systems from two different sources. We initially analyze the simplest case of the tripartite chain network, and extend our results to longer chains in a second step.

A. The tripartite chain

Let us begin with the tripartite chain network depicted in Fig. 1(a), also known as the bilocality network [18,25]. It comprises two bipartite sources, that distribute systems to three parties, in such a way that one of the parties receives a system from each of the sources. This is the simplest nontrivial network and thus has received substantial attention in the
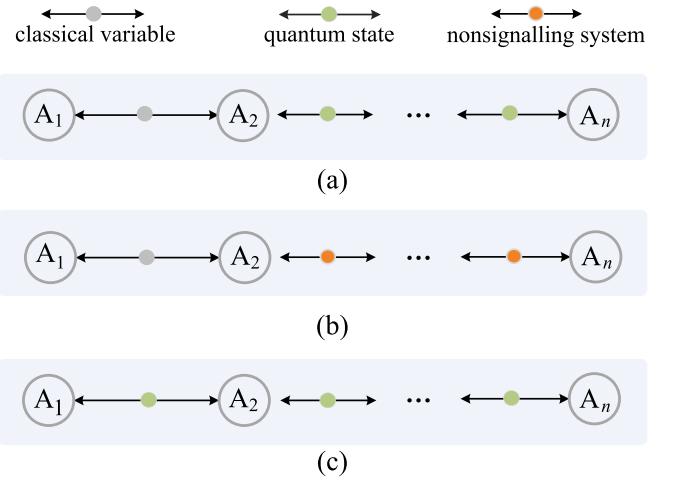

FIG. 2. The *n*-partite chain network. Correlations that do not admit a realization neither in the form of panel (a) nor with any other positioning of the classical variable are FQNN, and those that do not admit a realization neither in the form of panel (b) nor with any other positioning of the classical variable are FNN. There exist correlations of both types that can be produced in the setup depicted in panel (c).

literature [18,25–27,31]. When the parties have binary inputs and outputs, the inequality derived in Ref. [18], commonly known as the bilocality inequality or the *I-J* inequality, holds for bilocal correlations. This is, that correlations that admit an expression in terms of Eq. (1) for the bilocality network satisfy

$$\sqrt{|I|} + \sqrt{|J|} \leqslant 1, \qquad (4)$$

where I and J are defined by $I = \frac{1}{4}\langle A_0 B^0 C_0 + A_1 B^0 C_0 + A_0 B^0 C_1 + A_1 B^0 C_1 \rangle$ and $J = \frac{1}{4}\langle A_0 B^1 C_0 - A_1 B^1 C_0 - A_0 B^1 C_1 + A_1 B^1 C_1 \rangle$, A_i and C_k are the (dichotomic) measurement operators of the extremal parties, B denotes one measurement with two-bit outcomes, depicted $b^0 b^1$, and the correlator $\langle A_x B^y C_z \rangle$ is given by $\sum_{a,b^0 b^1,c}(-1)^{a+b^y+c}P(a, b^0 b^1, c|x, z)$.

In this section we compute bounds on the left-hand side of Eq. (4) under the models considered in Definitions 1 and 2. The case of $\ell = 2$ is, as discussed before, the standard notion of network nonlocality given by Eq. (1). Let us then consider the case of $\ell = 1$, i.e., when only one of the sources distributes classical systems. Without loss of generality, let us assume that the source distributing classical systems is that between Alice and Bob. This implies that

$$\sqrt{|I|} + \sqrt{|J|} = \frac{1}{2}\sqrt{|\langle A_0 B^0 C_0 + A_1 B^0 C_0 + A_0 B^0 C_1 + A_1 B^0 C_1 \rangle|} + \frac{1}{2}\sqrt{|\langle A_0 B^1 C_0 - A_1 B^1 C_0 - A_0 B^1 C_1 + A_1 B^1 C_1 \rangle|}$$

$$\leqslant \frac{1}{2}\sqrt{|\langle A_0 + A_1 \rangle||\langle B^0 C_0 + B^0 C_1 \rangle|} + \frac{1}{2}\sqrt{|\langle A_0 - A_1 \rangle||\langle B^1 C_0 - B^1 C_1 \rangle|}. \qquad (5)$$

Now, assume that $\langle A_0 + A_1 \rangle \geqslant 0$ and define $x = \langle A_0 + A_1 \rangle - 1$, so that $x \in [-1, 1]$. Note that $|\langle A_0 - A_1 \rangle| + |\langle A_0 + A_1 \rangle| \leqslant 2 \max\{|\langle A_0 \rangle|, |\langle A_1 \rangle|\} \leqslant 2$. Therefore, we have that $|\langle A_0 - A_1 \rangle| \leqslant 2 - |\langle A_0 + A_1 \rangle| = 1 - x$. Using this, Eq. (5) reads

$$\sqrt{|I|} + \sqrt{|J|} \leqslant \frac{1}{2}(\sqrt{(1+x)|\langle B^0 C_0 + B^0 C_1 \rangle|} + \sqrt{(1-x)|\langle B^1 C_0 - B^1 C_1 \rangle|})$$

$$= \frac{1}{\sqrt{2}}(\cos\theta\sqrt{|\langle B^0 C_0 + B^0 C_1 \rangle|} + \sin\theta\sqrt{|\langle B^1 C_0 - B^1 C_1 \rangle|}) \qquad (6)$$

$$\leqslant \frac{1}{\sqrt{2}}\sqrt{|\langle B^0 C_0 + B^0 C_1 \rangle| + |\langle B^1 C_0 - B^1 C_1 \rangle|}, \qquad (7)$$

where in Eq. (6) we have rewritten $1 + x := 2\cos^2\theta$ and $1 - x := 2\sin^2\theta$, and in Eq. (7) we have used that $a\cos\theta + b\sin\theta \leqslant (a^2 + b^2)^{1/2}$. If, instead, $\langle A_0 + A_1 \rangle < 0$, one can define $x = \langle A_0 + A_1 \rangle + 1$ and the same result follows.

Next, note that the object B^y only considers the yth bit of Bob's output. One can thus view the correlators $\langle B^y C_z \rangle$ as generated by the following simulated, two-input, two-output distribution:

$$P_{\text{sim}}(b, c|y, z) = P(b = b^y, c|z) = \sum_{b^0, b^1} \delta_{b, b^y} P(b^0 b^1, c|z),$$

which is obtained by Bob outputting only the yth bit upon choosing measurement y. Moreover, it is easy to see that $|\langle B^0 C_0 + B^0 C_1 \rangle| + |\langle B^1 C_0 - B^1 C_1 \rangle|$ is the largest of $|\langle B^0 C_0 + B^0 C_1 + (B^1 C_0 - B^1 C_1) \rangle|$ and $|\langle B^0 C_0 + B^0 C_1 - (B^1 C_0 - B^1 C_1) \rangle|$. These are two variants of the Clauser–Horne–Shimony–Holt (CHSH) operator under relabeling of Charlie's input. Thus, it follows that

$$\sqrt{|I|} + \sqrt{|J|} \leqslant \frac{1}{\sqrt{2}} \sqrt{\text{CHSH}(B, C)}.$$

Therefore, the bounds on $\sqrt{|I|} + \sqrt{|J|}$ for FNN and FQNN are connected to the maximum value of the CHSH inequality [42] that Bob and Charlie can achieve. When Bob and Charlie share a classical physical system, $\text{CHSH}(B, C) \leqslant 2$ and thus any value $\sqrt{|I|} + \sqrt{|J|} > \sqrt{2}/\sqrt{2} = 1$ implies standard network nonlocality, as derived in Refs. [18,25]. If they share an arbitrary system only limited by the no-signaling principle, then $\text{CHSH}(B, C) \leqslant 4$, and any value $\sqrt{|I|} + \sqrt{|J|} > \sqrt{4}/\sqrt{2} = \sqrt{2}$ implies FNN, as derived in Ref. [35]. Finally, if the parties share a quantum system, then $\text{CHSH}(B, C) \leqslant 2\sqrt{2}$ and any value $\sqrt{|I|} + \sqrt{|J|} > \sqrt{2\sqrt{2}}/\sqrt{2} = 2^{1/4}$ implies FQNN. This is a proof that the bound first provided in Ref. [43] is tight.

The same argumentation can be done for the case where the classical source is shared between Bob and Charlie, arriving to the same inequality, $\sqrt{|I|} + \sqrt{|J|} \leqslant 2^{1/4}$. Therefore, this is a single inequality whose violation guarantees that none of the sources in the scenario is classical. This is in contrast with earlier works on detection of FNN [35,38], which needed to violate as many inequalities as sources in the network in order to guarantee that all of them were nonclassical. Moreover, both bounds are tight: the bound for FQNN is saturated by the strategy in Ref. [43] and the bound for FNN is saturated by the strategy in Ref. [35]. In fact, the bound for FNN coincides with the algebraic maximum of $\sqrt{|I|} + \sqrt{|J|}$, computed in Ref. [35]. This means that $\sqrt{|I|} + \sqrt{|J|}$ cannot be used for detecting FNN, but it can be used for detecting FQNN. Indeed, take the sources to distribute pure states, $\rho_i = |\phi_i\rangle\langle\phi_i|$, given by

$$|\phi_i\rangle = \cos\theta_i |00\rangle + \sin\theta_i |11\rangle, \qquad (8)$$

where $\theta_i \in (0, \frac{\pi}{2})$. Let also $A_x = \cos\vartheta\sigma_3 + (-1)^x \sin\vartheta\sigma_1$ and likewise for C_z, and $B^{b^0 b^1} \in \{\Phi_0, \Phi_1, \Psi_0, \Psi_1\}$, where σ_1 and σ_3 are the Pauli matrices, and $\rho = |\rho\rangle\langle\rho|$ with $|\Phi_i\rangle = (|00\rangle + (-1)^i |11\rangle)/\sqrt{2}$ and $|\Psi_i\rangle = (|01\rangle + (-1)^i |10\rangle)/\sqrt{2}$ being the Bell states. A straightforward evaluation leads to

$$I = \cos^2\vartheta, \quad J = \sin^2\vartheta \sin 2\theta_1 \sin 2\theta_2,$$

and we thus have that

$$\sqrt{|I|} + \sqrt{|J|} = \cos\vartheta + \sin\vartheta\sqrt{\sin 2\theta_1 \sin 2\theta_2}.$$

This quantity exceeds $2^{1/4}$, for instance, for any θ_i satisfying $\sin 2\theta_1 \sin 2\theta_2 \geqslant \sqrt{2} - 1 \approx 0.4142$, whenever choosing ϑ such that $\cos\vartheta = (1 + \sin 2\theta_1 \sin 2\theta_2)^{-1/2}$. Moreover, the maximum value over all possible $\theta_1, \theta_2, \vartheta$ is $\sqrt{2}$, which is the algebraic maximum.

B. Long chains

The bilocality inequality can be extended to chain networks of arbitrary length by using the notion of k-independent sets of parties [31],[1] and with it building appropriate operators I and J in order to construct Bell-like inequalities whose violations witness ℓ-NN and ℓ-QNN. Indeed, Ref. [31] showed that a suitable generalization of the I and J operators is

$$I = \langle \Pi_{\text{odd } i} A^+_{x_i} \Pi_{\text{even } j} A_{x_j = 0} \rangle,$$

$$J = \langle \Pi_{\text{odd } i} A^-_{x_i} \Pi_{\text{even } j} A_{x_j = 1} \rangle,$$

with $A^\pm_{x_i} = (A_{x_i=0} \pm A_{x_i=1})/2$, and the correlators representing the combination of probabilities given by $\langle A_{x_1} \cdots A_{x_n} \rangle = \sum_{\vec{a}} (-1)^{\sum_{i=1}^n a_i} P(\vec{a}|\vec{x})$. Reference [31] proved that for any k-independent network the correlations generated by classical sources satisfy the Bell inequality $|I|^{\frac{1}{k}} + |J|^{\frac{1}{k}} \leqslant 1$, and that this inequality admits quantum violations using maximally entangled and GHZ states.

In the following we analyze the case where the network contains an odd number of parties and the nonclassical sources are required to follow quantum theory. In other words, we restrict to the case of ℓ-QNN in chain networks with an odd number of parties. We defer the analysis of the remaining cases, namely, calculations for ℓ-NN and for chain networks with an even number of parties, to Appendix A. We furthermore distinguish two subcases, namely, when the network contains $\ell \geqslant (n-1)/2$ classical sources, and when it contains $\ell < (n-1)/2$ classical sources.

Case $\ell \geqslant (n-1)/2$. In this case, there are more classical than quantum sources. This implies that there are no more than $(n-1)/2$ nonadjacent pairs of parties (A_i, A_{i+1}) who share a quantum system, and thus, the network is a k-independent network [31] with $k \leqslant (n-1)/2$. Let S consist of all odd integers, i, such that A_i receives only classical sources. The fact that there are more classical than quantum sources implies that $|S| \geqslant 1$, and from the classicality of the measurements of A_i when $i \in S$ it follows that

$$I^{\frac{2}{n+1}} + J^{\frac{2}{n+1}} = \left\langle \prod_{\text{odd } i} A^+_{x_i} \prod_{\text{even } j} A_{x_j = 0} \right\rangle^{\frac{2}{n+1}} + \left\langle \prod_{\text{odd } i} A^-_{x_i} \prod_{\text{even } j} A_{x_j = 1} \right\rangle^{\frac{2}{n+1}}$$

$$\leqslant \left\langle \prod_{i \in S} A^+_{x_i} \right\rangle^{\frac{2}{n+1}} \left\langle \prod_{\text{odd } i; i \notin S} A^+_{x_i} \prod_{\text{even } j} A_{x_j = 0} \right\rangle^{\frac{2}{n+1}}$$

$$+ \left\langle \prod_{i \in S} A^-_{x_i} \right\rangle^{\frac{2}{n+1}} \left\langle \prod_{\text{odd } i; i \notin S} A^-_{x_i} \prod_{\text{even } j} A_{x_j = 1} \right\rangle^{\frac{2}{n+1}}, \qquad (9)$$

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[1] As defined in Ref. [31], k independence indicates that the maximum number of parties in the network that share no source with each other is k.

where, recall, $A_{x_i}^{\pm} = \frac{1}{2}(A_{x_i=0} \pm A_{x_i=1})$.

Reference [31] gave a quantum bound in the k-independent network, which when $k = \frac{n-1}{2}$ reads

$$\left\langle \prod_{\text{odd } i; i \notin S} A_{x_i}^+ \prod_{\text{even } j} A_{x_j=0} \right\rangle^{\frac{2}{n-1}} + \left\langle \prod_{\text{odd } i; i \notin S} A_{x_i}^- \prod_{\text{even } j} A_{x_j=1} \right\rangle^{\frac{2}{n-1}} \leqslant \sqrt{2}.$$

Considering this, and taking into account that $0 \leqslant \langle \prod_{\text{odd } i; i \notin S} A_{x_i}^+ \prod_{\text{even } j} A_{x_j=0} \rangle^{\frac{2}{n-1}} \leqslant 1$ and that $0 \leqslant$

$\langle \prod_{\text{odd } i; i \notin S} A_{x_i}^- \prod_{\text{even } j} A_{x_j=1} \rangle^{\frac{2}{n-1}} \leqslant 1$, it is possible to write

$$\left\langle \prod_{\text{odd } i; i \notin S} A_{x_i}^+ \prod_{\text{even } j} A_{x_j=0} \right\rangle^{\frac{2}{n-1}} \leqslant \sqrt{2} \cos^2 \theta,$$

$$\left\langle \prod_{\text{odd } i; i \notin S} A_{x_i}^- \prod_{\text{even } j} A_{x_j=1} \right\rangle^{\frac{2}{n-1}} \leqslant \sqrt{2} \sin^2 \theta \qquad (10)$$

for some $\theta \in [0, \pi/2]$. Moreover, using the fact that $|\langle A_{x_i} \rangle| \leqslant 1$ when $i \in S$ it follows that

$$\left| \left\langle \prod_{i \in S} A_{x_i}^+ \pm \prod_{i \in S} A_{x_i}^- \right\rangle \right| \leqslant \frac{1}{2^{|S|-1}} \max \left\{ \left| \sum_{(x_i, i \in S); \oplus_{i \in S} x_i = 0} \left\langle \prod_{i \in S} A_{x_i} \right\rangle \right|, \left| \sum_{(x_i, i \in S); \oplus_{i \in S} x_i = 1} \left\langle \prod_{i \in S} A_{x_i} \right\rangle \right| \right\} \leqslant \frac{1}{2^{|S|-1}} \times 2^{|S|-1} = 1$$

because $|\{(x_i, i \in S) | \oplus_{i \in S} x_i = 0\}| = |\{(x_i, i \in S) | \oplus_{i \in S} x_i = 1\}| = 2^{|S|-1}$, the operation $\oplus$ representing addition modulo 2. It is thus possible to write $|\langle \prod_{i \in S} A_{x_i}^+ \rangle| \leqslant \cos^2 \vartheta$ and $|\langle \prod_{i \in S} A_{x_i}^- \rangle| \leqslant \sin^2 \vartheta$ for some $\vartheta \in [0, \pi/2]$. Substituting all the above in Eq. (9) it follows that

$$I^{\frac{2}{n+1}} + J^{\frac{2}{n+1}} \leqslant 2^{\frac{n-1}{2(n+1)}} \left[\left\langle \prod_{i \in S} A_{x_i}^+ \right\rangle^{\frac{2}{n+1}} \cos^{\frac{2n-2}{n+1}} \theta + \left\langle \prod_{i \in S} A_{x_i}^- \right\rangle^{\frac{2}{n+1}} \sin^{\frac{2n-2}{n+1}} \theta \right] \leqslant 2^{\frac{n-1}{2(n+1)}} \left(\cos^{\frac{4}{n+1}} \vartheta \cos^{\frac{2n-2}{n+1}} \theta + \sin^{\frac{4}{n+1}} \vartheta \sin^{\frac{2n-2}{n+1}} \theta \right). \quad (11)$$

The unique extreme point of the continuous function $\cos^{\frac{4}{n+1}} \vartheta \cos^{\frac{2n-2}{n+1}} \theta + \sin^{\frac{4}{n+1}} \vartheta \sin^{\frac{2n-2}{n+1}} \theta$ is given by $\tan \vartheta = \tan \theta$. By defining $\vartheta := \theta$, Eq. (11) reads

$$I^{\frac{2}{n+1}} + J^{\frac{2}{n+1}} \leqslant 2^{\frac{n-1}{2(n+1)}} (\cos^2 \theta + \sin^2 \theta) = 2^{\frac{n-1}{2(n+1)}}, \qquad (12)$$

where the equality holds if and only if $\vartheta = \theta = \pi/4$.

Case $\ell < (n-1)/2$. An example of this is, e.g., when the sources connecting parties A_{2i} and A_{2i+1} distribute classical variables, and the sources connecting parties A_{2i-1} and A_{2i} distribute quantum systems [recall that $i = 1, \ldots, (n+1)/2$]. This network, and in fact any network with fewer than half of its sources being classical, is a $(n+1)/2$-independent network [31], and so the joint correlations satisfy the inequality

$$I^{\frac{2}{n+1}} + J^{\frac{2}{n+1}} \leqslant \sqrt{2}, \qquad (13)$$

where the equality holds if all the bipartite quantum sources are maximally entangled EPR states [2].

In summary, we have that for chain networks composed of an odd number of parties, ℓ-QNL correlations satisfy the following inequality:

$$I^{\frac{2}{n+1}} + J^{\frac{2}{n+1}} \leqslant \begin{cases} 2^{\frac{n-1}{2(n+1)}} & \ell \geqslant \frac{n-1}{2} \\ \sqrt{2} & \ell < \frac{n-1}{2}. \end{cases} \qquad (14)$$

There are a few aspects to note of this inequality. First, the bounds obtained are valid for any ℓ within each of the ranges, and thus, a violation guarantees the strongest case (namely, any value above $2^{\frac{n-1}{2(n+1)}}$ guarantees $\frac{n-1}{2}$-QNN and any value above $\sqrt{2}$ guarantees FQNN). Second, the bound for $\ell \geqslant \frac{n-1}{2}$ increases with n, which means that it is more demanding to demonstrate ℓ-QNN with increasing n. Third, the gap between the two bounds closes in the limit of $n \to \infty$. Thus, in infinite chain networks demonstrating nonclassicality of even just one

of the sources is as demanding as demonstrating nonclassicality of all of them. Finally, $\sqrt{2}$ is the maximum value achievable with quantum sources [31], so Eq. (14) cannot be used to witness FQNN. In Appendix A we derive analogous inequalities in the case when the chain has an even number of parties, and that are satisfied by ℓ-NL correlations.

IV. NONLINEAR ℓ-(Q)NN HIERARCHIES FOR STAR NETWORKS

The bilocality inequality has also been generalized to star networks [26] [recall Fig. 1(c)], whereby n branch parties, $\mathsf{A}_1, \ldots, \mathsf{A}_n$, share each a bipartite system with a central node, B. The symmetry of this setup with respect to permutations of the branch parties allows for easier treatment, that leads to families of inequalities detecting ℓ-(Q)NN for each ℓ.

Let us begin with the analysis of ℓ-QNN for fixed ℓ. In this case, we set that the central party can perform two measurements B_y, $y \in \{0, 1\}$, with binary outcomes, $b \in \{0, 1\}$. In this setup, the quantities I and J are given by $I = \langle B_0 \Pi_{i=1}^n A_{x_i}^+ \rangle$ and $J = \langle B_1 \Pi_{i=1}^n A_{x_i}^- \rangle$ [26]. Consider thus the network in Fig. 3(a), where there are ℓ parties sharing classical variables with B while the remaining ones share quantum states. Without loss of generality, one can assume that said parties are $\mathsf{A}_1, \ldots, \mathsf{A}_\ell$. It follows that

$$|I|^{\frac{1}{n}} + |J|^{\frac{1}{n}} = \left| \left\langle \prod_{i=1}^\ell A_{x_i}^+ \right\rangle \right|^{\frac{1}{n}} \left| \left\langle B_0 \prod_{j=\ell+1}^n A_{x_j}^+ \right\rangle \right|^{\frac{1}{n}}$$
$$+ \left| \left\langle \prod_{i=1}^\ell A_{x_i}^- \right\rangle \right|^{\frac{1}{n}} \left| \left\langle B_1 \prod_{j=\ell+1}^n A_{x_j}^- \right\rangle \right|^{\frac{1}{n}}, \qquad (15)$$

where, as in previous sections, $A_{x_i}^{\pm} = (A_{x_i=0} \pm A_{x_i=1})/2$. For any set of quantum observables $A_{x_i}^{\pm}$, the following inequality
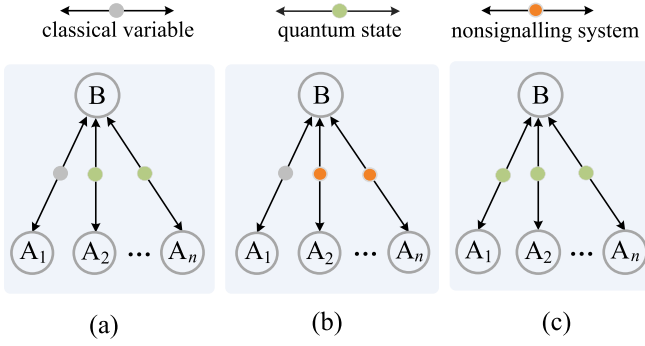
FIG. 3. The $(n + 1)$-partite star network. Correlations that do not admit a realization neither in the form of panel (a) nor with any other positioning of the classical variable are FQNN, and those that do not admit a realization neither in the form of panel (b) nor with any other positioning of the classical variable are FNN. There exist correlations of both types that can be produced in the setup depicted in panel (c).

holds:

$$\left| \left\langle B_0 \prod_{j=\ell+1}^{n} A_{x_j}^+ \right\rangle \right|^{\frac{1}{n-\ell}} + \left| \left\langle B_1 \prod_{j=\ell+1}^{n} A_{x_j}^- \right\rangle \right|^{\frac{1}{n-\ell}} \leqslant \sqrt{2}.$$

This follows from the quantum bound of the nonlinear Bell-type inequality for the $(n - k)$-independent quantum network in Ref. [31]. Therefore, one can write that

$$\left| \left\langle B_0 \prod_{j=\ell+1}^{n} A_{x_j}^+ \right\rangle \right|^{\frac{1}{n-\ell}} \leqslant \sqrt{2} \cos^2 \theta,$$

$$\left| \left\langle B_1 \prod_{j=\ell+1}^{n} A_{x_j}^- \right\rangle \right|^{\frac{1}{n-\ell}} \leqslant \sqrt{2} \sin^2 \theta \qquad (16)$$

for some $\theta \in [0, \pi/2]$. Additionally, since all the parties $A_1, \ldots, A_\ell$ are independent, we have the classical bounds $|\langle \prod_{i=1}^{\ell} A_{x_i}^{\pm} \rangle| \leqslant \prod_{i=1}^{\ell} |\langle A_{x_i}^{\pm} \rangle|$. In combination with $|\langle A_{x_i}^+ \pm A_{x_i}^- \rangle| = |\langle A_{x_i} \rangle| \leqslant 1$, one can define $|\langle \prod_{i=1}^{\ell} A_{x_i}^+ \rangle| \leqslant \prod_{i=1}^{\ell} \cos^2 \vartheta_i$ and $|\langle \prod_{i=1}^{\ell} A_{x_i}^- \rangle| \leqslant \prod_{i=1}^{\ell} \sin^2 \vartheta_i$ for some $\vartheta_i \in [0, \pi/2]$. Substituting this into Eq. (15) we have that

$$|I|^{\frac{1}{n}} + |J|^{\frac{1}{n}}$$
$$= 2^{\frac{n-\ell}{2n}} \left(\prod_{i=1}^{\ell} \cos^{\frac{2}{n}} \vartheta_i \cos^{\frac{2n-2\ell}{n}} \theta + \prod_{i=1}^{\ell} \sin^{\frac{2}{n}} \vartheta_i \sin^{\frac{2n-2\ell}{n}} \theta \right).$$

The maximum of the right-hand side is achieved at $\vartheta_1 = \cdots = \vartheta_\ell := \vartheta$. This implies that

$$|I|^{\frac{1}{n}} + |J|^{\frac{1}{n}} \leqslant 2^{\frac{n-\ell}{2n}} (\cos^{\frac{2\ell}{n}} \vartheta \cos^{\frac{2n-2\ell}{n}} \theta + \sin^{\frac{2\ell}{n}} \vartheta \sin^{\frac{2n-2\ell}{n}} \theta).$$

Finally, the continuous function $\cos^{\frac{2\ell}{n}} \vartheta \cos^{\frac{2n-2\ell}{n}} \theta + \sin^{\frac{2\ell}{n}} \vartheta \sin^{\frac{2n-2\ell}{n}} \theta$ has a unique extreme point, given by $\vartheta = \theta$. We thus have that

$$|I|^{\frac{1}{n}} + |J|^{\frac{1}{n}} \leqslant 2^{\frac{n-\ell}{2n}} (\cos^2 \theta + \sin^2 \theta) = 2^{\frac{n-\ell}{2n}}. \qquad (17)$$

The right-hand side of Eq. (17) has a different value for each ℓ, thereby giving a hierarchy of inequalities that detects ℓ-QNN for every $\ell \in [1, \ldots, m]$.

Consider now the situation when there are ℓ extremal parties (e.g., $A_1, \ldots, A_\ell$) that share classical randomness with B while the remaining parties share nonsignaling systems. In this case, the bounds in Eq. (16) are replaced by $|\langle B_0 \Pi_{i>\ell} A_{x_i}^+ \rangle| \leqslant 1$, $|\langle B_1 \Pi_{i>\ell} A_{x_i}^- \rangle| \leqslant 1$, since there exist distributions built out of PR boxes that saturate these bounds, in analogy with the construction in Appendix A 2 for chain networks. We thus have that

$$|I|^{\frac{1}{n}} + |J|^{\frac{1}{n}} \leqslant \left| \left\langle \prod_{i=1}^{\ell} A_{x_i}^+ \right\rangle \right|^{\frac{1}{n}} + \left| \left\langle \prod_{i=1}^{\ell} A_{x_i}^- \right\rangle \right|^{\frac{1}{n}}$$

$$\leqslant \prod_{i=1}^{\ell} \cos^{\frac{2}{n}} \vartheta_i + \prod_{i=1}^{\ell} \sin^{\frac{2}{n}} \vartheta_i \qquad (18)$$

$$:= \cos^{\frac{2\ell}{n}} \vartheta + \sin^{\frac{2\ell}{n}} \vartheta \qquad (19)$$

$$\leqslant 2^{\frac{n-\ell}{n}}, \qquad (20)$$

where we have used the fact that $|\langle A_{x_i}^+ \rangle| \leqslant \cos^2 \vartheta_i$ and $|\langle A_{x_i}^- \rangle| \leqslant \sin^2 \vartheta_i$ for some ϑ_i in Eq. (18), which stem from the inequality $|\langle A_{x_i}^+ \pm A_{x_i}^- \rangle| \leqslant 1$. Inequality (19) follows in a way analogous to that of ℓ-QNN studied above, and Eq. (20) follows from computing the unique extreme point of the continuous function $\cos^{\frac{2\ell}{n}} \vartheta + \sin^{\frac{2\ell}{n}} \vartheta$, which is $\vartheta = \pi/4$. We thus, again, obtain a hierarchy of bounds that detects ℓ-NN for every ℓ. Note also that, for any placement of the classical sources, the same inequality is obtained, and thus a violation of the single inequality (20) is a certification of ℓ-NN.

Both sets of inequalities, Eqs. (17) and (20), can be violated in quantum mechanics. If each of the sources distributes a bipartite quantum system in the state given by Eq. (8), and the parties perform the measurements given by $A_{x_i} = \cos \vartheta \sigma_3 + (-1)^{x_i} \sin \vartheta \sigma_1$ and $B_y = (1 - y)\sigma_3^{\otimes n} + y \sigma_1^{\otimes n}$ [31] one has that

$$|I|^{\frac{1}{n}} + |J|^{\frac{1}{n}} = \cos \vartheta + \sin \vartheta \prod_{i=1}^{n} \sin^{\frac{1}{n}} 2\theta_i.$$

If we choose ϑ such that $\cos \vartheta = (1 + \prod_{i=1}^{n} \sin^{\frac{2}{n}} 2\theta_i)^{-\frac{1}{2}}$, we have a violation of Eq. (17) whenever $\prod_{i=1}^{n} \sin^{\frac{2}{n}} 2\theta_i > 2^{\frac{n-\ell}{n}} - 1$, and a violation of Eq. (20) when $\prod_{i=1}^{n} \sin^{\frac{2}{n}} 2\theta_i > 2^{\frac{2n-2\ell}{n}} - 1$. Take as an example the case when the sources distribute maximally entangled EPR states, i.e., $\theta_i = \pi/4 \ \forall \ i$. In this case the value achieved is $|I|^{\frac{1}{n}} + |J|^{\frac{1}{n}} = \sqrt{2}$. This is larger than $2^{\frac{n-\ell}{n}}$ for all $\ell > n/2$, so Eq. (20) is useful for verifying ℓ-NN in such star networks only for $\ell > n/2$.

V. LINEAR FULL NETWORK NONLOCALITY INEQUALITIES FOR CHAINS

Up to now, we have used nonlinear inequalities to detect ℓ-NN and ℓ-QNN. On one hand, nonlinear inequalities are useful because they approximate better the boundaries of the relevant sets, which are known to be nonlinear [25]. However, the nonlinearities make it difficult to optimize these inequalities and to find explicit models. In this section and in the following one we present linear inequalities whose violations detect FNN. For deriving them, we exploit the fact that realizing entanglement swapping is not possible when parties

are connected by classical sources. It is known that at least some supra-quantum theories do not allow for entanglement swapping [44], and this fact has been exploited to develop NN inequalities in the tripartite chain scenario [45]. However, we are interested in guaranteeing nonclassical phenomena, so as long as there exists a nonclassical theory that allows for entanglement swapping (as it is the case of quantum mechanics), the inequalities below can, in principle, be used for detecting FNN.

A. The tripartite chain

Let us begin, as in Sec. III, with analyzing the tripartite chain scenario, depicted in Fig. 1(a). A way to guarantee that both sources are not classical is by completing entanglement swapping, so that after Bob's measurement the joint state of Alice and Charlie can violate the CHSH inequality. Using this fact, consider the following operator, where Alice can perform two dichotomic measurements, Bob performs a single, four-outcome measurement, and Charlie can perform three dichotomic measurements:

$$
\begin{aligned}
\mathcal{B}_3 := & A_0 B^0 C_0 + A_0 B^1 C_0 + A_0 B^0 C_1 - A_0 B^1 C_1 \\
& + A_1 B^0 C_0 + A_1 B^1 C_0 - A_1 B^0 C_1 + A_1 B^1 C_1 \\
& + A_0 B^2 C_1 - A_0 B^3 C_1 + A_0 B^2 C_2 + A_0 B^3 C_2 \\
& - A_1 B^2 C_1 + A_1 B^3 C_1 + A_1 B^2 C_2 + A_1 B^3 C_2.
\end{aligned}
$$

In the case where Bob and Charlie share classical randomness, the operator $\mathcal{B}_3$ is a linear combination of Charlie's operators, and since Charlie only receives classical shared randomness we can, without loss of generality, assume deterministic outputs and have $C_i \in \{\pm 1\}$. By substituting all possibilities for C_i, one arrives at

$$
\begin{aligned}
\langle \mathcal{B}_3 \rangle = & \langle C_0 (B^0 A_0 + B^0 A_1 + B^1 A_0 + B^1 A_1) \rangle \\
& + \langle C_1 (B^0 A_0 - B^0 A_1 - B^1 A_0 + B^1 A_1) \rangle \\
& + \langle C_1 (B^2 A_0 - B^2 A_1 - B^3 A_0 + B^3 A_1) \rangle \\
& + \langle C_2 (B^2 A_0 + B^2 A_1 + B^3 A_0 + B^3 A_1) \rangle \\
\leqslant & \max\{ \pm \langle 2(B^0 + B^2)A_0 + 2(B^1 + B^3)A_1 \rangle, \\
& \pm \langle 2(B^0 + B^2)A_1 + 2(B^1 + B^3)A_0 \rangle, \\
& \pm \langle 2(B^0 - B^3)A_1 + 2(B^1 - B^2)A_0 \rangle, \\
& \pm \langle 2(B^3 - B^0)A_1 + 2(B^2 - B^1)A_1 \rangle \} \\
\leqslant & 2,
\end{aligned}
$$

where the last bound can be found by individually obtaining the maximum value for each case, using $|\langle A_i \rangle| \leqslant 1$ and $\sum_{i=0}^{3} B^i = \mathbb{1}$.

Similarly, consider the alternative scenario, where Alice and Bob share the classical source. Since $\mathcal{B}_3$ is also linear in Alice's measurements, one can proceed analogously by assuming deterministic outputs for Alice, so $A_i \in \{\pm 1\}$, arriving at

$$
\begin{aligned}
\langle \mathcal{B}_3 \rangle = & \langle A_0 (B^0 C_0 + B^1 C_0 + B^0 C_1 - B^1 C_1) \rangle \\
& + \langle A_1 (B^0 C_0 + B^1 C_0 - B^0 C_1 + B^1 C_1) \rangle \\
& + \langle A_0 (B^2 C_1 - B^3 C_1 + B^2 C_2 + B^3 C_2) \rangle
\end{aligned}
$$

$$
\begin{aligned}
& + \langle A_1 (-B^2 C_1 + B^3 C_1 + B^2 C_2 + B^3 C_2) \rangle \\
\leqslant & \max\{ \pm \langle 2(B^0 + B^1)C_0 + 2(B^2 + B^3)C_2 \rangle, \\
& \pm \langle 2(B^0 - B^1)C_1 + 2(B^2 - B^3)C_2 \} \} \\
\leqslant & 2.
\end{aligned}
$$

This means that the inequality

$$
\begin{aligned}
& \langle A_0 B^0 C_0 + A_0 B^1 C_0 + A_0 B^0 C_1 - A_0 B^1 C_1 \rangle \\
& + \langle A_1 B^0 C_0 + A_1 B^1 C_0 - A_1 B^0 C_1 + A_1 B^1 C_1 \rangle \\
& + \langle A_0 B^2 C_1 - A_0 B^3 C_1 + A_0 B^2 C_2 + A_0 B^3 C_2 \rangle \\
& - \langle A_1 B^2 C_1 + A_1 B^3 C_1 + A_1 B^2 C_2 + A_1 B^3 C_2 \rangle \\
& \leqslant 2 \qquad\qquad (21)
\end{aligned}
$$

holds whenever one of the sources distributes classical systems, regardless of the nature of the other and of which specific source is the one distributing the classical systems. Its maximal quantum value is $2\sqrt{2}$, which can be derived directly from the upper bound to the CHSH inequality. Moreover, quantum strategies based on maximally entangled states saturate this bound (see Appendix B for an example realization).

B. Long chains

The reasoning above can be generalized to chain networks of arbitrary length, by exploiting the fact that nonclassical correlations between the extremal parties can only be established if all the sources are nonclassical. Consider thus the following operator:

$$
\begin{aligned}
\mathcal{B}_n := & A_0^{(1)} B^{00} A_0^{(n)} + A_0^{(1)} B^{01} A_0^{(n)} + A_0^{(1)} B^{00} A_1^{(n)} \\
& - A_0^{(1)} B^{01} A_1^{(n)} + A_1^{(1)} B^{00} A_0^{(n)} + A_1^{(1)} B^{01} A_0^{(n)} \\
& - A_1^{(1)} B^{00} A_1^{(n)} + A_1^{(1)} B^{01} A_1^{(n)} + A_0^{(1)} B^{10} A_1^{(n)} \\
& - A_0^{(1)} B^{11} A_1^{(n)} + A_0^{(1)} B^{10} A_2^{(n)} + A_0^{(1)} B^{11} A_2^{(n)} \\
& - A_1^{(1)} B^{10} A_1^{(n)} + A_1^{(1)} B^{11} A_1^{(n)} + A_1^{(1)} B^{10} A_2^{(n)} \\
& + A_1^{(1)} B^{11} A_2^{(n)},
\end{aligned}
$$

where the extremal parties are A_1 and A_n, which have operators $A_i^{(1)}$ and $A_i^{(n)}$, respectively, and B^{st} are products of operators of all the nonextremal parties $\mathsf{A}_2, \ldots, \mathsf{A}_{n-1}$, given by $B^{st} = \sum_{\oplus_{k=2}^{n-1} i_k j_k = st} M^{i_2 j_2} \cdots M^{i_{n-1} j_{n-1}}$, with $\oplus$ denoting the bitwise XOR operation, i.e., $xy \oplus uv = (x \oplus u, y \oplus v)$, and $M^{i_k j_k}$ are the projectors corresponding to the single measurement of party A_k. Similarly to Eq. (21), the extremal party A_n has three dichotomic measurements.

Let us begin by assuming that the classical source is that connecting A_1 and A_2. This network can be understood as a tripartite chain network where the central party is composed of parties $\mathsf{A}_2, \ldots, \mathsf{A}_{n-1}$. Then, following the previous section, we can assume deterministic outputs for $A_{x_1}^{(1)} \in \{\pm 1\}$, finding that

$$
\begin{aligned}
\langle \mathcal{B}_n \rangle = & \left\langle A_0^{(1)} \left(B^{00} A_0^{(n)} + B^{01} A_0^{(n)} + B^{00} A_1^{(n)} - B^{01} A_1^{(n)} \right) \right\rangle \\
& + \left\langle A_1^{(1)} \left(B^{00} A_0^{(n)} + B^{01} A_0^{(n)} - B^{00} A_1^{(n)} + B^{01} A_1^{(n)} \right) \right\rangle \\
& + \left\langle A_0^{(1)} \left(B^{10} A_1^{(n)} - B^{11} A_1^{(n)} + B^{10} A_2^{(n)} + B^{11} A_2^{(n)} \right) \right\rangle \\
& + \left\langle A_1^{(1)} \left(-B^{10} A_1^{(n)} + B^{11} A_1^{(n)} + B^{10} A_2^{(n)} + B^{11} A_2^{(n)} \right) \right\rangle
\end{aligned}
$$

$$\leqslant \max\left\{ \pm\langle 2(B^{00}+B^{01})A_0^{(n)}+2(B^{10}+B^{11})A_2^{(n)}\rangle,\right.$$
$$\left.\pm\langle 2(B^{00}-B^{01})A_1^{(n)}+2(B^{10}-B^{11})A_2^{(n)}\rangle\right\}$$
$$\leqslant 2,$$

where the terms in the maximization are found by applying the triangle inequality and forming groups that can be jointly maximized. Then, the bound follows from optimizing individually each of the elements under the constraints $\sum_{b_0,b_1} B^{b_0 b_1} = \mathbb{1}$ and $|\langle A_i^{(n)}\rangle| \leqslant 1$.

Now consider the case where the classical source is not in the extreme of the chain. In that case, we can rewrite $\mathcal{B}_n$ as

$$\mathcal{B}_n = B^{00}\big(A_0^{(1)}A_0^{(n)} + A_0^{(1)}A_1^{(n)} + A_1^{(1)}A_0^{(n)} - A_1^{(1)}A_1^{(n)}\big)$$
$$+ B^{01}\big(A_0^{(1)}A_0^{(n)} - A_0^{(1)}A_1^{(n)} + A_1^{(1)}A_0^{(n)} + A_1^{(1)}A_1^{(n)}\big)$$
$$+ B^{10}\big(A_0^{(1)}A_1^{(n)} - A_0^{(1)}A_2^{(n)} + A_1^{(1)}A_1^{(n)} + A_1^{(1)}A_2^{(n)}\big)$$
$$+ B^{11}\big(-A_0^{(1)}A_1^{(n)} + A_0^{(1)}A_2^{(n)} + A_1^{(1)}A_1^{(n)} + A_1^{(1)}A_2^{(n)}\big).$$

Each line above corresponds to a different CHSH test, weighted by a different probability of obtaining certain outcomes in the central parties (recall that the B^{st} are defined by $B^{st} = \sum_{\oplus_{\ell=2}^{n-1} i_\ell j_\ell = st} M^{i_2 j_2} \cdots M^{i_{n-1} j_{n-1}}$, with $\oplus$ being the bitwise XOR). Therefore, the expectation value of each of the lines above can be bounded by $2 \sum_{\oplus_{i=2}^{n-1} a_i=00} p(a_2, \ldots, a_{n-1})$, and using the fact that $\sum_{i=2}^{n-1} \sum_{a_i} p(a_2, \ldots, a_{n-1}) = 1$ the upper bound of 2 follows. We are thus left with the inequality

$$\langle\mathcal{B}_n\rangle = \big\langle A_0^{(1)}\big[(B^{00}+B^{01})A_0^{(n)} + (B^{00}-B^{01})A_1^{(n)}\big]\big\rangle$$
$$+ \big\langle A_1^{(1)}\big[(B^{00}+B^{01})A_0^{(n)} + (B^{00}-B^{01})A_1^{(n)}\big]\big\rangle$$
$$+ \big\langle A_0^{(1)}\big[(B^{10}-B^{11})A_1^{(n)} + (B^{10}+B^{11})A_2^{(n)}\big]\big\rangle$$
$$+ \big\langle A_1^{(1)}\big[(-B^{10}+B^{11})A_1^{(n)} + (B^{10}+B^{11})A_2^{(n)}\big]\big\rangle$$
$$\leqslant 2. \tag{22}$$

The maximal quantum bound for Eq. (22) is, as before, $2\sqrt{2}$ from the CHSH inequality. As in the previous cases, this is a single inequality whose violation certifies that all sources in the network are not classical, i.e., it detects FNN in chain networks. Moreover, it can be violated within quantum mechanics, as we also show in Appendix B.

VI. LINEAR FULL NETWORK NONLOCALITY INEQUALITIES FOR STARS

It is possible to use the results of the previous section to witness FNN in star networks, in two different ways. The first one is by decomposing the star network into a collection of tripartite chain subnetworks comprised by the parties $(A_i, A_{j\neq i}, B)$, and testing Eq. (21) on each of them. This leads to witnesses in scenarios where the central party performs a single, four-outcome measurement, and the branch parties can perform one of either two or three measurements, depending on the particular inequalities that are used for the testing.

The second one is by exploiting the same principle, namely, that only measurements on nonclassical systems can produce nonlocal correlations. In this case, we are interested in detecting genuinely multipartite nonlocality in the branch parties after the measurement of the central party. One way of

detecting genuine multipartite nonlocality for n parties is via the Mermin-Svetlichny inequality [9,46,47]:

$$\left\langle\mathcal{S}\left(A_{x_1}^{(\vec{i})}, \ldots, A_{x_n}^{(\vec{i})}\right)\right\rangle \leqslant 2^{n-1}, \tag{23}$$

where the operator $\mathcal{S}(A_1, \ldots, A_n) \equiv S_n$ is built recursively via

$$S_n = (A_{x_n=0} + A_{x_n=1})S_{n-1} + (A_{x_n=0} - A_{x_n=1})S'_{n-1},$$

with $S_1 = A_{x_1=0}$, and S'_k being obtained from S_k by flipping all the inputs of all the parties. By creating one such operator for every outcome of the central party B, one obtains the inequality

$$\left\langle\sum_{\vec{i}} M^{\vec{i}}\mathcal{S}^{(\vec{i})}\left(A_{x_1}^{(\vec{i})}, \ldots, A_{x_n}^{(\vec{i})}\right)\right\rangle \leqslant 2^{n-1}, \tag{24}$$

where $\{M^{\vec{i}}\}$ denotes the projection measurement of B with outcome $\vec{i} := i_1, \ldots, i_n$, and $A_{x_j}^{(\vec{i})}$ represents the corresponding measurement for party A_j. If one of the sources in the network is classical, then the measurement of the central party projects the joint state of the extremal parties into a biseparable-type state, thus satisfying Eq. (23) for any measurements $A_{x_1}^{(\vec{i})}, \ldots, A_{x_n}^{(\vec{i})}$ [9,46,47], and thus satisfying Eq. (24).

Equivalently, from the quantum bound of the Mermin-Svetlichny operator it is easy to see that the maximum quantum value of Eq. (24) is $2^{n-1}\sqrt{2}$. A violation of Eq. (24) can be achieved, for instance, having the sources distribute states of the form in Eq. (8). Take the measurement of B to be given by $\{M^{\vec{i}}\}$ with

$$M^{\vec{i}} = |\phi_{0i_{[n-1]}}^{i_1}\rangle\langle\phi_{0i_{[n-1]}}^{i_1}|,$$

where $i_{[n-1]} := i_2 \ldots i_n$, $|\phi_{0i_{[n-1]}}^{i_1}\rangle = \frac{1}{\sqrt{2}}(|0, i_{[n-1]}\rangle + (-1)^{i_1}|1, j_{[n-1]}\rangle)$, and $j_{[n-1]}$ denotes the complement bit series of $i_{[n-1]}$, i.e., $i_{[n-1]} \oplus j_{[n-1]} = 1 \ldots 1$ with $\oplus$ being the bitwise XOR. For each measurement outcome $\vec{i}$, with probability $p(\vec{i})$, the joint system of particles owned by all A_js will collapse into

$$|\Phi_{0i_{[n-1]}}\rangle = \gamma_{0i_{[n-1]}}|0, i_{[n-1]}\rangle + (-1)^{i_1}\delta_{1i_{[n-1]}}|1, j_{[n-1]}\rangle,$$

where $\gamma_{0i_{[n-1]}}$ and $\delta_{1i_{[n-1]}}$ are defined by

$$\gamma_{0i_{[n-1]}} = \frac{1}{\sqrt{p(\vec{i})}}\alpha_0\prod_{j=2}^{n}\alpha_{i_j}, \quad \delta_{1i_{[n-1]}} = \frac{1}{\sqrt{p(\vec{i})}}\alpha_1\prod_{j=2}^{n}\alpha_{\bar{i}_j},$$

with $p(\vec{i}) = \alpha_0^2\prod_{j=2}^{n}\alpha_{i_j}^2 + \alpha_1^2\prod_{j=2}^{n}\alpha_{\bar{i}_j}^2$, $\alpha_{i_j=0} = \cos\theta_j$, and $\alpha_{i_j=1} = \sin\theta_j$, $j = 1, \ldots, n$. This type of state can be transformed into generalized GHZ states [48] by using local unitary operations $U_{i_2}, \ldots, U_{i_n}$, i.e., $|\text{GHZ}_{0i_{[n-1]}}\rangle := \otimes_{j=1}^{n}U_{i_j}|\Phi_{0i_{[n-1]}}\rangle$.

Now, define the quantum observables of A_j conditional on the outcome $\vec{i}$ as $A_{a_j=i_j}^{(i_{[n-1]})} \in \{U_{i_j}^{\dagger}(\cos\vartheta_{\vec{i}}\sigma_1 \pm \sin\vartheta_{\vec{i}}\sigma_y)U_{i_j}\}$, with $\vartheta_{\vec{i}} \in (0, \pi)$ and $j = 1, \ldots, n$. From Ref. [49], we have that by choosing proper phases $\vartheta_{\vec{i}}$ one can obtain that $\langle\text{GHZ}_{0i_{[n-1]}}|\mathcal{S}(A_1^{(\vec{i})}, \ldots, A_n^{(\vec{i})})|\text{GHZ}_{0i_{[n-1]}}\rangle$ exceeds 2^{n-1} for some θ_j satisfying $2\gamma_{0i_{[n-1]}}\delta_{1i_{[n-1]}} > 2^{-(n-1)/2}$. Therefore, using the necessary phases $\vartheta_{\vec{i}}$, we have a violation of Eq. (24) whenever the θ_j satisfy the condition $\min_{i_{[n-1]}}\{2\gamma_{0i_{[n-1]}}\delta_{0i_{[n-1]}}\} > 2^{-(n-1)/2}$.
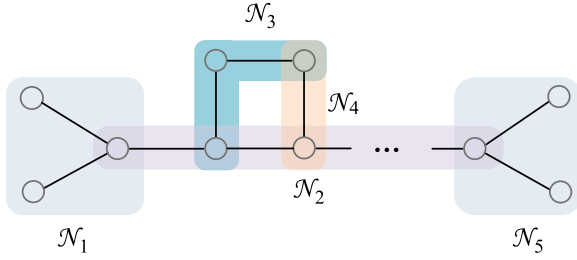
FIG. 4. Decomposition of an arbitrary network into stars and chains. Here, the network can be decomposed into two star subnetworks ($\mathcal{N}_1$ and $\mathcal{N}_5$) and three chain subnetworks ($\mathcal{N}_2$, $\mathcal{N}_3$, and $\mathcal{N}_4$).

VII. INEQUALITIES FOR GENERAL NETWORKS

In general, one can use all the inequalities derived in this work to test for FQNN and FNN in arbitrary networks. For the case of networks composed exclusively of bipartite sources, one can decompose the network in a set of chain and star networks (see Fig. 4 for an example), and test F(Q)NN in the different subnetworks using Eqs. (14), (17), (20), (22), and (24). In particular, cycles can be decomposed in terms of chain networks. Since all sources belong to at least one subnetwork, observation of F(Q)NN in all subnetworks implies that the full correlation is F(Q)NN.

When the networks contain sources that distribute systems to more than two parties, one can use the strategy that we used for star networks, namely, building a Svetlichny-type inequality of the form of Eq. (24) for all the parties that share states with parties that receive from the multipartite source, considering as the operators M the projections on all possible outcomes of the parties that are connected by the multipartite source. In this case, if there are k parties that share states with parties that receive from the multipartite source, non-FNN correlations satisfy

$$\left\langle \sum_{\vec{t}} M^{\vec{t}} \mathcal{S}^{(\vec{t})}\left(B_{x_{i_1}}^{(\vec{t})}, \ldots, B_{x_{i_k}}^{(\vec{t})}\right) \right\rangle \leqslant 2^{k-1}. \quad (25)$$

VIII. NOISE ROBUSTNESS

To conclude, we present a few remarks on the robustness to noise of the inequalities developed in this work. To analyze the robustness to experimental imperfections of the inequalities in the paper, we consider quantum realizations based on Werner states, ρ_{v_i}, given by

$$\rho_{v_i} = v_i |\phi_i\rangle\langle\phi_i| + \frac{1 - v_i}{d^2} I_{d^2},$$

where $|\phi_i\rangle$ are arbitrary bipartite entangled pure states with local dimension d, I_d is the $d \times d$ identity matrix, and $0 \leqslant v_i \leqslant 1$, $i = 1, \ldots, m$.

Using the states $|\phi_i\rangle$ and the measurements that achieve the quantum violations described throughout the paper (see, e.g., Sec. IV and Appendix A 3), the corresponding quantities I and J get multiplied by a factor $\prod_{i=1}^{m} v_i$. This implies that, in order to observe a violation, the visibilities of the sources must satisfy $\prod_{i=1}^{m} v_i > \frac{1}{\sqrt{2}}$ when using maximally entangled states. Similarly, using maximally entangled states for $|\phi_i\rangle$ in the $n + 1$-partite star network, ℓ-NN correlations can be

detected via the violation of Eq. (20) for $\prod_{i=1}^{m} v_i > 2^{-(\ell - \frac{n}{2})}$, while ℓ-QNN correlations can be detected via the violation of Eq. (17) whenever $\prod_{i=1}^{m} v_i > 2^{-\frac{\ell}{2}}$. As discussed in Sec. IV, this shows that Eq. (20) cannot be used for detecting FNN (although it can be used for detecting ℓ-NN as long as $\ell > n/2$), while Eq. (17) can be used for detecting ℓ-QNN in a noise-robust manner independently of the number of sources in the network, since the critical visibility only depends on ℓ.

For the case of the linear inequalities for detecting FNN, both allow for violations when using Werner states as long as $\prod_{i=1}^{m} v_i > 1/\sqrt{2}$. When instead using noisy generalized EPR states (8), the critical visibilities needed for observing a violation of Eq. (22) relate to the angles θ_i via

$$\prod_{i=1}^{n} v_i > \frac{1}{\sqrt{1 + \prod_{j=1}^{n-1} \sin^2 2\theta_j}}.$$

Analogously, when decomposing star networks into sets of tripartite chain networks, FNN can in principle be guaranteed when every pair of visibilities satisfies

$$v_i v_j > \frac{1}{\sqrt{1 + \sin^2 2\theta_i \sin^2 2\theta_j}}.$$

Note how, in both cases, the lowest requirements on the visibilities are obtained for $\theta_i = \pi/4 \ \forall \ i$, i.e., for maximally entangled states.

IX. DISCUSSION

In this work we have provided Bell-like inequalities that witness genuinely quantum phenomena in networks, in the sense that their violations by particular distributions guarantee that these cannot be reproduced when at least a number of the sources in the network are classical. We have done so using two different assumptions on the nonclassical sources. First we assumed that they are only constrained by the no-signaling principle. This leads to a hierarchy of levels of network nonlocality, that interpolates from the standard definition of network nonlocality [23] to that of full network nonlocality [35]. Second, we assumed that the sources distribute systems described by quantum mechanics. This gives rise to a hierarchy analogous to the previous one, that has milder requirements for experimental observations and is thus more suitable for applications in quantum communication. More concretely, we have provided Bell-like inequalities whose violation guarantees (i) $n/2$-QNN and ℓ-NN in the n-partite chain network [Eq. (14) and Eq. (A5) in Appendix A, respectively], (ii) ℓ-NN and ℓ-QNN for any ℓ in the $(n + 1)$-partite star network [Eqs. (20) and (17), respectively], and (iii) FNN in arbitrary chain and star networks [Eqs. (22) and (24), respectively].

Being able to guarantee, from observed statistics, the absence of classical sources in a network is an important step in the development of network-based quantum information protocols. Concrete protocols exist, for instance for quantum conference key agreement [50–52] or quantum secret sharing [53,54]. However, their proofs of security rely on guaranteeing global multipartite nonclassicality, which leads to demanding requirements. Our work paves the way to the development of protocols and security proofs that exploit the network structure, which has received limited attention so

far [54–56]. Indeed, consider the $(n + 1)$-partite star network. Following Refs. [54,55] one can bound the predictive power of an adversary for all outcomes $\vec{a}$ by the value obtained in Eq. (17) via

$$D\left(P(\vec{e}\,|\vec{a};\vec{x}, y, \vec{z}), \prod_{i=1}^{n} P(e_i|z_i)\right) \leqslant (2 - \mathcal{S}_\ell)n,$$

where $D(p, q)$ is the total variation distance between distributions p and q, and $\mathcal{S}_\ell$ is the value achieved for Eq. (17). By substituting the bounds for $\mathcal{S}_\ell$ in Eq. (17), it is possible to bound the predictive power of an eavesdropper if the existence of ℓ classical sources in the network cannot be discarded. This, and more importantly, the demonstrations of quantum realizations that lead to FNN and FQNN that we show in this work, strongly motivate the development of novel quantum information protocols tailored to networks.

Having inequalities for detecting ℓ-NN and ℓ-QNN allows us to easily see how inserting the assumption that nature is governed by quantum mechanics results in milder conditions to guarantee nonclassicality, which is relevant for experimental realizations. For instance, detection of FNN ($\ell = 1$) in the star network with $n = 3$ via Eq. (20) is not possible with noisy versions of the generalized EPR states considered throughout for any visibility, while detection of FQNN is possible when the product of the states' visibilities exceeds $2^{-1/6} \approx 0.891$.

All the inequalities we have presented require some of the parties to have a choice of input. However, a fundamental aspect of network nonlocality is that it can be realized in setups without input choices [19,24,57–59]. Seeing whether the hierarchies of nonlocality developed here are also present in scenarios without inputs is an interesting problem that we leave open, and that can have consequences useful in applications such as quantum random number generation. Numerical tools allow us to easily address these scenarios for arbitrary networks (see, e.g., Ref. [41] or the computational Appendix of Ref. [35]), thus in principle permitting the exploration of ℓ-(Q)NN beyond the situations discussed here.

APPENDIX A: NONLINEAR INEQUALITIES FOR CHAIN NETWORKS

In this appendix we analyze the cases that are missing in Sec. III B, namely, ℓ-QNN in chain networks with an even number of parties, and ℓ-NN in chain networks with an arbitrary number of parties. The core of the derivations is based on the concept of k-independent sets of sources [31].

1. ℓ-QNL bounds for even number of parties

Let us begin discussing the case when the nonclassical sources distribute quantum systems. An inequality for the case of chains with an odd number of parties was derived in Sec. III B, namely,

$$I^{\frac{2}{n+1}} + J^{\frac{2}{n+1}} \leqslant \begin{cases} 2^{\frac{n-1}{2(n+1)}} & \ell > \frac{n}{2} \\ \sqrt{2} & \ell < \frac{n}{2}. \end{cases}$$

For the case of even n, begin considering that the chain network contains ℓ classical sources with $\ell \geqslant n/2$. Consider the two subnetworks $\mathcal{N}_1$ and $\mathcal{N}_2$, where $\mathcal{N}_1$ contains the parties $A_1, \ldots, A_{n-1}$, and $\mathcal{N}_2$ contains the parties $A_2, \ldots, A_n$. Either subnetwork is a network with an odd number of parties which has, at least, $\ell - 1$ classical sources. From Eqs. (9)–(12), the correlations in $\mathcal{N}_1$ and $\mathcal{N}_2$ satisfy the inequality (12) for $n - 1$ parties, i.e., $I^{\frac{2}{n}} + J^{\frac{2}{n}} \leqslant 2^{\frac{n-2}{2n}}$. One can thus test this inequality in the first $n - 1$ parties and in the last $n - 1$ parties to verify ℓ-QNN with $\ell \geqslant n/2$.

If less than half of the sources in the network are classical, the situation is similar to that in Eq. (13). Namely, the joint correlations generated by either $\mathcal{N}_1$ or $\mathcal{N}_2$ achieve the maximal bound of $\sqrt{2}$ for the inequality (12) with $n - 1$ parties, i.e., $I^{\frac{2}{n}} + J^{\frac{2}{n}} \leqslant \sqrt{2}$. This means that, in the case of an even number of parties and less than half of the sources being classical, it is not currently possible to witness ℓ-QNN with $\ell < n/2$.

2. ℓ-NN bounds

Now we focus on the bounds that are achieved when the only constraint on the nonclassical sources is that they distribute systems that satisfy the no-signaling principle. As in the previous section, we distinguish the cases of networks with odd and even number of parties.

Case 1. Odd n with $n \geqslant 3$. Suppose that the chain network consists of ℓ classical variables with $\ell \geqslant 1$ while the other sources distribute arbitrary nonsignaling systems, as shown in Fig. 2(b). Equation (9) holds for this kind of networks. However, in contrast with the derivation in Sec. III B, for sources distributing nonsignaling systems the objects in Eq. (10) are bounded by

$$\left|\left\langle \prod_{\text{odd } i; i \notin S} A_{x_i}^+ \prod_{\text{even } j} A_{x_j=0} \right\rangle\right| \leqslant 1,$$

$$\left|\left\langle \prod_{\text{odd } i; i \notin S} A_{x_i}^- \prod_{\text{even } j} A_{x_j=1} \right\rangle\right| \leqslant 1. \tag{A1}$$

Take $n = 5$ as an example, and $S = \{1\}$, i.e., that only the source connecting A_1 and A_2 distributes classical randomness. Given a PR box, defined by $\text{PR}(a, b|x, y) = 1/2$ if $a \oplus b = xy$

and 0 otherwise, and defining the total correlation as

$$P(a_2, \ldots, a_5 | x_2, \ldots, x_5) = \mathrm{PR}(a_2, a_3 | x_2, x_3) \mathrm{PR}(a_4, a_5 | x_4, x_5),$$

straightforward evaluation leads to saturating the upper bounds, i.e., $\langle A_{x_2=0}^+ A_{x_3=0}^+ A_{x_4=0}^+ \rangle = 1$ and $\langle A_{x_2=1}^- A_{x_3=1}^- A_{x_5}^- \rangle = 1$. Similar results hold for general $n \geqslant 3$.

Now, let S consist of all odd integers i such that A_i shares classical variables with both A_{i-1} and A_{i+1}. Combining Eq. (9) with the inequalities in Eq. (A1), one finds

$$I^{\frac{2}{n+1}} + J^{\frac{2}{n+1}} \leqslant \left\langle \prod_{i \in S} A_{x_i}^+ \right\rangle^{\frac{2}{n+1}} + \left\langle \prod_{i \in S} A_{x_i}^- \right\rangle^{\frac{2}{n+1}}$$

$$\leqslant \prod_{i \in S} \cos^{\frac{4}{n+1}} \vartheta_i + \prod_{i \in S} \sin^{\frac{4}{n+1}} \vartheta_i \qquad (A2)$$

$$\leqslant \sin^{\frac{4|S|}{n+1}} \left(\frac{\pi}{2} - \frac{1}{|S|} \sum_{i \in S} \vartheta_i \right) + \sin^{\frac{4|S|}{n+1}} \left(\frac{1}{|S|} \sum_{i \in S} \vartheta_i \right) \qquad (A3)$$

$$:= \cos^{\frac{4|S|}{n+1}} \vartheta + \sin^{\frac{4|S|}{n+1}} \vartheta \qquad (A4)$$

$$\leqslant 2^{1 - \frac{2|S|}{n+1}}, \qquad (A5)$$

where in Eq. (A2) we have used that $|\langle A_{x_i}^+ \rangle| \leqslant \cos^2 \vartheta_i$ and $|\langle A_{x_i}^- \rangle| \leqslant \sin^2 \vartheta_i$ for $\vartheta_i \in [0, \pi/2]$ as in the derivation of Eq. (11). The inequality (A3) is from Ref. [31, Lemma 1], namely, $\prod_{i=1}^{k} \sin^{\frac{1}{k}} \vartheta_i \leqslant \sin(k^{-1} \sum_{i=1}^{k} \vartheta_i)$, where $k = |S|$. Note that, in this case, the relevant quantity for the bound is not the number of classical sources in the network, ℓ, but the number of parties which only receive classical systems, $|S|$. In Eq. (A4), we have defined $\vartheta = |S|^{-1} \sum_{i \in S} \vartheta_i$. The last inequality is from the fact that the unique extreme point of the continuous function $\cos^{\frac{4k}{n+1}} \vartheta + \sin^{\frac{4k}{n+1}} \vartheta$ for $\vartheta \in [0, \pi/2]$ is $\vartheta = \pi/4$.

Case 2. Even n with $n \geqslant 4$. As in the case in the previous section, a chain network with an even number of parties can be decomposed into subnetworks consisting of $n-1$ parties. If there are ℓ classical sources, each subnetwork contains no less than $\ell - 1$ classical sources, and thus the $n-1$-partite joint correlations satisfy the inequality (A5) for $n-1$ parties.

3. Quantum violations

Now we prove that it is possible to violate Eq. (14) with realizations consisting of quantum states for models with $\ell \geqslant \frac{n-1}{2}$. In quantum scenarios, suppose that each pair of parties, A_i and A_{i+1}, shares one generalized EPR state defined in Eq. (8), as shown in Fig. 2(c).

For odd $n \geqslant 3$, Ref. [31] gave quantum observables $A_{x_1}, \ldots, A_{x_n}$ such that

$$I^{\frac{2}{n+1}} + J^{\frac{2}{n+1}} = \left(1 + \prod_{i=1}^{n-1} \sin^{\frac{4}{n+1}} 2\theta_i \right)^{1/2}. \qquad (A6)$$

These measurements exceed the bound in Eq. (12) whenever the θ_i satisfy $\Pi_{i=1}^{n-1} \sin 2\theta_i > (2^{\frac{n-1}{n+1}} - 1)^{\frac{n+1}{4}}$, thus certifying ℓ-QNN for long chain quantum networks with $\ell \geqslant (n-1)/2$. Moreover, as explained above, inequality (14) cannot guarantee ℓ-QNN of any quantum network if $\ell < (n-1)/2$.

For even n, consider two connected $n-1$-partite subnetworks $\mathcal{N}_1$ and $\mathcal{N}_2$, where $\mathcal{N}_1$ consists of $A_1, \ldots, A_{n-1}$ and $\mathcal{N}_2$ consists of $A_2, \ldots, A_n$. The strategy of Ref. [31] gives ℓ-QNN correlations in each subnetwork, since it also gives the value in Eq. (A6) for $\ell \geqslant n/2 - 1$.

Note that the bound in the inequality (A5) is larger than the maximal quantum bound [31] if $|S| \leqslant (n+1)/2$ for odd n. From the definition of S before Eq. (A2), if there are at most $\lfloor (n-1)/4 \rfloor$ sources of nonsignaling systems the bound in the inequality (A5) is smaller than the bound in the inequality (14), where $\lfloor x \rfloor$ denotes the largest integer smaller than x. This means that ℓ-NN in chain networks with $\ell \geqslant n - \lfloor (n-1)/4 \rfloor = \lceil (3n-1)/4 \rceil$ may be verified by violating the inequality (14), where $\lceil x \rceil$ denotes the smallest integer larger than x. However, with our results it is not possible to verify FNN in any chain network if it has less than $\lceil (3n-1)/4 \rceil$ classical sources. This result can be extended for n-partite chain networks with even n, again, by using two $(n-1)$-partite connected subnetworks.

APPENDIX B: QUANTUM VIOLATIONS OF LINEAR FULL NETWORK NONLOCALITY INEQUALITIES FOR CHAIN NETWORKS

In this appendix we show quantum strategies that violate the inequalities (21) and (22).

1. The tripartite chain

In quantum scenarios, consider the case where each source distributes generalized EPR states given by Eq. (8). The total state is thus given by

$$|\Phi\rangle = \cos\theta_1 \cos\theta_2 |0000\rangle + \cos\theta_1 \sin\theta_2 |0011\rangle$$
$$+ \sin\theta_1 \cos\theta_2 |1100\rangle + \sin\theta_1 \sin\theta_2 |1111\rangle. \qquad (B1)$$

Let $A_0 = \cos\vartheta \sigma_3 + \sin\vartheta \sigma_1$, $A_1 = \cos\vartheta \sigma_3 - \sin\vartheta \sigma_1$, $B^0 = |\phi^+\rangle\langle\phi^+|$, $B^1 = |\phi^-\rangle\langle\phi^-|$, $B^2 = |\psi^+\rangle\langle\psi^+|$, $B^3 = |\psi^-\rangle\langle\psi^-|$, $C_0 = \sigma_3$, $C_1 = \sigma_1$, and $C_2 = \sigma_1\sigma_3\sigma_1$, where $|\phi^\pm\rangle = \frac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle)$ and $|\psi^\pm\rangle = \frac{1}{\sqrt{2}}(|01\rangle \pm |10\rangle)$. After Bob's local measurement, the joint system shared by Alice and Charlie is, depending on Bob's outcome, one of the following states:

$$|\Psi_0\rangle = r_1(\cos\theta_1 \cos\theta_2 |00\rangle + \sin\theta_1 \sin\theta_2 |11\rangle),$$
$$|\Psi_1\rangle = r_1(\cos\theta_1 \cos\theta_2 |00\rangle - \sin\theta_1 \sin\theta_2 |11\rangle),$$
$$|\Psi_2\rangle = r_2(\cos\theta_1 \sin\theta_2 |01\rangle + \sin\theta_1 \cos\theta_2 |10\rangle),$$
$$|\Psi_3\rangle = r_2(\cos\theta_1 \sin\theta_2 |01\rangle - \sin\theta_1 \cos\theta_2 |10\rangle),$$

with $r_1 = (\cos^2\theta_1 \cos^2\theta_2 + \sin^2\theta_1 \sin^2\theta_2)^{-1/2}$ and $r_2 = (\cos^2\theta_1 \sin^2\theta_2 + \sin^2\theta_1 \cos^2\theta_2)^{-1/2}$. Moreover, we have $p(b=0) = p(b=1) = 1/(2r_1^2)$ and $p(b=2) = p(b=3) = 1/(2r_2^2)$.

Note that $\mathcal{B}_3$ can be written as a different CHSH game for each outcome of Bob. Thus, we have that

$$\text{CHSH}_0 := \langle\Psi_0|A_0C_0 + A_0C_1 + A_1C_0 - A_1C_1|\Psi_0\rangle$$
$$= 2\cos\vartheta + r_1^2\sin\vartheta\sin 2\theta_1\sin 2\theta_2, \qquad \text{(B2)}$$

$$\text{CHSH}_1 := \langle\Psi_1|A_0C_0 - A_0C_1 + A_1C_0 + A_1C_1|\Psi_1\rangle$$
$$= 2\cos\vartheta + r_1^2\sin\vartheta\sin 2\theta_1\sin 2\theta_2, \qquad \text{(B3)}$$

$$\text{CHSH}_2 := \langle\Psi_2|A_0C_1 + A_0C_2 - A_1C_1 + A_1C_2|\Psi_2\rangle$$
$$= 2\cos\vartheta + r_2^2\sin\vartheta\sin 2\theta_1\sin 2\theta_2, \qquad \text{(B4)}$$

$$\text{CHSH}_3 := \langle\Psi_3| - A_0C_1 + A_0C_2 + A_1C_1 + A_1C_2|\Psi_3\rangle$$
$$= 2\cos\vartheta + r_2^2\sin\vartheta\sin 2\theta_1\sin 2\theta_2. \qquad \text{(B5)}$$

Substituting Eqs. (B2)–(B5) into Eq. (21) gives

$$\langle\Phi|\mathcal{B}_3|\Phi\rangle = \sum_i p(b=i)\text{CHSH}_i$$
$$= 2(\cos\vartheta + \sin\vartheta\sin 2\theta_1\sin 2\theta_2).$$

If, for instance, we choose ϑ such that it satisfies $\cos\vartheta = (1 + \sin^2 2\theta_1\sin^2 2\theta_2)^{-1/2}$, we have that $\langle\Phi|\mathcal{B}_3|\Phi\rangle = 2(1 + \sin^2 2\theta_1\sin^2 2\theta_2)^{1/2}$, therefore violating Eq. (21) for any $\theta_1, \theta_2 \in (0, \frac{\pi}{2})$.

2. Long chains

Consider the quantum realization in which all $n-1$ sources in the chain distribute states of the form (8). The total state of the system is

$$|\Phi\rangle = \sum_{i_1,,i_n}\left(\prod_{j=1}^n \alpha_{i_j}\right)|i_1, i_1 i_2, \ldots, i_{n-1} i_n, i_n\rangle$$

$$= \sum_{i_2,\ldots,i_n}\left[\left(\prod_{j=2}^{n-1}\alpha_{i_j}\right)|0, 0i_2, \ldots, i_{n-1}0, 0\rangle + \left(\prod_{j=2}^{n-1}\alpha_{\bar{i}_j}\right)|1, 1i_2, \ldots, i_{n-1}1, 1\rangle + \left(\prod_{j=2}^{n-1}\alpha_{i_j}\right)|0, 0i_2, \ldots, i_{n-1}1, 1\rangle\right.$$

$$\left. + \left(\prod_{j=2}^{n-1}\alpha_{\bar{i}_j}\right)|1, 1i_2, \ldots, i_{n-1}0, 0\rangle\right],$$

where $\bar{j} = 1 \oplus j$, $\alpha_{i_j=0} = \cos\theta_j$, and $\alpha_{i_j=1} = \sin\theta_j$. Moreover, define the quantum measurement observables

$$A_0^{(1)} = \cos\vartheta\sigma_3 + \sin\vartheta\sigma_1, \quad A_1 = \cos\vartheta\sigma_3 - \sin\vartheta\sigma_1,$$
$$M^{0,0} = |\phi^+\rangle\langle\phi^+|, \quad M^{1,1} = |\phi^-\rangle\langle\phi^-|,$$
$$M^{0,1} = |\psi^+\rangle\langle\psi^+|, \quad M^{1,0} = |\psi^-\rangle\langle\psi^-|,$$
$$A_0^{(n)} = \sigma_3, \quad A_1^{(n)} = \sigma_1, \quad A_2^{(n)} = \sigma_1\sigma_3\sigma_1.$$

After all the projection measurements of $A_2, \ldots, A_{n-1}$, each of which has a two-bit outcome $a_2, \ldots, a_{n-1} \in \{00, \ldots, 11\}$, the joint system of the particles held by A_1 and A_n collapses into a state in one of the following four sets:

$$S_{00} := \left\{r_{\bar{i}}\left(\prod_{j=1}^n\alpha_{i_j}|00\rangle + \prod_{j=1}^n\alpha_{\bar{i}_j}|11\rangle\right)\forall a_2, \ldots, a_{n-1} \text{ s.t. } \oplus_{j=2}^{n-1}a_j = 00\right\},$$

$$S_{01} := \left\{r_{\bar{i}}\left(\prod_{j=1}^n\alpha_{i_j}|01\rangle + \prod_{j=1}^n\alpha_{\bar{i}_j}|10\rangle\right)\forall a_2, \ldots, a_{n-1} \text{ s.t. } \oplus_{j=2}^{n-1}a_j = 01\right\},$$

$$S_{10} := \left\{r_{\bar{i}}\left(\prod_{j=1}^n\alpha_{i_j}|01\rangle - \prod_{j=1}^n\alpha_{\bar{i}_j}|10\rangle\right)\forall a_2, \ldots, a_{n-1} \text{ s.t. } \oplus_{j=2}^{n-1}a_j = 10\right\},$$

$$S_{11} := \left\{r_{\bar{i}}\left(\prod_{j=1}^n\alpha_{i_j}|00\rangle - \prod_{j=1}^n\alpha_{\bar{i}_j}|11\rangle\right)\forall a_2, \ldots, a_{n-1} \text{ s.t. } \oplus_{j=2}^{n-1}a_j = 11\right\},$$

where $\oplus$ denotes the bit-wise XOR, and $r_{\bar{i}} = (\Pi_{j=1}^n\alpha_{i_j}^2 + \Pi_{j=1}^n\alpha_{\bar{i}_j}^2)^{-1/2}$.

Let $p_{ij} = \sum_{\oplus_{s=2}^{n-1}a_s=ij}p(a_2, \ldots, a_{n-1})$, i.e., the total probability of all the measurements of $A_2, \ldots, A_{n-1}$ such that the collapsed state is in the set S_{ij}. Similarly to Eqs. (B2)–(B5), for any state $|\phi\rangle$ in one subset S_{0i} with $i = 0, 1$, there exists ϑ

independent of $|\phi\rangle$ such that

$$\mathrm{CHSH}_{|\phi\rangle} := \left\langle A_0^{(1)} A_{2i}^{(n)} + A_0^{(1)} A_{2i+1}^{(n)} + A_1^{(1)} A_{2i}^{(n)} - A_1^{(1)} A_{2i+1}^{(n)} \right\rangle_\phi = 2\cos\vartheta + 2\sin\vartheta\, r_{\bar{i}} \prod_{j=1}^{n} \alpha_{i_j} \geqslant 2\cos\vartheta + 2\sin\vartheta \min_{\oplus_{i=1}^{n-1} a_i = 0i} \left\{ r_{\bar{i}} \prod_{j=1}^{n} \alpha_{i_j} \right\}.$$

(B6)

Now, if we define ϑ such that $\sin\vartheta = \min_{\oplus_{i=1}^{n-1} a_i = 0i} (1 + r_{\bar{i}}^2 \Pi_{j=1}^{n} \alpha_{i_j}^2)^{-1/2}$, it follows from inequality (B6) that

$$\mathrm{CHSH}_{|\phi\rangle} \geqslant 2\sqrt{1 + \min_{\oplus_{i=1}^{n-1} a_i = 0i} \left\{ r_{\bar{i}}^2 \prod_{j=1}^{n} \alpha_{i_j}^2 \right\}} = \min_{\oplus_{i=1}^{n-1} a_i = 0i} \left\{ 2\sqrt{1 + r_{\bar{i}}^2 \prod_{j=1}^{n} \alpha_{i_j}^2} \right\}.$$

Similarly, for any state $|\phi\rangle$ in one subset S_{1i}, it follows the same result as above, just with $-\vartheta$ instead of ϑ. Substituting both in Eq. (22) one has

$$\mathcal{B}_n = \sum_{i,j=0,1} p_{ij} \mathrm{CHSH}_{|\phi\rangle \in S_{ij}} \geqslant 2(p_{00} + p_{10}) \min_{\oplus_{i=1}^{n-1} a_i = 00} \left\{ \sqrt{1 + r_{\bar{i}}^2 \prod_{j=1}^{n} \alpha_{i_j}^2} \right\} + 2(p_{01} + p_{11}) \min_{\oplus_{i=1}^{n-1} a_i = 01} \left\{ \sqrt{1 + r_{\bar{i}}^2 \prod_{j=1}^{n} \alpha_{i_j}^2} \right\}$$

$$> 2 \sum_{i,j=0,1} p_{ij} = 2,$$

violating Eq. (22). We note that this result can be extended for general bipartite entangled pure states [3], using the subspace spanned by two basis states following Ref. [31].

[1] J. S. Bell, On the Einstein Podolsky Rosen paradox, Physics **1**, 195 (1964).

[2] A. Einstein, B. Podolsky, and N. Rosen, Can quantum-mechanical description of physical reality be considered complete? Phys. Rev. **47**, 777 (1935).

[3] N. Gisin, Bell's inequality holds for all non-product states, Phys. Lett. A **154**, 201 (1991).

[4] D. Mayers and, A. Yao, Quantum cryptography with imperfect apparatus, arXiv:quant-ph/9809039.

[5] N. Brunner, D. Cavalcanti, S. Pironio, V. Scarani, and S. Wehner, Bell nonlocality, Rev. Mod. Phys. **86**, 419 (2014).

[6] A. K. Ekert, Quantum cryptography based on Bell's theorem, Phys. Rev. Lett. **67**, 661 (1991).

[7] A. Acín, N. Brunner, N. Gisin, S. Massar, S. Pironio, and V. Scarani, Device-independent security of quantum cryptography against collective attacks, Phys. Rev. Lett. **98**, 230501 (2007).

[8] S. Pironio, A. Acín, S. Massar, A. Boyer de La Giroday, D. N. Matsukevich, P. Maunz, S. Olmschenk, D. Hayes, L. Luo, T. A. Manning, and C. Monroe, Random numbers certified by Bell's theorem, Nature (London) **464**, 1021 (2010).

[9] G. Svetlichny, Distinguishing three-body from two-body non-separability by a Bell-type inequality, Phys. Rev. D **35**, 3066 (1987).

[10] R. Gallego, L. E. Würflinger, A. Acín, and M. Navascués, Operational framework for nonlocality, Phys. Rev. Lett. **109**, 070401 (2012).

[11] J.-D. Bancal, J. Barrett, N. Gisin, and S. Pironio, Definitions of multipartite nonlocality, Phys. Rev. A **88**, 014102 (2013).

[12] A. Acín, J. I. Cirac, and M. Lewenstein, Entanglement percolation in quantum networks, Nat. Phys. **3**, 256 (2007).

[13] H.-J. Kimble, The quantum internet, Nature (London) **453**, 1023 (2008).

[14] N. Sangouard, C. Simon, H. de Riedmatten, and N. Gisin, Quantum repeaters based on atomic ensembles and linear optics, Rev. Mod. Phys. **83**, 33 (2011).

[15] S. Wehner, D. Elkouss, and R. Hanson, Quantum internet: A vision for the road ahead, Science **362**, eaam9288 (2018).

[16] M.-O. Renou, D. Trillo, M. Weilenmann, T. P. Le, A. Tavakoli, N. Gisin, A. Acín, and M. Navascués, Quantum theory based on real numbers can be experimentally falsified, Nature (London) **600**, 625 (2021).

[17] S. Popescu, Bell's inequalities and density matrices: Revealing "hidden" nonlocality, Phys. Rev. Lett. **74**, 2619 (1995).

[18] C. Branciard, N. Gisin, and S. Pironio, Characterizing the nonlocal correlations created via entanglement swapping, Phys. Rev. Lett. **104**, 170401 (2010).

[19] T. Fritz, Beyond Bell's theorem: Correlation scenarios, New J. Phys. **14**, 103001 (2012).

[20] R. Chaves, C. Majenz, and D. Gross, Information-theoretic implications of quantum causal structures, Nat. Commun. **6**, 5766 (2015).

[21] J.-M. A. Allen, J. Barrett, D. C. Horsman, C. M. Lee, and R. W. Spekkens, Quantum common causes and quantum causal models, Phys. Rev. X **7**, 031021 (2017).

[22] J. Åberg, R. Nery, C. Duarte, and R. Chaves, Semidefinite tests for quantum network topologies, Phys. Rev. Lett. **125**, 110505 (2020).

[23] A. Tavakoli, A. Pozas-Kerstjens, M.-X. Luo, and M.-O. Renou, Bell nonlocality in networks, Rep. Prog. Phys. **85**, 056001 (2022).

[24] P. Abiuso, T. Kriváchy, E.-C. Boghiu, M.-O. Renou, A. Pozas-Kerstjens, and A. Acín, Single-photon nonlocality in quantum networks, Phys. Rev. Res. **4**, L012041 (2022).

[25] C. Branciard, D. Rosset, N. Gisin, and S. Pironio, Bilocal versus nonbilocal correlations in entanglement-swapping experiments, Phys. Rev. A **85**, 032119 (2012).

[26] A. Tavakoli, P. Skrzypczyk, D. Cavalcanti, and A. Acín, Nonlocal correlations in the star-network configuration, Phys. Rev. A **90**, 062109 (2014).

[27] C. Branciard, N. Brunner, H. Buhrman, R. Cleve, N. Gisin, S. Portmann, D. Rosset, and M. Szegedy, Classical simulation of entanglement swapping with bounded communication, Phys. Rev. Lett. **109**, 100401 (2012).

[28] R. Chaves, Polynomial Bell inequalities, Phys. Rev. Lett. **116**, 010402 (2016).

[29] D. Rosset, C. Branciard, T. J. Barnea, G. Pütz, N. Brunner, and N. Gisin, Nonlinear Bell inequalities tailored for quantum networks, Phys. Rev. Lett. **116**, 010403 (2016).

[30] N. Gisin, Q. Mei, A. Tavakoli, M.-O. Renou, and N. Brunner, All entangled pure quantum states violate the bilocality inequality, Phys. Rev. A **96**, 020304 (2017).

[31] M.-X. Luo, Computationally efficient nonlinear Bell inequalities for quantum networks, Phys. Rev. Lett. **120**, 140402 (2018).

[32] A. Tavakoli, N. Gisin, and C. Branciard, Bilocal Bell inequalities violated by the quantum elegant joint measurement, Phys. Rev. Lett. **126**, 220401 (2021).

[33] A. Pozas-Kerstjens, R. Rabelo, Ł. Rudnicki, R. Chaves, D. Cavalcanti, M. Navascués, and A. Acín, Bounding the sets of classical and quantum correlations in networks, Phys. Rev. Lett. **123**, 140503 (2019).

[34] I. Šupić, J.-D. Bancal, Y. Cai, and N. Brunner, Genuine network quantum nonlocality and self-testing, Phys. Rev. A **105**, 022206 (2022).

[35] A. Pozas-Kerstjens, N. Gisin, and A. Tavakoli, Full network nonlocality, Phys. Rev. Lett. **128**, 010403 (2022).

[36] E. Håkansson, A. Piveteau, S. Muhammad, and M. Bourennane, Experimental demonstration of full network nonlocality in the bilocal scenario, arXiv:2201.06361.

[37] C.-X. Huang, X.-M. Hu, Y. Guo, C. Zhang, B.-H. Liu, Y.-F. Huang, C.-F. Li, G.-C. Guo, N. Gisin, C. Branciard, and A. Tavakoli, Entanglement swapping and quantum correlations via symmetric joint measurements, Phys. Rev. Lett. **129**, 030502 (2022).

[38] N.-N. Wang, A. Pozas-Kerstjens, C. Zhang, B.-H. Liu, Y.-F. Huang, C.-F. Li, G.-C. Guo, N. Gisin, and A. Tavakoli, Certification of non-classicality in all links of a photonic star network without assuming quantum mechanics, Nat. Commun. **14**, 2153 (2023).

[39] X.-M. Gu, L. Huang, A. Pozas-Kerstjens, Y.-F. Jiang, D. Wu, B. Bai, Q.-C. Sun, M.-C. Chen, J. Zhang, S. Yu, Q. Zhang, C.-Y. Lu, and J.-W. Pan, Experimental full network nonlocality with independent sources and strict locality constraints, Phys. Rev. Lett. **130**, 190201 (2023).

[40] N. Gisin, J.-D. Bancal, Y. Cai, P. Remy, A. Tavakoli, E. Z. Cruzeiro, S. Popescu, and N. Brunner, Constraints on nonlocality in networks from no-signaling and independence, Nat. Commun. **11**, 2378 (2020).

[41] E.-C. Boghiu, E. Wolfe, and A. Pozas-Kerstjens, Inflation: A Python library for classical and quantum causal compatibility, Quantum **7**, 996 (2023).

[42] J. F. Clauser, M. A. Horne, A. Shimony, and R. A. Holt, Proposed experiment to test local hidden-variable theories, Phys. Rev. Lett. **23**, 880 (1969).

[43] D. J. Saunders, A. J. Bennet, C. Branciard, and G. J. Pryde, Experimental demonstration of nonbilocal quantum correlations, Sci. Adv. **3**, e1602743 (2017).

[44] A. J. Short, S. Popescu, and N. Gisin, Entanglement swapping for generalized nonlocal correlations, Phys. Rev. A **73**, 012101 (2006).

[45] M. Weilenmann and R. Colbeck, Self-testing of physical theories, or, is quantum theory optimal with respect to some information-processing task? Phys. Rev. Lett. **125**, 060406 (2020).

[46] N.-D. Mermin, Extreme quantum entanglement in a superposition of macroscopically distinct states, Phys. Rev. Lett. **65**, 1838 (1990).

[47] D. Collins, N. Gisin, S. Popescu, D. Roberts, and V. Scarani, Bell-type inequalities to detect true *n*-body nonseparability, Phys. Rev. Lett. **88**, 170405 (2002).

[48] D. M. Greenberger, M. A. Horne, and A. Zeilinger, *Bell's Theorem, Quantum Theory, and Conceptions of the Universe* (Springer, Dordrecht, 1989), Chap. 10, pp. 69–72.

[49] M. Żukowski, Č. Brukner, W. Laskowski, and M. Wieśniak, Do all pure entangled states violate Bell's inequalities for correlation functions? Phys. Rev. Lett. **88**, 210402 (2002).

[50] F. Hahn, J. de Jong, and A. Pappa, Anonymous quantum conference key agreement, PRX Quantum **1**, 020325 (2020).

[51] S. Das, S. Bäuml, M. Winczewski, and K. Horodecki, Universal limitations on quantum key distribution over a network, Phys. Rev. X **11**, 041016 (2021).

[52] A. Pickston, J. Ho, A. Ulibarrena, F. Grasselli, M. Proietti, C. L. Morrison, P. Barrow, F. Graffitti, and A. Fedrizzi, Conference key agreement in a quantum network, npj Quantum Inf. **9**, 82 (2023).

[53] M. G. M. Moreno, S. Brito, R. V. Nery, and R. Chaves, Device-independent secret sharing and a stronger form of Bell nonlocality, Phys. Rev. A **101**, 052339 (2020).

[54] M.-X. Luo, Fully device-independent model on quantum networks, Phys. Rev. Res. **4**, 013203 (2022).

[55] C. M. Lee and M. J. Hoban, Towards device-independent information processing on general quantum networks, Phys. Rev. Lett. **120**, 020504 (2018).

[56] E. Wolfe, A. Pozas-Kerstjens, M. Grinberg, D. Rosset, A. Acín, and M. Navascués, Quantum inflation: A general approach to quantum causal compatibility, Phys. Rev. X **11**, 021043 (2021).

[57] M.-O. Renou, E. Bäumer, S. Boreiri, N. Brunner, N. Gisin, and S. Beigi, Genuine quantum nonlocality in the triangle network, Phys. Rev. Lett. **123**, 140401 (2019).

[58] S. Boreiri, A. Girardin, B. Ulu, Patryk Lipka-Bartosik, N. Brunner, and P. Sekatski, Towards a minimal example of quantum nonlocality without inputs, Phys. Rev. A **107**, 062413 (2023).

[59] A. Pozas-Kerstjens, A. Girardin, T. Kriváchy, A. Tavakoli, and N. Gisin, Post-quantum nonlocality in the minimal triangle scenario, New J. Phys. **25**, 113037 (2023).